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CHARACTERIZATIONS AND PROPERTIES OF WEAK CORE INVERSES IN RINGS WITH INVOLUTION

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ABSTRACT. In a ring with involution, we first investigate some necessary and sufficient conditions under which Jacobson's lemma for weak core inverse holds true. Then, we present reverse order laws of weak core inverses and some equivalent conditions under which absorption laws of weak core inverses hold true. Finally, some equivalent characterizations of a^* commuting with the weak core inverse of a are shown, which improve the relevant result of Zhou et al. [Weak group inverses and partial isometries in proper $*$-rings. Linear Multilinear Algebra (2021)].

1. Introduction

Moore-Penrose inverses [36] and Drazin inverses [12] are two types of classical generalized inverses and have been thoroughly studied since they were defined (see, e.g., [3–5, 9, 11, 23, 30]). Afterwards, some new kinds of generalized inverses, such as core inverses [1], core-EP inverses [29], pseudo core inverses [20], DMP inverses [28], weak group inverses [41, 43] and m-weak group inverses [46], were introduced and have attracted widespread attention (for more details, see, e.g., [7, 16–18, 27, 34]).

The subject of this article is to investigate some characterizations and properties of the weak core inverse in a ring with involution. The concept of weak core inverses of complex matrices was first introduced by Ferreyra et al. [15] and later was generalized to a ring with involution by Zhou and Chen [45]. The weak core inverse is a new extension of the concept of the core inverse and different from other generalized inverses (see [15, Example 3.10]). This is an interesting research topic and it deserves further study. For example, Mosić and Stanimirović [35] provided various novel expressions in terms of Moore-Penrose inverses, integral and limit representations as well as perturbation formulae of weak core inverses for complex matrices. Fu et al. [19] investigated some new characterizations of the weak core inverse by using ranges, null spaces and matrix equations.

Throughout the paper, R is a unitary ring with involution $*$. The motivations and outline of this paper are as follows.

In Section 2, we give some definitions of relevant generalized inverses and necessary lemmas.

Given any $a, b \in R$, it is well-known as Jacobson's lemma that if $1-ab$ is invertible, then so is $1-ba$. Moreover, these two inverses are related by the following formula

$$(1-ba)^{-1} = 1 + b(1-ab)^{-1}a.$$

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It is natural to ask whether Jacobson's lemma for various kinds of generalized inverses is valid and many scholars paid attention to this topic. To be specific, Jacobson's lemma for regular elements holds with a related expression, and that for reflexive inverses (see, e.g., [6, Theorem 3.4]), group inverses (see, e.g., [6, Theorem 3.5]) and Drazin inverses (see, e.g., [6, Theorem 3.6] and [10, Theorem 2.2]) were established, respectively. Lam and Nielsen [26] also investigated Jacobson's lemma for Drazin inverses and expressed a simple formula. However, neither Jacobson's lemma for Moore-Penrose inverses nor that for pseudo core inverses hold (see [6, Example 3.10] and [38, Example 3.7]). This inspires scholars to consider under what conditions Jacobson's lemma for these generalized inverses holds true. For example, Shi et al. [38] presented several necessary and sufficient conditions under which $1-ba$ is Moore-Penrose invertible when $1-ab$ has a Moore-Penrose inverse in a ring with involution. Additionally, they also investigated some equivalent conditions under which Jacobson's lemma for pseudo core inverses is valid. Motivated by these discussion, we aim to study some necessary and sufficient conditions under which Jacobson's lemma for weak core inverse holds true, and express a similar related formula in Section 3.

If $a,b \in R$ are invertible, then we have the following two properties:

$$(ab)^{-1} = b^{-1}a^{-1}$$

is known as the reverse order law and

$$a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}$$

is known as the absorption law. However, these properties for generalized inverses, such as Moore-Penrose inverses, Drazin inverses, pseudo core inverses, weak group inverses, may not hold in general. Many scholars are devoted to finding some conditions which guarantee reverse order laws and absorption laws for these generalized inverses to hold. For example, reverse order laws of Moore-Penrose inverses were investigated in [14, 25, 33]. Gao et al. [20, 22] studied reverse order laws and absorption laws of pseudo core inverses, absorption laws of Drazin inverses. Wang [40] illustrated reverse order laws of Drazin inverses. Zhou et al. [44] demonstrated reverse order laws of weak group inverses. Inspired by the discussion above, we investigate reverse order laws and absorption laws for weak core inverses in Section 4.

In Section 5, we are committed to investigating the case of $a^* \in R$ commuting with the generalized inverse of a. This idea originates from the study of $a^* \in R$ commuting with some generalized inverses. For example, Hartwig and Spindelböck [24] investigated the class of complex star-dagger matrices for which A^* and $A^\dagger$ commute. Mosić and Djordjević [31] presented sufficient conditions for Moore-Penrose invertible element in a ring with involution to be star-dagger. Additionally, Zhou et al. [44] provided equivalent characterizations for a^* commuting with weak group inverses in proper $*$-rings (i.e., R is a proper $*$-ring if $a^*a = 0$ implies $a = 0$ for any $a \in R$).

2. Preliminaries

Throughout this paper, we use $\mathbb{N}$ and $\mathbb{N}^+$ to denote the sets of all nonnegative integers and positive integers, respectively. In this section, we present some definitions of relevant generalized inverses and auxiliary lemmas.

Definition 2.1. [36] *Let $a \in R$. Then a is said to be Moore-Penrose invertible if there exists $x \in R$ such that the following four equations*

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ (xa)^* = xa$$

hold. Such x is unique when it exists, and is called the Moore-Penrose inverse of a, denoted by $a^\dagger$.

An element x is called an outer inverse of a if there exists $x \in R$ satisfying Equation (2). An element $a \in R$ is said to be $\{1,3\}$-invertible if there is $a^{(1,3)} \in R$ satisfying Equations (1) and (3), in which case, $a^{(1,3)}$ is called a $\{1,3\}$-inverse of a. Similarly, the $\{1,4\}$-inverse of a is defined. We use the symbols $a\{1,3\}$, $a\{1,4\}$ to denote the sets of all $\{1,3\}$-inverses and $\{1,4\}$-inverses of a, respectively. In addition, the symbols $R^{\{1,3\}}$ and $R^{\{1,4\}}$ denote the sets of all $\{1,3\}$-invertible and $\{1,4\}$-invertible elements of R, respectively.

Lemma 2.2. [23] *Let $a \in R$. Then $a \in R^{\{1,3\}}$ with a $\{1,3\}$-inverse x if and only if $x^*a^*a = a$.*

Definition 2.3. [12] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k,\ ax^2 = x,\ ax = xa, \tag{2.1}$$

then a is said to be Drazin invertible. Such x is unique when it exists, and is called the Drazin inverse of a, denoted by a^D.

If k is the smallest positive integer such that Equations (2.1) hold, then k is called the Drazin index of a and denoted by $\mathrm{ind}(a)$.

Definition 2.4. [20, Definition 1.1] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k,\ ax^2 = x,\ (ax)^* = ax, \tag{2.2}$$

then a is said to be pseudo core invertible. Such x is unique when it exists, and is called the pseudo core inverse of a, denoted by $a^{\textcircled{D}}$.

The smallest positive integer k satisfying Equations (2.2) is called the pseudo core index of a, which coincides with its Drazin index, and still denoted by $\mathrm{ind}(a)$. In particular, if $\mathrm{ind}(a) = 1$, then x is called the core inverse of a, denoted by $a^{\textcircled{\#}}$.

