Bifurcation from interval for discrete boundary value problem involving p-Laplacian

Fumei Ye[a,b]

[a] School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China
[b] School of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China

Abstract

In this paper, we are concerned with the global bifurcation results for p-Laplacian discrete problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + a(t)f(t,u(t),\lambda) + g(t,u(t),\lambda), & t \in [1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $a : [1,T]_Z \to [0,\infty)$, $f, g \in C([0,T+1]_Z \times \mathbb{R}^2, \mathbb{R})$, $\Delta u(t) = u(t+1) - u(t)$ is the forward difference operator, $\varphi_p(s) = |s|^{p-2}s (1 < p < +\infty)$. We shall show that there are two distinct unbounded continua $\mathcal{C}^+$ and $\mathcal{C}^-$, consisting of the bifurcation branch $\mathcal{C}$ if f is not necessarily differentiable at the origin with respect to $\varphi_p(u)$, and there are two distinct unbounded continua $\mathcal{D}^+$ and $\mathcal{D}^-$, consisting of the bifurcation branch $\mathcal{D}$ if f is not necessarily differentiable at infinity with respect to $\varphi_p(u)$.

As the applications of the above result, we shall obtain that there exist at least a positive solution and a negative one for the half-quasilinear problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \mu a(t)F(u(t)) + \alpha(t)\varphi_p(u^+(t)) + \beta(t)\varphi_p(u^-(t)), & t \in [1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\mu \neq 0$ is a parameter, $a : [1,T]_Z \to (0,+\infty), \alpha, \beta : [1,T]_Z \to \mathbb{R}, u^+ = \max\{u,0\}, u^- = -\min\{u,0\}$, $F \in C(\mathbb{R},\mathbb{R})$ satisfies $sF(s) > 0$.

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1. Introduction

In [3], Berestycki considered the following nonlinear Sturm-Liouville problem

$$\begin{cases} -(pu')' + qu = \lambda a(t)u + F(t,u,u',\lambda), & t \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0, \end{cases} \tag{1}$$

where p, q are continuous function on $[0,1]$ and b_i, c_i are real numbers such that $|b_i|+|c_i| \neq 0, i = 0,1$. Moreover, the nonlinear term has the form $F = f_1 + f_2$, f_1, f_2 satisfying the following conditions:

Email address: `yfm_nwnu@163.com` (Fumei Ye)

(A1) $f_1 \in C([0,1]\times\mathbb{R}^3,\mathbb{R})$ and $\left|\frac{f_1(t,u,s,\lambda)}{u}\right| \le M$, for all $t\in[0,1], 0<|u|\le 1, |s|\le 1$ and all $\lambda\in\mathbb{R}$, where M is a positive constant;

(A2) $f_2 \in C([0,1]\times\mathbb{R}^3,\mathbb{R})$ and $f_2(t,u,s,\lambda) = o(|u|+|s|)$, near $(u,s)=(0,0)$, uniformly in $t\in[0,1]$ and λ on bounded sets.

Using the result of Rabinowitz [18], the author obtained that there are two unbounded connected branches of problem (1) with bifurcation from an interval. The proof of the main result of (1) strictly depends on the linear property of operator.

In [13], Ma and Dai extended Berestycki's result to show a unilateral global bifurcation result for (1) under the assumptions (A1) and (A2).

Many authors have discussed the existence and multiplicity of solutions for discrete boundary value problems involving the p-Laplacian difference operator, we refer to [1, 2, 6, 8, 10, 14, 15] and references therein. These results were usually obtained by the applicability of the topological method such as the upper and lower solutions technique, critical theory, variational methods, Leray-Schauder degree, fixed point theory, etc. Moreover, a great attention has been paid to discrete equations using variational methods and critical point theory, we refer the reader to [7, 9, 12] and the related results mentioned there.