Definition 2.5. [43, Definition 3.1] *Let $a \in R$. Then a is said to be weak group invertible if there exist $x \in R$ and $k \in \mathbb{N}^+$ satisfying*

$$xa^{k+1} = a^k,\ ax^2 = x,\ (a^k)^*a^2x = (a^k)^*a.$$

Any such x is called the weak group inverse of a.

Definition 2.6. [46, Definition 4.1] *Let $m \in \mathbb{N}$. An element $a \in R$ is said to be m-weak group invertible if there exist $x \in R$ and $k \in \mathbb{N}^+$ satisfying*

$$xa^{k+1} = a^k,\ ax^2 = x,\ (a^k)^*a^{m+1}x = (a^k)^*a^m. \tag{2.3}$$

Any such x is called the m-weak group inverse of a.

When the m-weak group inverse (resp., weak group inverse) of a is unique, we use $a^{ⓦ_m}$ (resp., $a^{ⓦ}$) to denote the unique m-weak group inverse (resp., weak group inverse) of a (see Lemma 2.9).

Comparing Definition 2.5 with Definition 2.6, the definition of the 1-weak group inverse is exactly that of the weak group inverse. If k is the smallest positive integer such that Equations (2.3) hold, then k is called the m-weak group index of a. If a is m-weak group invertible, then a is Drazin invertible and the m-weak group index of a is equal to the Drazin index of a. Therefore, we still use $\mathrm{ind}(a)$ to denote the m-weak group index of a.

Throughout this paper, the symbols $R^{\#\!\!\!\bigcirc}, R^D, R^{Ⓓ}, R^{ⓦ}$ denote the sets of all core invertible, Drazin invertible, pseudo core invertible and weak group invertible elements of R, respectively.

Lemma 2.7. [38, Theorem 3.3] *If $a \in R^D$, then $a \in R^{Ⓓ}$ if and only if $aa^D \in R^{\{1,3\}}$. In this case, $aa^{Ⓓ} \in (aa^D)\{1,3\}$ and $a^{Ⓓ} = a^D(aa^D)^{(1,3)}$ for any $(aa^D)^{(1,3)} \in (aa^D)\{1,3\}$.*

Lemma 2.8. ([21, Theorem 3.1], **Core-EP decomposition**) *Let $a \in R^{Ⓓ}$. Then $a = a_1 + a_2$, where*

(i) *$a_1^{\#\!\!\!\bigcirc}$ exists.*
(ii) *$a_2^m = 0$ for some $m \in \mathbb{N}^+$.*
(iii) *$a_1^* a_2 = a_2 a_1 = 0$.*

In this case, $a_1^{\#\!\!\!\bigcirc} = a^{Ⓓ}$, $a_1^{\#} = (a^{Ⓓ})^2 a$, $a_1 = aa^{Ⓓ}a$ and $a_2 = a - aa^{Ⓓ}a$.

In the following of this paper, unless specifically noted, we will restrict $a_1 = aa^{Ⓓ}a$ and $a_2 = a - aa^{Ⓓ}a$ when $a \in R^{Ⓓ}$ according to Lemma 2.8.

Lemma 2.9. [46, Corollary 4.11] *Let $a \in R$ and $m \in \mathbb{N}^+$. If $a \in R^{Ⓓ}$, then a has a unique m-weak group inverse.*

In addition, it was shown in [46, Corollary 4.3] that $a \in R^{Ⓓ}$ if and only if $a \in R^{ⓦ_0}$, in this case, $a^{ⓦ_0} = a^{Ⓓ}$. Furthermore, $a^{ⓦ} = (a^{Ⓓ})^2 a$ when $a \in R^{Ⓓ}$ according to [46, Proposition 4.8]

Definition 2.10. [45, Definition 3.6] *Let $a \in R$. If $a \in R^{ⓦ} \cap R^{\{1,3\}}$, then a is said to be weak core invertible. The unique $x \in R$ satisfying the following equations*

$$xax = x, \ ax = aa^{ⓦ}aa^{(1,3)}, \ xa = a^{ⓦ}a$$

is called the weak core inverse of a and denoted by a^{wC}.

We use R^{wC} to denote the set of all weak core invertible elements of R. From [45], we know that

$$R^{wC} \subseteq R^{Ⓓ} \subseteq R^{ⓦ} \subseteq R^D \text{ and } a^{wC} = a^{ⓦ}aa^{(1,3)} = (a^{Ⓓ})^2 a^2 a^{(1,3)}.$$

Lemma 2.11. [45, Corollary 3.2] *Let $a \in R$. Then $a \in R^{ⓦ} \cap R^{\{1,3\}}$ if and only if $a \in R^{Ⓓ}$ and $a_2 \in R^{\{1,3\}}$.*

Let $a \in R^D$. In [45], Zhou and Chen wrote

$$T_l(a) = \{x \in R : xa^{k+1} = a^k, ax^2 = x \text{ for some } k \in \mathbb{N}^+\},$$

which is also equal to $\{x \in R : xa^{\mathrm{ind}(a)+1} = a^{\mathrm{ind}(a)}, ax^2 = x\}$. According to [20, Lemma 2.1], if $a \in R^D$ and $x \in T_l(a)$, then we get that

$$xax = x \text{ and } ax = a^m x^m \text{ for arbitrary } m \in \mathbb{N}^+.$$

Lemma 2.12. [45, Lemma 2.2] *Let $a \in R^D$, $k_1,\dots,k_n,s_1,\dots,s_n \in \mathbb{N}$ and $x_1,\dots,x_n \in T_l(a)$. If $s_n \neq 0$, then*

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s,$$

where $k = \sum\limits_{i=1}^{n} k_i$ and $s = \sum\limits_{i=1}^{n} s_i$.

3. The relation between $1-ab \in R^{wC}$ and $1-ba \in R^{wC}$

In [38, Theorem 3.10], Shi et al. presented some necessary and sufficient conditions under which $1-ba$ has a pseudo core inverse when $1-ab$ is pseudo core invertible, and gave a formula for $(1-ba)^{\textcircled{D}}$ in terms of $(1-ab)^{\textcircled{D}}$. As follows in Lemma 3.1, we improve this result and give a new formula.

Lemma 3.1. *Let $a,b \in R$. If $\alpha = 1-ab \in R^{\textcircled{D}}$, then the following conditions are equivalent.*

(i) $\beta = 1-ba \in R^{\textcircled{D}}$.

(ii) $b(1-\alpha\alpha^D)ra \in R^{\{1,4\}}$, *where $r = 1+\alpha+\cdots+\alpha^{k-1}$ and $k = \mathrm{ind}(\alpha)$.*

(iii) $u = (1-\alpha\alpha^{\textcircled{D}})aa^* + \alpha\alpha^{\textcircled{D}}$ *is invertible.*

In this case, $\beta^{\textcircled{D}} = (1+b\alpha^D a)\left(1-a^*u^{-1}(1-\alpha\alpha^{\textcircled{D}})a\right)$.

Proof. (i) $\Leftrightarrow$ (ii). It can be found in [38, Theorem 3.10].

(ii) $\Leftrightarrow$(iii). By a similar method to the proof of (ii) $\Leftrightarrow$(iii) in [38, Theorem 3.10], we can get

$$a\left(br(1-\alpha\alpha^D)\right) = (1-\alpha)r(1-\alpha\alpha^D) = (1-\alpha^k)(1-\alpha\alpha^D) = (1-\alpha\alpha^D),$$
$$\left((1-\alpha\alpha^D)a\right)br = (1-\alpha\alpha^D)(1-\alpha)r = (1-\alpha\alpha^D)(1-\alpha^k) = (1-\alpha\alpha^D).$$

Since $\alpha\alpha^D = \alpha^D\alpha$, we have $b(1-\alpha\alpha^D)ra = br(1-\alpha\alpha^D)a$. From $1-\alpha\alpha^{\textcircled{D}} \in (1-\alpha\alpha^D)\{1,4\}$, it follows that $b(1-\alpha\alpha^D)ra \in R^{\{1,4\}}$ if and only if $u = (1-\alpha\alpha^{\textcircled{D}})aa^* + \alpha\alpha^{\textcircled{D}}$ is invertible by [38, Theorem 3.8]. Then a similar argument can derive

$$\beta^{\textcircled{D}} = (1+b\alpha^D a)\left(1-a^*u^{-1}(1-\alpha\alpha^{\textcircled{D}})a\right).$$

□

Next, we present some necessary and sufficient conditions under which $1-ba \in R^{wC}$ when $1-ab \in R^{wC}$, and also give the formulae of $(1-ba)^{wC}$.

Theorem 3.2. *Let $a,b \in R$. If $\alpha = 1-ab \in R^{wC}$, then the following conditions are equivalent.*

(i) $\beta = 1-ba \in R^{wC}$.

(ii) $b(1-\alpha\alpha^D)ra \in R^{\{1,4\}}$ *and $b\alpha_r^\pi a \in R^{\{1,4\}}$, where $r = 1+\alpha+\cdots+\alpha^{k-1}$ and $k = \mathrm{ind}(\alpha)$.*

(iii) $u = \alpha^\pi aa^* + 1 - \alpha^\pi$ *and $v = \alpha_r^\pi aa^* + 1 - \alpha_r^\pi$ are invertible.*

In this case,

$$\begin{aligned}\beta^{wC} &= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^\pi\alpha a\right)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\ &= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^2(\alpha^{(1,3)}-\alpha^{wC})a\right)\left(1-a^*v^{-1}\alpha_r^\pi a\right),\end{aligned}$$

where $\alpha^\pi = 1-\alpha\alpha^{\textcircled{D}}$ and $\alpha_r^\pi = 1-\alpha\alpha^{(1,3)}$.

Proof. According to Lemma 3.1 and [38, Theorem 5.6], it is easy to derive that (i)⇔(ii)⇔(iii).

For the expressions of β^{wC}, we first calculate $(\beta^{\mathbb{D}})^2$. Write $\alpha^\pi = 1-\alpha\alpha^{\mathbb{D}}$ and $\alpha_r^\pi = 1-\alpha\alpha^{(1,3)}$. Since $\alpha^\pi u = \alpha^\pi aa^*$ and $\alpha_r^\pi v = \alpha_r^\pi aa^*$, we get

$$\alpha^\pi = \alpha^\pi aa^* u^{-1} \text{ and } \alpha_r^\pi = \alpha_r^\pi aa^* v^{-1}.$$

Then also by Lemma 2.12, we can obtain that

$$\begin{aligned}
(\beta^{\mathbb{D}})^2 &= (1+b\alpha^D a)\left(1-a^*u^{-1}\alpha^\pi a\right)(1+b\alpha^D a)\left(1-a^*u^{-1}\alpha^\pi a\right)\\
&= (1+b\alpha^D a)\left(1+b\alpha^D a - a^*u^{-1}\alpha^\pi a - a^*u^{-1}\alpha^\pi(1-\alpha)\alpha^D a\right)\left(1-a^*u^{-1}\alpha^\pi a\right)\\
&= (1+b\alpha^D a)\left(1+b\alpha^D a - a^*u^{-1}\alpha^\pi a - a^*u^{-1}\alpha^\pi\alpha^D(1-\alpha)a\right)\left(1-a^*u^{-1}\alpha^\pi a\right)\\
&= (1+b\alpha^D a)\left(1+b\alpha^D a - a^*u^{-1}\alpha^\pi a\right)\left(1-a^*u^{-1}\alpha^\pi a\right)\\
&= (1+b\alpha^D a)^2\left(1-a^*u^{-1}\alpha^\pi a\right) - (1+b\alpha^D a)\left(a^*u^{-1}\alpha^\pi a\right)\left(1-a^*u^{-1}\alpha^\pi a\right)\\
&= (1+b\alpha^D a)^2\left(1-a^*u^{-1}\alpha^\pi a\right) - (1+b\alpha^D a)a^*u^{-1}\left(\alpha^\pi - \alpha^\pi aa^*u^{-1}\alpha^\pi\right)a\\
&= (1+b\alpha^D a)^2\left(1-a^*u^{-1}\alpha^\pi a\right).
\end{aligned}$$

Therefore, combing [38, Theorem 5.6], it follows that

$$\begin{aligned}
\beta^{wC} &= (\beta^{\mathbb{D}})^2\beta^2\beta^{(1,3)}\\
&= (1+b\alpha^D a)^2\left(1-a^*u^{-1}\alpha^\pi a\right)(1-ba)^2(1+b\alpha^{(1,3)}a)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^\pi\alpha a\right)(1-b\alpha_r^\pi a)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^\pi\alpha a\right)\left(\left(1-a^*v^{-1}\alpha_r^\pi a\right) - b\alpha_r^\pi a\left(1-a^*v^{-1}\alpha_r^\pi a\right)\right)\\
&= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^\pi\alpha a\right)\left(\left(1-a^*v^{-1}\alpha_r^\pi a\right) - b\left(\alpha_r^\pi - \alpha_r^\pi aa^*v^{-1}\alpha_r^\pi\right)a\right)\\
&= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^\pi\alpha a\right)\left(1-a^*v^{-1}\alpha_r^\pi a\right).
\end{aligned}$$

In addition, we can give another formula of β^{wC}, i.e.,

$$\begin{aligned}
\beta^{wC} &= (\beta^{\mathbb{D}})^2\beta^2\beta^{(1,3)}\\
&= (1+b\alpha^D a)^2\left(1-a^*u^{-1}\alpha^\pi a\right)(1-ba)^2(1+b\alpha^{(1,3)}a)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\left((1-ba)^2 - a^*u^{-1}\alpha^\pi\alpha^2 a\right)(1+b\alpha^{(1,3)}a)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\Big((1-ba)^2(1+b\alpha^{(1,3)}a) - a^*u^{-1}\alpha^\pi\alpha^2 a(1+b\alpha^{(1,3)}a)\Big)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\Big(1-b(1+\alpha\alpha_r^\pi)a - a^*u^{-1}(\alpha^2-\alpha\alpha^{\mathbb{D}}\alpha^2)(\alpha^{(1,3)}+\alpha_r^\pi)a\Big)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\Big(1-b(1+\alpha\alpha_r^\pi)a - a^*u^{-1}(\alpha^2-\alpha^2\alpha^{wC}\alpha)(\alpha^{(1,3)}+\alpha_r^\pi)a\Big)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\Big(1-b(1+\alpha\alpha_r^\pi)a - a^*u^{-1}\alpha^2(1-\alpha^{wC}\alpha)(\alpha^{(1,3)}+\alpha_r^\pi)a\Big)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&= (1+b\alpha^D a)^2\Big(1-ba-a^*u^{-1}\alpha^2(1-\alpha^{wC}\alpha)\alpha^{(1,3)}a\Big)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\
&\quad - (1+b\alpha^D a)^2\left(b\alpha + a^*u^{-1}\alpha^2(1-\alpha^{wC}\alpha)\right)\alpha_r^\pi a\left(1-a^*v^{-1}\alpha_r^\pi a\right).
\end{aligned}$$