Therefore, in this article, we extend the results of [13] to the p-Laplacian difference case. We will establish the unilateral interval bifurcation results for the problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + a(t)f(t,u(t),\lambda) + g(t,u(t),\lambda), \quad t\in[1,T]_Z, \\ u(0)=u(T+1)=0, \end{cases} \tag{2}$$

where $a:[1,T]_Z\to[0,\infty)$ and $a(t)\not\equiv 0$ on any subinterval of $[1,T]_Z$, $T>2$ is integer. Let Z denote the integer set, for $m,n\in Z$ with $m<n$, $[m,n]_Z := \{m, m+1,\cdots,n\}$. $f,g\in C([1,T]_Z\times\mathbb{R}^2,\mathbb{R})$ and we make the following assumptions:

$(\mathbf{H_1})$ There exist $f_0, f^0\in\mathbb{R}$ with $f_0\ne f^0$, where

$$f_0 = \liminf_{|s|\to 0^+}\frac{f(t,s,\lambda)}{|s|^{p-1}}, \qquad f^0 = \limsup_{|s|\to 0^+}\frac{f(t,s,\lambda)}{|s|^{p-1}}$$

for any $t\in[1,T]_Z, 0<|s|\le 1$ and for all $\lambda\in\mathbb{R}$.

$(\mathbf{H_2})$ $g(t,s,\lambda)=o(|s|^{p-1})$, near $s=0$, uniformly in $t\in[1,T]_Z$ and in every bounded interval of λ.

$(\mathbf{H_3})$ There exist $f_\infty, f^\infty\in\mathbb{R}$ with $f_\infty\ne f^\infty$, where

$$f_\infty = \liminf_{|s|\to +\infty}\frac{f(t,s,\lambda)}{|s|^{p-1}}, \qquad f^\infty = \limsup_{|s|\to +\infty}\frac{f(t,s,\lambda)}{|s|^{p-1}}$$

for any $t\in[1,T]_Z, |s|\ge C$ for some positive constant C large enough and for all $\lambda\in\mathbb{R}$.

$(\mathbf{H_4})$ $g(t,s,\lambda)=o(|s|^{p-1})$, near $s=\infty$, uniformly in $t\in[1,T]_Z$ and in every bounded interval of λ.

In the differential case, the spectrum of the one-dimensional half-quasilinear p-Laplacian problem has been clearly determined, but in the difference case it is not known. In fact, the spectrum of half-quasilinear eigenvalue problems needs the help of interval bifurcation theory. Therefore, based on the obtained global interval bifurcation theory, we obtain the existence of principal half-eigenvalue for half-quasilinear discrete eigenvalue problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + \alpha(t)\varphi_p(u^+(t)) + \beta(t)\varphi_p(u^-(t)), \quad t\in[1,T]_Z, \\ u(0)=u(T+1)=0, \end{cases}$$

where $a:[1,T]_Z\to(0,+\infty),\alpha,\beta:[1,T]_Z\to\mathbb{R},u^+=\max\{u,0\},u^-=-\min\{u,0\}$. We proved that the above problem has two principal half-eigenvalues λ_+ and λ_-, aside from λ_+ and λ_-, there is no other half-eigenvalue has positive or negative eigenfunction. As the applications of the above result, it is the purpose of this paper to determine the interval of μ in which

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))]=\mu a(t)F(u(t))+\alpha(t)\varphi_p(u^+(t))+\beta(t)\varphi_p(u^-(t)), & t\in[1,T]_Z,\\ u(0)=u(T+1)=0 \end{cases}$$

has a positive or negative solution, where $\mu\neq 0$ is a parameter, $F\in C(\mathbb{R},\mathbb{R})$.

This paper is organized as follows: In Section 2, we state some notations and preliminary results. Sections 3-4 are devoted to study the bifurcation phenomena from the trivial solution axis or from infinity for (2) which are not necessarily linearizable, respectively. In Section 5, we will discuss the properties of the eigenvalue of the half-quasilinear discrete problem and the global structure of one-sign solution sets for the corresponding nonlinear problems in detail.

2. Preliminaries

In this section, we introduce some well-known results which will be used in the subsequent section.

Set $E=\{u:[0,T+1]_Z\to\mathbb{R}:u(0)=u(T+1)=0\}$ with the norm $\|u\|=\max\limits_{t\in[0,T+1]_Z}|u(t)|$. Let $Y=\big\{u\big|u:[1,T]_Z\to\mathbb{R}\big\}$ with the norm $||u||_Y=\max\limits_{t\in[1,T]_Z}\big|u(t)\big|$.