Since $\alpha_r^\pi = \alpha_r^\pi aa^*v^{-1}$, we have $\alpha_r^\pi a\left(1-a^*v^{-1}\alpha_r^\pi a\right)=0$. Then it follows that

$$\begin{aligned}\beta^{wC} &= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^2(1-\alpha^{wC}\alpha)\alpha^{(1,3)}a\right)\left(1-a^*v^{-1}\alpha_r^\pi a\right)\\ &= (1+b\alpha^D a)^2\left(1-ba-a^*u^{-1}\alpha^2(\alpha^{(1,3)}-\alpha^{wC})a\right)\left(1-a^*v^{-1}\alpha_r^\pi a\right).\end{aligned}$$

□

4. Reverse order laws and absorption laws of weak core inverses

Let $a,b\in R$ with $ab=ba$ and $ab^*=b^*a$. Gao and Chen [20, Theorem 4.3] proved that if $a,b\in R^{\mathbb{D}}$, then $(ab)^{\mathbb{D}}=a^{\mathbb{D}}b^{\mathbb{D}}=b^{\mathbb{D}}a^{\mathbb{D}}$. Zhou et al. [44, Theorem 5.2] showed that $(ab)^{\circledW}=a^{\circledW}b^{\circledW}=b^{\circledW}a^{\circledW}$ when $a,b\in R^{\circledW}$ in a proper $*$-ring. In this section, we first investigate the reverse order law of weak core inverses in R.

Example 4.1. *Let $R=\mathbb{C}^{2\times 2}$ with the transpose as the involution. Take $a=\begin{pmatrix}1&1\\0&0\end{pmatrix}\in R$, $b=a^*$. Then it is easy to verify that $ab^*=b^*a$ and $(ab)^\dagger=b^\dagger a^\dagger$ but $ab\neq ba$. In addition, by computation, we get $a,b\in R^{wC}$ with $a^{wC}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $b^{wC}=\begin{pmatrix}\frac12&\frac12\\\frac12&\frac12\end{pmatrix}$. Moreover, we obtain $ab\in R^{wC}$ with $(ab)^{wC}=\begin{pmatrix}\frac12&0\\0&0\end{pmatrix}$. However $(ab)^{wC}\neq b^{wC}a^{wC}$.*

Example 4.1 shows that the commutativity property $ab=ba$ is required for the reverse order law of weak core inverses. That is also to say, only satisfying the condition that $(ab)^\dagger=b^\dagger a^\dagger$ does not guarantee the reverse order law of weak core inverses to hold.

Lemma 4.2. [8, Lemma 3.1], [47, Proposition 5.11] *Let $a,b,x\in R$ with $ax=xb$ and $a^*x=xb^*$.*
(i) *If $a,b\in R^{\{1,3\}}$, then $aa^{(1,3)}x=xbb^{(1,3)}$.*
(ii) *If $a,b\in R^{\mathbb{D}}$, then $a^{\mathbb{D}}x=xb^{\mathbb{D}}$.*

In Lemma 4.2, by induction, it follows that $(a^{\mathbb{D}})^m x=x(b^{\mathbb{D}})^m$ for arbitrary $m\in\mathbb{N}^+$ when $a,b\in R^{\mathbb{D}}$.

Proposition 4.3. *Let $a,b,x\in R$ with $ax=xb$ and $a^*x=xb^*$. If $a,b\in R^{wC}$, then $a^{wC}x=xb^{wC}$.*

Proof. Since $a,b\in R^{wC}$, we get $a,b\in R^{\mathbb{D}}$. By Lemma 4.2, it follows that

$$\begin{aligned}a^{wC}x &= (a^{\mathbb{D}})^2a^2a^{(1,3)}x=(a^{\mathbb{D}})^2axbb^{(1,3)}\\ &= (a^{\mathbb{D}})^2xb^2b^{(1,3)}=x(b^{\mathbb{D}})^2b^2b^{(1,3)}=xb^{wC}.\end{aligned}$$

□

Corollary 4.4. *Let $a,b\in R$ with $ab=ba$ and $ab^*=b^*a$. If $b\in R^{wC}$, then $ab^{wC}=b^{wC}a$.*

Theorem 4.5. *Let $a,b\in R^{wC}$ with $ab=ba$ and $ab^*=b^*a$. Then $ab\in R^{wC}$ and*

$$(ab)^{wC}=a^{wC}b^{wC}=b^{wC}a^{wC}.$$

Proof. By Corollary 4.4, we have $b^{wC}a=ab^{wC}$ and $a^{wC}b=ba^{wC}$. Since $b^*a^*=a^*b^*$ and $ab^*=b^*a$, we obtain that $a^{wC}b^*=b^*a^{wC}$, which together with $a^{wC}b=ba^{wC}$, implies $a^{wC}b^{wC}=b^{wC}a^{wC}$.

Also by Lemma 4.2, we can get $aa^{(1,3)}bb^{(1,3)}=bb^{(1,3)}aa^{(1,3)}$ similarly. Then it follows that

$$(ab)(b^{(1,3)}a^{(1,3)})(ab) = bb^{(1,3)}aa^{(1,3)}ab = bb^{(1,3)}ab = ab,$$

$$(abb^{(1,3)}a^{(1,3)})^* = (bb^{(1,3)}aa^{(1,3)})^* = aa^{(1,3)}bb^{(1,3)} = bb^{(1,3)}aa^{(1,3)} = abb^{(1,3)}a^{(1,3)}.$$

Hence $ab \in R^{\{1,3\}}$ with $b^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$.

Since $ab \in R^{\textcircled{\tiny D}}$, it follows that $ab \in R^{\textcircled{\tiny w}}$, and hence $ab \in R^{wC}$. In addition, $bb^{(1,3)}a = abb^{(1,3)}$ implies $bb^{(1,3)}a^* = a^*bb^{(1,3)}$, then we get $bb^{(1,3)}a^{\textcircled{\tiny D}} = a^{\textcircled{\tiny D}}bb^{(1,3)}$. Hence by Lemma 4.2 and [20, Theorem 4.3], we can get that

$$\begin{aligned}(ab)^{wC} &= \left((ab)^{\textcircled{\tiny D}}\right)^2 (ab)^2 (ab)^{(1,3)} = \left(b^{\textcircled{\tiny D}}a^{\textcircled{\tiny D}}\right)^2 (ab)^2 b^{(1,3)}a^{(1,3)} \\ &= (b^{\textcircled{\tiny D}})^2 b^2 b^{(1,3)} (a^{\textcircled{\tiny D}})^2 a^2 a^{(1,3)} = b^{wC}a^{wC}.\end{aligned}$$

□

Remark 4.6. *In [22], Gao et al. investigated the reverse order law of pseudo core inverses under a weaker condition that $a, b \in R^{\textcircled{\tiny D}}$ with $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$. However, when $a, b \in R^{wC}$ with $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$, ab may not be weak core invertible. Thus, we do not consider the reverse order law of weak core inverses in this weaker condition. For example, let $R = \mathbb{Z}^{3\times 3}$ with the transpose as the involution. Take $a = \begin{pmatrix}1&0&0\\0&0&0\\1&0&1\end{pmatrix}$, $b = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$. Then it is easy to check that $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$ however $ab \neq ba$ and $ab^* \neq b^*a$. Moreover, $a, b \in R^{wC}$. But $ab = \begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix} \notin R^{\{1,3\}}$ by Lemma 2.2, and hence $ab \notin R^{wC}$.*