Lemma 2.1. *(See. [4, Propositions1.8-1.10]) The eigenvalue problem*

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))]=\lambda a(t)\varphi_p(u(t)), & t\in[1,T]_Z,\\ u(0)=u(T+1)=0 \end{cases}\tag{3}$$

has the first simple eigenvalue λ_1, let ϕ_1 be the eigenfunction corresponding to λ_1. Moreover, ϕ_1 does not change sign in $[1,T]_Z$.

Next, we will give the generalized Picone type identity, which will be used in the proof of the main theorem.

Lemma 2.2. *(See. [16]) Define*

$$l_p(x(t))=\Delta[\varphi_p(\Delta x(t))]-P(t)\varphi_p(x(t+1))$$

and

$$L_p(y(t))=\Delta[\varphi_p(\Delta y(t))]-Q(t)\varphi_p(y(t+1)),$$

where $t \in [m,n]_Z, m,n \in \mathbb{Z}, m \leq n$, and $P(t), Q(t)$ are real-valued sequences defined on $[m,n]_Z$. Let $x(t), y(t)$ be defined on $[m,n+2]_Z$ and let $y(t) \neq 0$ for $t \in [m,n+1]_Z$. Then the equality

$$\begin{aligned}
&\Delta\Big\{\frac{x(t)}{\varphi_p(y(t))}[\varphi_p(y(t))\varphi_p(\Delta x(t)) - \varphi_p(x(t))\varphi_p(\Delta y(t))]\Big\} \\
&= (P(t) - Q(t))|x(t+1)|^p + x(t+1)l_p(x(t)) - \frac{|x(t+1)|^p}{\varphi_p(y(t+1))}L_p(y(t)) \\
&+ [|\Delta x(t)|^p + \frac{\varphi_p(\Delta y(t))}{\varphi_p(y(t))}|x(t)|^p - \frac{\varphi_p(\Delta y(t))}{\varphi_p(y(t+1))}|x(t+1)|^p]
\end{aligned}$$

holds for $t \in [m,n]_Z$.

Lemma 2.3. *(Young's inequality) If $a, b \geq 0, p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, and the equality holds if and only if $b = a^{p-1}$.*

Definition 2.4. (See. [11]) Let $u : [1,T]_Z \to \mathbb{R}$. If $u(t_0) = 0$, then t_0 is a zero of u. If $u(t_0) = 0$ and $u(t_0 - 1)u(t_0 + 1) < 0$ for some $t_0 \in [2, T-1]_Z$, then t_0 is a simple zero of u.

Consider the following auxiliary problem

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] = h(t), \quad t \in [1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases} \tag{4}$$

where $h : [1,T]_Z \to \mathbb{R}$. Clearly the problem (4) is equivalent to

$$u(t) = G_p(h)(t) := u(1) + \sum_{s=1}^{t-1} \varphi_p^{-1}[\varphi_p(u(1)) + \sum_{\tau=1}^{s} h(\tau)], \quad t \in [1, T+1]_Z,$$

It is known that $G_p : Y \to E$ is continuous and maps bounded sets of $\mathbb{R}$ into relatively compacts of E.

We define the operator $T_\lambda^p : E \to E$ by

$$\begin{aligned}
T_\lambda^p(u)(t) &= u(1) + \sum_{s=1}^{t-1} \varphi_p^{-1}[\varphi_p(u(1)) - \sum_{\tau=1}^{s} \lambda a \varphi_p(u)(\tau)] \\
&= G_p\big(-\lambda a \varphi_p(u)\big)(t),
\end{aligned}$$

Then $T_\lambda^p : E \to E$ is compact.

Define the Nemitskii operators $F : \mathbb{R} \times E \to Y$ by

$$F(\lambda, u)(t) = -\lambda a(t)\varphi_p(u(t)) - a(t)f(t, u(t), \lambda) - g(t, u(t), \lambda).$$

Thus it is obvious that F is continuous operator which maps bounded sets of $\mathbb{R} \times E$ into the bounded sets of Y and problem (1.1) can be equivalently written as

$$u = G_p \circ F(\lambda, u) = A_p(\lambda, u).$$

A is completely continuous in $\mathbb{R} \times E \to E$ and for any $\lambda \in \mathbb{R}$, $A(\lambda, 0) = 0$.