From [20, Theorem 4.4], it was shown that if $a, b \in R^{\textcircled{\tiny D}}$ with $ab = ba = 0$ and $a^*b = 0$, then $a + b \in R^{\textcircled{\tiny D}}$ with $(a+b)^{\textcircled{\tiny D}} = a^{\textcircled{\tiny D}} + b^{\textcircled{\tiny D}}$. In [44, Theorem 5.3], Zhou et al. also proved the relevant result for weak group inverses, i.e., if R is a proper $*$-ring and $a, b \in R^{\textcircled{\tiny w}}$ with $ab = ba = 0$ and $a^*b = 0$, then $a + b \in R^{\textcircled{\tiny w}}$ with $(a+b)^{\textcircled{\tiny w}} = a^{\textcircled{\tiny w}} + b^{\textcircled{\tiny w}}$. However, this property may not hold for weak core inverses, under the condition that $a, b \in R^{wC}$ with $ab = ba = 0$ and $a^*b = 0$ (see Example 4.7).

Example 4.7. *Let $R = \mathbb{Z}^{3\times 3}$ with the transpose as the involution. Take*

$$a = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\ b = \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}.$$

Then it is easy to check that $a, b \in R^{wC}$ with $ab = ba = 0$ and $a^*b = 0$. By computation, we get $a + b = \begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$ and $(a+b)^*(a+b) = \begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}$. Since $a + b \notin R(a+b)^*(a+b)$, we have $a + b \notin R^{\{1,3\}}$ by Lemma 2.2. Hence $a + b$ is not weak core invertible.

Example 4.8. *Let $R = \mathbb{Z}_6$ with the involution induced from the identity involution on R. Take $a = 4$ and $b = 1$. Then $a^{wC} = 4$ and $b^{wC} = 1$. Hence $a^{wC}(a+b)b^{wC} \neq a^{wC} + b^{wC}$.*

Example 4.8 shows that the absorption law of weak core inverses is not valid, which motivates us to find some necessary and sufficient conditions under which the absorption laws for weak core inverses hold.

Denote $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$. The right (resp., left) annihilator of a is defined by $a^\circ = \{x \in R : ax = 0\}$ (resp., ${}^\circ a = \{x \in R : xa = 0\}$).

Lemma 4.9. [37, Lemma 2.5] *Let $a, b \in R$.*

(i) *If $aR \subseteq bR$, then ${}^\circ b \subseteq {}^\circ a$.*
(ii) *If $Ra \subseteq Rb$, then $b^\circ \subseteq a^\circ$.*

In the following, we first give a general case of absorption laws when $a, b \in R^D$ with $x \in T_l(a)$ and $y \in T_l(b)$.

Theorem 4.10. *Let $a, b \in R^D$ with $k = \max\{\mathrm{ind}(a), \mathrm{ind}(b)\}$, $x \in T_l(a)$ and $y \in T_l(b)$. Then the following conditions are equivalent.*

(i) $x(a+b)y = x + y$.
(ii) $ax = by$.
(iii) $a^kR = b^kR$ *and* $Rx = Ry$.
(iv) $a^kR \subseteq b^kR$ *and* $Ry \subseteq Rx$.
(v) ${}^\circ(a^k) = {}^\circ(b^k)$ *and* $x^\circ = y^\circ$.
(vi) ${}^\circ(a^k) \subseteq {}^\circ(b^k)$ *and* $y^\circ \subseteq x^\circ$.

Proof. (i)⇒(ii). Multiplying on the left side of $x(a+b)y = x+y$ by axa, we get $axby = ax$.

Again, multiplying on the right side of $x(a+b)y = x+y$ by b^2y, we get $xaby = by$ by Lemma 2.12. It follows that $xaby = axby$ by multiplying on the left side by ax. Hence $axby = by$. Then we get $ax = by$.

(ii)⇒(i). Since $ax = by$, we get that $xby = x$ and

$$xay = xaby^2 = xaaxy = axy = by^2 = y.$$

Hence $x(a+b)y = xay + xby = x + y$.

(ii)⇒(iii). From $ax = by$, we get

$$b^k = byb^k = axb^k = a^kx^kb^k.$$

Then $b^kR \subseteq a^kR$. Similarly we also have $a^kR \subseteq b^kR$. Hence $a^kR = b^kR$.

Again, we can get $x = xax = xby$ since $ax = by$. Then $Rx \subseteq Ry$. Analogously, we have $Ry \subseteq Rx$. Then $Rx = Ry$.

(iii)⇒(iv). It is obvious.

(iv)⇒(ii). Since $a^kR \subseteq b^kR$, there exists $s \in R$ such that $a^k = b^ks$. Then $a^k = byb^ks = bya^k$, which implies that

$$ax = a^kx^k = bya^kx^k = byax.$$

Also, $Ry \subseteq Rx$ implies that there exists $t \in R$ satisfying $y = tx$. Then

$$by = btx = btxax = byax.$$

Therefore $ax = by$.

(iii)$\Rightarrow$(v). It is obtained by Lemma 4.9.

(v)$\Rightarrow$(vi). It is clear.

(vi)$\Rightarrow$(ii). Since ${}^{\circ}(a^k) \subseteq {}^{\circ}(b^k)$, we get $(1-ax)b^k = 0$, it follows that $by = axby$. From $y^{\circ} \subseteq x^{\circ}$, we have $x(1-by) = 0$, it follows that $ax = axby$. Hence $ax = by$. $\square$

In [2], Bapat et al. characterized the absorption laws of outer inverses. Here we can also present a special case that $a, b \in R^D$ with $x \in T_l(a)$ and $y \in T_l(b)$ and give a brief proof.

Theorem 4.11. *Let $a, b \in R^D$ with $x \in T_l(a)$ and $y \in T_l(b)$. Then the following conditions are equivalent.*

(i) $x(a+b)y = x + y$.

(ii) $Rx = R(xby)$ *and* $yR = (xay)R$.

(iii) $Rx \subseteq Ry$ *and* $yR \subseteq xR$.

Proof. (i)$\Rightarrow$(ii). Multiplying on the left side of $x(a+b)y = x+y$ by xa, we can derive that $xby = x$ and $xay = y$. Therefore $Rx = R(xby)$ and $yR = (xay)R$.

(ii)$\Rightarrow$(iii). It is clear.

(iii)$\Rightarrow$(i). Since $Rx \subseteq Ry$, we can get that $x = ty$ for some $t \in R$. Then

$$x = ty = tyby = xby.$$

A similar argument for $yR \subseteq xR$ can imply $xay = y$. Hence we have $x(a+b)y = xay + xby = x + y$. $\square$

From Theorems 4.10 and 4.11, we can get some equivalent conditions under which absorption laws hold true for Drazin inverses ([22, Theorem 2.2]), pseudo core inverses ([22, Theorems 2.4 and 2.7]), DMP inverses ([22, Theorems 2.5 and 2.8]) and weak core inverses, respectively. Here we present these equivalent characterizations of the absorption law holding for weak core inverses.

Theorem 4.12. *Let $a, b \in R^{wC}$ with $k = \max\{\mathrm{ind}(a), \mathrm{ind}(b)\}$. Then the following conditions are equivalent.*

(i) $a^{wC}(a+b)b^{wC} = a^{wC} + b^{wC}$.