We use the terminology of Rabinowitz [20]. Let us denote $S^+ = \{u \in E \big| u(t) > 0, t \in [1,T]_Z\}$, and let $S^- = -S^+$ and $S = S^+ \cup S^-$. S^+ and S^- are disjoint and open in E. Furthermore, we use $\mathscr{C}$ to denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (2) under assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$. $\mathscr{C}^{\pm}$ denote the subset of $\mathscr{C}$ with $u \in S^{\pm}$, and $\mathscr{C} = \mathscr{C}^+ \cup \mathscr{C}^-$.

In addition, we use the terminology of Rynne [17]. For any $\lambda \in \mathbb{R}$, we say that a subset $C' \subset \mathscr{C}$ meets $(\lambda, 0)$ (similarly, (λ, ∞)) if there is a sequence $(\lambda_n, u_n) \in C' (n = 1, 2, \cdots)$ such that $\lambda_n \to \lambda, \|u_n\| \to 0$ (similarly, $\|u_n\| \to \infty$) as $n \to +\infty$. Furthermore, we will say that $C' \subset \mathscr{C}$ meets $(\lambda, 0)$ throught $\mathbb{R} \times E$ if the sequence $(\lambda_n, u_n) \in C'(n = 1, 2, \cdots)$ can be chosen such that $u_n \in E$ for all n. If $I \subset \mathbb{R}$ is a bounded interval we say that $C' \subset \mathscr{C}$ meets $I \times \{0\}$ (similarly, $I \times \{\infty\}$) if C' meets $(\lambda, 0)$ (similarly, (λ, ∞)) for some $\lambda \in I$. Similarly, we can define C' meets $I \times \{0\}$ or $I \times \{\infty\}$ through $\mathbb{R} \times E$.

3. Interval bifurcation from trivial solution axis

In this section, we shall study the unilateral global bifurcation phenomena of problem (2) which bifurcates from trivial solution axis or from infinity. In order to obtain the main result, the generalized Picone identity plays a key role.

The following lemmas will be needed in our further considerations.

Lemma 3.1. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Let* (λ, u) *is a solution of (2), if there is* $t_0 \in [1, T]_Z$ *such that one of the following two situations hold:*

(i) $u(t_0) = 0, \Delta u(t_0) = 0$;

(ii) $u(t_0) = 0, u(t_0 - 1)u(t_0 + 1) \geq 0$.

Then $u \equiv 0$.

Proof. **(i)** In view of (2), we have

$$\varphi_p(\Delta u(t_0 - 1)) - \varphi_p(\Delta u(t_0)) = \lambda a(t)\varphi_p(u(t_0)) + a(t)f(t, u(t_0), \lambda) + g(t, u(t_0), \lambda).$$

Connecting the assumptions $(\mathbf{H_1})$-$(\mathbf{H_2})$ with $u(t_0) = 0, \Delta u(t_0) = 0$, we obtain that $\varphi_p(\Delta u(t_0 - 1)) = 0$, which ensures $\Delta u(t_0 - 1) = 0$. Since $u(t_0) = 0$, thus $u(t_0 - 1) = 0$. Repeat this step and we can find: $u(t) \equiv 0$ for every $t \leq t_0, t \in [1, T]_Z$.

Similarly, by virtue of $\varphi_p(\Delta u(t_0)) - \varphi_p(\Delta u(t_0 + 1)) = 0$, this yields $\Delta u(t_0 + 1) = 0$. Hence, $u(t_0 + 2) = 0$. Repeat this step and we can find: $u(t) \equiv 0$ for every $t \geq t_0, t \in [1, T]_Z$.

(ii) Similar to the calculation in **(i)**, it follows that $\varphi_p(-u(t_0 - 1)) - \varphi_p(u(t_0 + 1)) = 0$. Combining the conclusion of **(i)** and $u(t_0 - 1)u(t_0 + 1) \geq 0$, we obtain that $u \equiv 0$. □

To get the main results, we introduce the following approximate problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + a(t)f(t, u(t)|u(t)|^{\varepsilon}, \lambda) + g(t, u(t), \lambda), & t \in [1, T]_Z, \\ u(0) = u(T + 1) = 0. \end{cases} \tag{5}$$

Note that it follows from the condition $(\mathbf{H_1})$, for any $\varepsilon > 0$, the function $f(t, u(t)|u(t)|^{\varepsilon}, \lambda)$ satisfies

$$f(t, u(t)|u(t)|^{\varepsilon}, \lambda) = o(|u|^{p-1})$$

near $u = 0$, uniformly for $t \in [1, T]_Z$ and λ on bounded sets.

Lemma 3.2. *Let $I^0 = [\lambda_1 - f^0, \lambda_1 - f_0]$, let $\varepsilon_n \to 0,\ 0 < \varepsilon_n < 1$. If there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times S^{\sigma}$ such that (λ_n, u_n) a nontrivial solution of (5) corresponding to $\varepsilon = \varepsilon_n$, and $(\lambda_n, u_n) \to (\lambda, 0)$ in $\mathbb{R} \times E$. Then $\lambda \in I^0$, where $\sigma = +$ or $-$.*

Proof. Without loss of generality, let $\|u_n\| \le 1$ and $v_n = \frac{u_n}{\|u_n\|}$. So v_n satisfies

$$\begin{cases} -\Delta[\varphi_p(\Delta v_n(t-1))] = \lambda a(t)\varphi_p(v_n(t)) + a(t)f_n(t) + g_n(t), \quad t \in [1, T]_Z, \\ \\ v_n(0) = v_n(T+1) = 0, \end{cases} \tag{6}$$

where $f_n(t) = \frac{f(t, u_n|u_n|^{\varepsilon_n}, \lambda_n)}{\|u_n\|^{p-1}}$, $g_n(t) = \frac{g(t, u_n, \lambda_n)}{\|u_n\|^{p-1}}$.

Setting $\overline{g}(t, u, \lambda) = \max\limits_{0 \le |s| \le u} \big|g(t, s, \lambda)\big|$ for any $t \in [1, T]_Z$. According to $(\mathbf{H_2})$, $\overline{g}$ is nondecreasing with respect to u and

$$\lim_{u \to 0} \frac{\overline{g}(t, u, \lambda)}{|u|^{p-1}} = 0 \tag{7}$$

uniformly for $t \in [1, T]_Z$ and in every bounded interval of λ. Furthermore, it can be obtained from (7)

$$\frac{|g(t, u, \lambda)|}{\|u\|^{p-1}} \le \frac{\overline{g}(t, u, \lambda)}{\|u\|^{p-1}} \le \frac{\overline{g}(t, \|u\|, \lambda)}{\|u\|^{p-1}} \to 0, \qquad u \to 0 \tag{8}$$

uniformly for $t \in [1, T]_Z$ and in every bounded interval of λ. Obviously, in view of $(\mathbf{H_1})$, there is

$$\begin{aligned} f_n(t) &= \tfrac{f(t, u_n|u_n|^{\varepsilon_n}, \lambda_n)}{\varphi_p(u_n|u_n|^{\varepsilon_n})} \tfrac{\varphi_p(u_n|u_n|^{\varepsilon_n})}{\|u_n\|^{p-1}} \\ &\le f^0 \|u_n\|^{\varepsilon_n(p-1)} \\ &\to f^0, \quad n \to +\infty \end{aligned} \tag{9}$$

for any $t \in [1, T]_Z$. Connecting (6), (8) with (9), we may assume that $v_n \to v$ and $\|v\| = 1$. Therefore, v lies in the closure of S^{σ}.

Let us prove that in fact $v \in S^{\sigma}$. If $v \notin S^{\sigma}$, then $v \in \partial S^{\sigma}$. Hence v has at least one double zero in $[1, T]_Z$. We assume that there exists $t_0 \in [1, T]_Z$ such that either $v_n(t_0) \to 0, \Delta v_n(t_0) \to 0$ or $v_n(t_0) \to 0, v_n(t_0 - 1)v_n(t_0 + 1) \ge 0$ as $n \to +\infty$. By Lemma 3.1, we can see that $v_n \equiv 0$, which contradicts $\|v\| = 1$. Hence $v \in S^{\sigma}$.