(ii) $aa^{wC} = bb^{wC}$.

(iii) $a^kR = b^kR$ *and* $Ra^{wC} = Rb^{wC}$.

(iv) $a^kR \subseteq b^kR$ *and* $Rb^{wC} \subseteq Ra^{wC}$.

(v) ${}^{\circ}(a^k) = {}^{\circ}(b^k)$ *and* $(a^{wC})^{\circ} = (b^{wC})^{\circ}$.

(vi) ${}^{\circ}(a^k) \subseteq {}^{\circ}(b^k)$ *and* $(b^{wC})^{\circ} \subseteq (a^{wC})^{\circ}$.

Theorem 4.13. *Let $a, b \in R^{wC}$. Then the following conditions are equivalent.*

(i) $a^{wC}(a+b)b^{wC} = a^{wC} + b^{wC}$.

(ii) $Ra^{wC} = R(a^{wC}bb^{wC})$ *and* $b^{wC}R = (a^{wC}ab^{wC})R$.

(iii) $Ra^{wC} \subseteq Rb^{wC}$ *and* $b^{wC}R \subseteq a^{wC}R$.

5. Characterizations of $a^*a^{wC} = a^{wC}a^*$

Let $a \in R$. Recall that an element a is called star-dagger if a is Moore-Penrose invertible and $a^*a^{\dagger} = a^{\dagger}a^*$ [24]. Later, Mosić and Djordjević [31] gave some characterizations of star-dagger elements in R. Zhou et al. [44] provided equivalent conditions for $a^*a^{\textcircled{w}} = a^{\textcircled{w}}a^*$. It is noted that in [44, Theorem

6.3], Zhou et al. only obtained equivalent conditions for $a^*a^{D} = a^{D}a^*$. In fact, the condition required on $a_1 = a^2a^{ⓦ}$ (i.e., a_1 is an EP element) is very strong. In this section, we investigate the case of $a^*a^{wC} = a^{wC}a^*$, which improves the relevant results of Zhou et al. [44].

Let $p, q \in R$ be idempotent. If $x \in R$, then x can be represented as a sum $x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q)$ or as a formal matrix

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_{p\times q}, \tag{5.1}$$

where $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$ and $x_{22} = (1-p)x(1-q)$, which is well-known as Peirce decomposition.

Suppose that $a \in R^{D}$ with $\mathrm{ind}(a) = k$. Write $p = aa^{D}$. According to Peirce decomposition, the element a can be represented in the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times p}, \tag{5.2}$$

where $a_{11} = a^2a^{D}$, $a_{12} = a_1 - a_{11}$ and a_2 is nilpotent of index k. It follows that

$$a^k = \begin{pmatrix} a_{11}^k & \widetilde{a_{12}} \\ 0 & 0 \end{pmatrix}_{p\times p},$$

where $\widetilde{a_{12}} = \sum_{j=0}^{k-1} a_{11}^j a_{12} a_2^{k-1-j}$.

Lemma 5.1. *Let $a \in R^{D}$ with $\mathrm{ind}(a) = k$ and $p = aa^{D}$. If $x \in T_l(a)$, then $x = \begin{pmatrix} a^{D} & x_{12} \\ 0 & 0 \end{pmatrix}_{p\times p}$, where $x_{12} \in pR(1-p)$.*

Proof. Suppose that $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_{p\times p}$. Since $x \in T_l(a)$, we get that $ax^2 = x$ and $xa^{k+1} = a^k$. From $xa^{k+1} = a^k$, we conclude by Lemma 2.12 that

$$\begin{cases} x_{11}a_{11}^{k+1} = a_{11}^k \\ x_{11}\sum_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = \widetilde{a_{12}} \\ x_{21}a_{11}^{k+1} = 0 \\ x_{21}\sum_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = 0 \end{cases} \Rightarrow \begin{cases} x_{11}aa^{D} = a^{D} \\ x_{11}a_{12}a_2^k + x_{11}a_{11}\widetilde{a_{12}} = \widetilde{a_{12}} \\ x_{21}aa^{D} = 0 \\ x_{21}\sum_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = 0. \end{cases}$$

Then $x_{21} = x_{21}aa^{D} = 0$ and $x_{11} = x_{11}aa^{D} = a^{D}$. From $ax^2 = x$, we can get

$$\begin{cases} a_{11}(a^{D})^2 = a^{D} \\ a_{11}a^{D}x_{12} + (a_{11}x_{12} + a_{12}x_{22})x_{22} = x_{12} \\ a_2x_{22}^2 = x_{22}. \end{cases}$$

Obviously, $a_{11}(a^{\textcircled{D}})^2 = a^{\textcircled{D}}$.

From $a_2 x_{22}^2 = x_{22}$ and a_2 is nilpotent of index k, we can get $a_2^{k-1}x_{22} = a_2^k x_{22}^2 = 0$ by multiplying on the left side by a_2^{k-1}. Then multiplying on the left side by a_2^{k-2}, we can get $a_2^{k-2}x_{22} = a_2^{k-1}x_{22}^2 = 0$. By induction, we can obtain $a_2 x_{22} = 0$, which implies $x_{22} = 0$. Then $a_{11}a^{\textcircled{D}}x_{12} + (a_{11}x_{12} + a_{12}x_{22})x_{22} = x_{12}$ is clear. Hence $x = \begin{pmatrix} a^{\textcircled{D}} & x_{12} \\ 0 & 0 \end{pmatrix}_{p\times p}$, where $x_{12} \in pR(1-p)$. □

Remark 5.2. *If $a \in R^{wC}$, then $a^D, a^{\textcircled{D}}, a^{\textcircled{w}_m}$ and $a^{wC} \in T_l(a)$ according to [45, Remark 3.7]. Take $p = aa^{\textcircled{D}}$. Then their expressions can be given as follows:*

$$a^D = \begin{pmatrix} a^{\textcircled{D}} & (a^{\textcircled{D}})^{k+1}\widetilde{a_{12}} \\ 0 & 0 \end{pmatrix}_{p\times p}, \quad a^{\textcircled{D}} = \begin{pmatrix} a^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p},$$

$$a^{\textcircled{w}_m} = \begin{pmatrix} a^{\textcircled{D}} & (a^{\textcircled{D}})^{m+1}\sum\limits_{j=0}^{m-1} a_{11}^j a_{12} a_2^{m-1-j} \\ 0 & 0 \end{pmatrix}_{p\times p}, \quad a^{wC} = \begin{pmatrix} a^{\textcircled{D}} & (a^{\textcircled{D}})^2 a_{12} a_2 a_2^{(1,3)} \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Recall from [21, Lemma 2.3], $a \in R$ is $*$-DMP if and only if $a \in R^{\textcircled{D}}$ and $a^{\textcircled{D}} = a^D$.

Proposition 5.3. *Let $a \in R^{\textcircled{D}}$ with $\operatorname{ind}(a) = k$. If $x \in T_l(a)$, then the following conditions are equivalent.*

(i) $a^*x = xa^*$.

(ii) *$x = a^{\textcircled{D}}$, a is $*$-DMP and $a_1^*a_1 = a_1a_1^*$.*

Proof. According to Lemma 5.1, we can get that $a^*x = xa^*$ if and only if

$$\begin{cases} a_{11}^*a^{\textcircled{D}} = a^{\textcircled{D}}a_{11}^* + x_{12}a_{12}^* \\ a_{12}^*a^{\textcircled{D}} = 0 \\ a_{11}^*x_{12} = x_{12}a_2^* \\ a_{12}^*x_{22} = 0 \end{cases} \Leftrightarrow \begin{cases} a_{11}^*a^{\textcircled{D}} = a^{\textcircled{D}}a_{11}^* \\ a_{12} = 0 \\ a_{11}^*x_{12} = x_{12}a_2^*. \end{cases} \tag{5.3}$$

Now it suffices to prove Equations (5.3) ⇔ (ii).