In order to obtain the interval of λ, we will now compare v and ϕ_1^{σ} in the spirit of the Picone's identity (cf. [16], Lemma 1). We know that v_n satisfies

$$\Delta[\varphi_p(\Delta v_n(t-1))] + \Big(\lambda_n a(t) + a(t)\frac{f_n(t)}{\varphi_p(v_n(t))} + \frac{g_n(t)}{\varphi_p(v_n(t))}\Big)\varphi_p(v_n(t)) = 0,$$

and ϕ_1^σ satisfies

$$\Delta[\varphi_p(\Delta\phi_1^\sigma(t-1))] + \lambda_1 a(t)\varphi_p(\phi_1^\sigma(t)) = 0.$$

Since $v_n \in S^\sigma, \phi_1^\sigma \in S^\sigma$. In view of Lemma 2.2, we can obtain that

$$\begin{aligned}
&\Delta\Big\{v_n(t-1)\varphi_p(\Delta v_n(t-1)) - \frac{|v_n(t-1)|^p}{\varphi_p(\phi_1^\sigma(t-1))}\varphi_p(\Delta\phi_1^\sigma(t-1))\Big\}\\
&= v_n(t)\{\Delta[\varphi_p(\Delta v_n(t-1))]\} - \frac{|v_n(t)|^p}{\varphi_p(\phi_1^\sigma(t))}\{\Delta[\varphi_p(\Delta\phi_1^\sigma(t-1))]\}\\
&+ [|\Delta v_n(t-1)|^p + \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t-1))}|v_n(t-1)|^p - \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t))}\big(v_n(t)\big)^p]\\
&= \big(\lambda_1 a(t) - \lambda_n a(t) - a(t)\frac{f_n(t)}{\varphi_p(v_n(t))} - \frac{g_n(t)}{\varphi_p(v_n(t))}\big)\big(v_n(t)\big)^p\\
&+ |\Delta v_n(t-1)|^p + \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t-1))}|v_n(t-1)|^p - \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t))}\big(v_n(t)\big)^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{t=1}^{T}\Delta\Big\{v_n(t-1)\varphi_p(\Delta v_n(t-1)) - \frac{|v_n(t-1)|^p}{\varphi_p(\phi_1^\sigma(t-1))}\varphi_p(\Delta\phi_1^\sigma(t-1))\Big\}\\
&= \sum_{t=1}^{T}\big(\lambda_1 a(t) - \lambda_n a(t) - a(t)\frac{f_n(t)}{\varphi_p(v_n(t))} - \frac{g_n(t)}{\varphi_p(v_n(t))}\big)\big(v_n(t)\big)^p\\
&+ \sum_{t=1}^{T}\Big[|\Delta v_n(t-1)|^p + \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t-1))}|v_n(t-1)|^p - \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t))}\big(v_n(t)\big)^p\Big].
\end{aligned}$$

By calculation, we know that

$$\sum_{t=1}^{T}\Delta\Big\{v_n(t-1)\varphi_p(\Delta v_n(t-1)) - \frac{|v_n(t-1)|^p}{\varphi_p(\phi_1^\sigma(t-1))}\varphi_p(\Delta\phi_1^\sigma(t-1))\Big\} = 0.$$

Since

$$|\Delta v_n(t-1)|^p + \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t-1))}|v_n(t-1)|^p - \frac{\varphi_p(\Delta\phi_1^\sigma(t-1))}{\varphi_p(\phi_1^\sigma(t))}\big(v_n(t)\big)^p \geq 0, \quad t \in [1,T]_Z,$$

and the equality holds if and only if $\Delta v_n(t-1) = \frac{v_n(t-1)\Delta\phi_1^\sigma(t-1)}{\phi_1^\sigma(t-1)}$. Combining the above conclusion, we conclude that

$$\sum_{t=1}^{T}\big(\lambda_1 a(t) - \lambda_n a(t) - a(t)\frac{f_n(t)}{\varphi_p(v_n(t))} - \frac{g_n(t)}{\varphi_p(v_n(t))}\big)\big(v_n(t)\big)^p \leq 0. \tag{10}$$