(a). We first prove that $a_{11}^*x_{12} = x_{12}a_2^*$ is equivalent to $x = a^{\textcircled{D}}$. From $a_{11}^*x_{12} = x_{12}a_2^*$ and a_2 is nilpotent of index k, we have $(a_{11}^*)^k x_{12} = x_{12}(a_2^*)^k = 0$. Then $(a^{k+1}a^{\textcircled{D}})^*x_{12} = 0$, which implies $aa^{\textcircled{D}}x_{12} = 0$ by multiplying on the left side by $\left((a^{\textcircled{D}})^k\right)^*$. Hence $x_{12} = aa^{\textcircled{D}}x_{12} = 0$. Then we have $x = a^{\textcircled{D}}$. Conversely, if $x = a^{\textcircled{D}}$, then $x_{12} = 0$ by Lemma 5.1, which is obvious to indicate $a_{11}^*x_{12} = x_{12}a_2^*$.

(b). Next we prove that $a_{12} = 0$ and $a_{11}^*a^{\textcircled{D}} = a^{\textcircled{D}}a_{11}^*$ are equivalent to the conditions that a is $*$-DMP and $a_1^*a_1 = a_1a_1^*$. Following Remark 5.2, it is easy to check that $a_{12} = 0$ if and only if $a^D = a^{\textcircled{D}}$, which is equivalent to that a is $*$-DMP. When $a^D = a^{\textcircled{D}}$, we have $a_{11}^*a^{\textcircled{D}} = a^{\textcircled{D}}a_{11}^*$ if and only if $a_1^*a_1^\# = a_1^\#a_1^*$, which is also equivalent to $a_1^*a_1 = a_1a_1^*$ by [13, Theorem 2.2]. □

Example 5.4. *However, the condition $x = a^{\textcircled{D}}$ of Proposition 5.3 can not be dropped. For example, let $R = \mathbb{C}^{2\times 2}$ with the transpose as the involution. Take $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in T_l(a)$. Obviously, $a^D = a^{\textcircled{D}} = a = a_1$ and $a_1^*a_1 = a_1a_1^*$. However $a^*x \neq xa^*$.*

Remark 5.5. *Let $a \in R^D$. If there exists $x \in T_l(a)$ such that $a^*x = xa^*$, then a may not be pseudo core invertible.*

For example, let $R = \mathbb{C}^{2\times 2}$ with the transpose as the involution. Take $a = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}, x = \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix} \in R$. *Then $ax^2 = x$, $xa^2 = a$ and $a^*x = xa^*$. However, since $a \notin Ra^*a$, we know that $a \notin R^{\{1,3\}}$ by Lemma 2.2. Then $a \notin R^{\textcircled{\#}}$ according to [42, Theorem 2.6], hence a is not pseudo core invertible.*

Recall from [32], an element $a \in R$ satisfying $a^*a^n = a^na^*$ for some $n \in \mathbb{N}^+$ will be called generalized normal. In the following, we give some equivalent characterizations of $a^*a^{wC} = a^{wC}a^*$.

Theorem 5.6. *Let $a \in R^{wC}$ with $\text{ind}(a) = k$ and $m \in \mathbb{N}^+$. Then the following conditions are equivalent.*

(i) $a^*a^D = a^Da^*$.
(ii) $a^*a^{\textcircled{D}} = a^{\textcircled{D}}a^*$.
(iii) $a^*a^{\textcircled{w}_m} = a^{\textcircled{w}_m}a^*$.
(iv) $a^*a^{wC} = a^{wC}a^*$.
(v) *a is $*$-DMP and* $a_1^*a_1 = a_1a_1^*$.

In this case, a is generalized normal and $a^D = a^{\textcircled{D}} = a^{\textcircled{w}_m} = a^{wC}$.

Proof. Note that $a_{12} = 0$ if a is $*$-DMP. Then it follows that the proofs are obtained according to Remark 5.2 and Proposition 5.3. In this case, we can get $a^*a^k = a^ka^*$, which implies that a is generalized normal. □

Remark 5.7. *In [44, Theorem 6.3], Zhou et al. proved that in a proper $*$-ring R, if $a \in R^{\textcircled{W}}$ and $a_1 = a^2a^{\textcircled{w}}$ is EP, then*

$$a^*a^{\textcircled{D}} = a^{\textcircled{D}}a^* \Leftrightarrow a^*a^{\textcircled{w}} = a^{\textcircled{w}}a^* \Leftrightarrow a_1^*a_1 = a_1a_1^*.$$

In fact, the condition that a_1 is EP can imply that a is $*$-DMP. We can reduce this condition to $a_1 \in R^{\textcircled{\#}}$, under which (i), (ii), (iii) and (v) in Theorem 5.6 are equivalent.

Example 5.8. *However, if a is generalized normal, then $a_1^*a_1 = a_1a_1^*$ may not be true. For example, let $R = \mathbb{C}^{2\times 2}$ with the transpose as the involution. Take* $a = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in R$. *Then* $a^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *it follows $a^*a^2 = a^2a^*$, which implies that a is generalized normal. However, $a_1^*a_1 \neq a_1a_1^*$ since $a_1 = a$.*

According to [39, Theorem 2.2], every matrix $A \in \mathbb{C}^{n\times n}$ of index k can be represented in the form

$$A = U \begin{pmatrix} T & S \\ 0 & N \end{pmatrix} U^*, \tag{5.4}$$

where T is nonsingular with rank(T)=rank(A^k), N is nilpotent of index k and U is unitary. Then we can give the corresponding results for complex matrices and omit their proofs.

Proposition 5.9. *Let $A \in \mathbb{C}^{n\times n}$ of index k be written as in (5.4) and $X \in T_l(A)$. Then the following conditions are equivalent.*

(i) $A^*X = XA^*$.
(ii) $X = U \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$, *$T^*T = TT^*$ and $S = 0$.*

Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k. The DMP inverse of A, denoted by $A^{d,\dagger}$, is defined to be the matrix $A^{d,\dagger} = A^D A A^\dagger$ [28]. Moreover, it is proved that $A^{d,\dagger} \in T_l(A)$.

Proposition 5.10. *Let $A \in \mathbb{C}^{n\times n}$ of index k and $m \in \mathbb{N}^+$. Then the following conditions are equivalent.*

(i) $A^*A^D = A^D A^*$.
(ii) $A^*A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^*$.
(iii) $A^*A^{\textcircled{w}_m} = A^{\textcircled{w}_m}A^*$.
(iv) $A^*A^{wC} = A^{wC}A^*$.
(v) $A^*A^{d,\dagger} = A^{d,\dagger}A^*$.

In this case, A is generalized normal and $A^D = A^{\textcircled{\dagger}} = A^{\textcircled{w}_m} = A^{wC} = A^{d,\dagger}$.

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References

[1] Baksalary, O.M., Trenkler, G.: Core inverse of matrices. Linear Multilinear Algebra 58(6), 681–697 (2010)
[2] Bapat, R.B., Jain, S.K., Karantha, M.P., Raj, M. D.: Outer inverses: Characterization and applications. Linear Algebra Appl. 528, 171–184 (2017)
[3] Ben-Israel, A., Greville, T.N.E.: Generalized inverses: theory and applications. 2nd ed. Springer, New York (2003)
[4] Campbell, S.L., Meyer, C.D.: Generalized inverses of linear transformations. SIAM, Philadelphia (2009)
[5] Castro-González, N., Hartwig, R.E.: Perturbation results and the forward order law for the Moore-Penrose inverse of a product. Electron. J. Linear Algebra 34, 514–525 (2008)
[6] Castro-González, N., Mendes-Araújo, C., Patrício, P.: Generalized inverses of a sum in rings. Bull. Aust. Math. Soc. 82(1), 156–164 (2010)
[7] Chen, J.L., Li, W.D., Zhou, M.M.: Pseudo core inverses of a sum of morphisms. Quaest. Math. 44, 1321–1332 (2021)
[8] Chen, J.L., Zhu, H.H., Patrício, P., Zhang, Y.L.: Characterizations and representations of core and dual core inverses. Canad. Math. Bull. 60, 269–282 (2017)
[9] Cvetković-Ilić, D.S.: New additive results on Drazin inverse and its applications. Appl. Math. Comput. 218, 3019–3024 (2011)
[10] Cvetković-Ilić, D., Harte, R.: On Jacobson's lemma and Drazin invertibility. Appl. Math. Lett. 23, 417–420 (2010)
[11] Cvetković-Ilić, D.S., Wei, Y.M.: Algebraic properties of generalized inverses. Springer, Singapore (2017)
[12] Drazin, M.P.: Pseudo-inverse in associative rings and semigroups. Amer. Math. Monthly 65, 506–514 (1958)
[13] Drazin, M.P.: Commuting properties of generalized inverses. Linear Multilinear Algebra 61(12), 1675–1681 (2013)
[14] Djordjević, D.S., Dinčić, N.Č.: Reverse order law for the Moore-Penrose inverse. J. Math. Anal. Appl. 361, 252–261 (2010)
[15] Ferreyra, D.E., Levis, F.E., Priori, A.N., Thome, N.: The weak core inverse. Aequat Math. 95(2), 351–373 (2021)
[16] Ferreyra, D.E., Levis, F.E., Thome, N.: Characterizations of k-commutative equalities for some outer generalized inverses. Linear Multilinear Algebra 68(1), 177–192 (2018)
[17] Ferreyra, D.E., Levis, F.E., Thome, N.: Revisiting the core-EP inverse and its extension to rectangular matrices. Quaest Math. 41(2), 265–281 (2018)
[18] Ferreyra, D.E., Orquera, V., Thome, N.: A weak group inverse for rectangular matrices. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113, 3727–3740 (2019)
[19] Fu, Z.M., Zuo, K.Z., Chen, Y.: Further characterizations of the weak core inverse of matrices and the weak core matrix. AIMS Math. 7, 3630–3647 (2022)
[20] Gao, Y.F., Chen, J.L.: Pseudo core inverses in rings with involution. Comm. Algebra 46(1), 38–50 (2018)
[21] Gao, Y.F., Chen, J.L., Ke, Y.Y.: $*$-DMP elements in $*$-semigroups and $*$-rings. Filomat 39, 3073–3085 (2018)

[22] Gao, Y.F., Chen, J.L., Wang, L., Zou, H.L.: Absorption laws and reverse order laws for generalized core inverses. Comm. Algebra 49, 3241–3254 (2021)

[23] Hartwig, R.E.: Block generalized inverses. Arch. Ration. Mech. Anal. 61(3), 197–251 (1976)

[24] Hartwig, R.E., Spindelböck, K.: Matrices for which A^* and $A^\dagger$ commute. Linear Multilinear Algebra 14, 241–256 (1983)

[25] Koliha, J.J., Djordjević, D., Cvetković, D.: Moore-Penrose inverse in rings with involution. Linear Algebra Appl. 426, 371–381 (2007)

[26] Lam, T.Y., Nielsen, P.P.: Jacobson's lemma for Drazin inverses. Contemp. Math. 609, 185–195 (2014)

[27] Ma, H.F., Stanimirović, P.S.: Characterizations, approximation and perturbations of the core-EP inverse. Appl Math Comput. 359, 404–417 (2019)

[28] Malik, S.B., Thome, N.: On a new generalized inverse for matrices of an arbitrary index. Appl Math Comput. 226, 575–580 (2014)

[29] Manjunatha Prasad, K., Mohana, K.S.: Core-EP inverse. Linear Multilinear Algebra 62(6), 792–802 (2014)

[30] Meyer, C.D., Rose, N.J.: The index and the Drazin inverse of block triangular matrices. SIAM J. A. Math. 33(1), 1-7 (1977)

[31] Mosić, D., Djordjević, D.S.: Partial isometries and EP elements in rings with involution. Electron. J. Linear Algebra 18, 761–772 (2009)

[32] Mosić, D., Djordjević, D.S.: New characterizations of EP, generalized normal and generalized Hermitian elements in rings. Appl. Math. Comput. 218, 6702–6710 (2012)

[33] Mosić, D., Djordjević, D.S.: Further results on the reverse order law for the Moore-Penrose inverse in rings with involution. Appl. Math. Comput. 218, 1478–1483 (2011)

[34] Mosić, D., Stanimirović, P.S.: Representations for the weak group inverse. Appl. Math. Comput. 397, 125957 (2021)

[35] Mosić, D., Stanimirović, P.S.: Expressions and properties of weak core inverse. Appl. Math. Comput. 415, 126704 (2022)

[36] Penrose, R.: A generalized inverse for matrices. Proc Cambridge Philos Soc. 51, 406–413 (1955)

[37] Rakić, D.S., Dinčić, N.Č., Djordjević, D.S.: Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl. 463, 115–133 (2014)

[38] Shi, G.Q., Chen, J.L., Li, T.T., Zhou, M.M.: Jacobson's lemma and Cline's formula for generalized inverses in a ring with involution. Comm Algebra 48(9), 3948–3961 (2020)

[39] Wang, H.X.: Core-EP decomposition and its applications. Linear Algebra Appl. 508(1), 289–300 (2016)

[40] Wang, G.R.: The reverse order law for the Drazin inverses of multiple matrix products. Linear Algebra Appl. 348, 265–272 (2002)

[41] Wang, H.X., Chen, J.L.: Weak group inverse. Open Math. 16, 1218–1232 (2018)

[42] Xu, S.Z., Chen, J.L., Zhang, X.X.: New characterizations for core inverses in rings with involution. Front Math China 12(1), 231–246 (2017)

[43] Zhou, M.M., Chen, J.L., Zhou, Y.K.: Weak group inverses in proper $*$-rings. J Algebra Appl. 19(12), 2050238 (2020)

[44] Zhou, M.M., Chen, J.L., Zhou, Y.K., Thome, N.: Weak group inverses and partial isometries in proper $*$-rings. Linear Multilinear Algebra (2021) DOI: 10.1080/03081087.2021.1884639

[45] Zhou, Y.K., Chen, J.L.: Weak core inverses and pseudo core inverses in a ring with involution. Linear Multilinear Algebra (2021) DOI: 10.1080/03081087.2021.1971151.

[46] Zhou, Y.K., Chen, J.L., Zhou, M.M.: m-weak group inverses in a ring with involution. Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM 115(1), 2 (2021)

[47] Zhu, H.H., Patrício, P.: Characterizations for pseudo core inverses in a ring with involution. Linear Multilinear Algebra 67, 1109–1120 (2019)

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