Similarly, by swapping v_n and ϕ_1^σ, we can obtain that

$$\sum_{t=1}^{T}\big(\lambda_1 a(t) - \lambda_n a(t) - a(t)\frac{f_n(t)}{\varphi_p(v_n(t))} - \frac{g_n(t)}{\varphi_p(v_n(t))}\big)\big(\phi_1^\sigma(t)\big)^p \geq 0. \tag{11}$$

In view of (10), $(\mathbf{H_1})$ $(\mathbf{H_2})$, we see that

$$\begin{aligned}\sum_{t=1}^T \big(\lambda_1-\lambda)a(t)\big(v_n(t)\big)^p &\le \lim_{n\to+\infty}\sum_{t=1}^T a(t)\frac{f(t,u_n|u_n|^{\varepsilon_n},\lambda_n)}{\|u_n\|^{p-1}}\big(v_n(t)\big)^p\\ &\le \lim_{n\to+\infty}\sum_{t=1}^T a(t)\frac{f(s,u_n|u_n|^{\varepsilon_n},\lambda_n)}{|\varphi_p(u_n|u_n|^{\varepsilon_n})|}\cdot|\varphi_p(|u_n|^{\varepsilon_n})|\big(v_n(t)\big)^p\\ &\le \sum_{t=1}^T a(t)f^0\big(v_n(t)\big)^p.\end{aligned}$$

It follows that $\lambda\ge\lambda_1-f^0$.

In view of (11), $(\mathbf{H_1})$ $(\mathbf{H_2})$, we see that

$$\begin{aligned}\sum_{t=1}^T \big(\lambda_1-\lambda)a(t)\big(\phi_1^\sigma(t)\big)^p &\ge \lim_{n\to+\infty}\sum_{t=1}^T a(t)\frac{f(t,u_n|u_n|^{\varepsilon_n},\lambda_n)}{\|u_n\|^{p-1}}\big(\phi_1^\sigma(t)\big)^p\\ &\ge \lim_{n\to+\infty}\sum_{t=1}^T a(t)\frac{f(s,u_n|u_n|^{\varepsilon_n},\lambda_n)}{|\varphi_p(u_n|u_n|^{\varepsilon_n})|}\cdot|\varphi_p(|u_n|^{\varepsilon_n})|\big(\phi_1^\sigma(t)\big)^p\\ &\ge \sum_{t=1}^T a(t)f_0\big(\phi_1^\sigma(t)\big)^p.\end{aligned}$$

It follows that $\lambda\le\lambda_1-f_0$.

Hence, $\lambda\in I^0$.

□

Based on the analysis above, we have the following interval bifurcation result for the problem (2).

Theorem 3.3. *Assume* $(\mathbf{H_1})$-$(\mathbf{H_2})$ *hold. The connected component* $\mathcal{C}^+$ *of* $\mathscr{C}^+\cup(I^0\times\{0\})$, *containing* $I^0\times\{0\}$ *is unbounded and* $\mathcal{C}^+\subset(\mathbb{R}\times S^+)\cup(I^0\times\{0\})$, *the connected component* $\mathcal{C}^-$ *of* $\mathscr{C}^-\cup(I^0\times\{0\})$, *containing* $I^0\times\{0\}$ *is unbounded and* $\mathcal{C}^-\subset(\mathbb{R}\times S^-)\cup(I^0\times\{0\})$.

Proof. Without loss of generality, we only prove the case of $\mathcal{C}^+$, and we can prove $\mathcal{C}^-$ in the same method. Let $\mathcal{C}^+$ be the component of $\mathscr{C}^+\cup(I^0\times\{0\})$ containing $I^0\times\{0\}$. We divide the proof into the following two steps.

Step 1. Firstly, we prove that $\mathcal{C}^+\subset(\mathbb{R}\times S^+)\cup(I^0\times\{0\})$;

For any $(\lambda_*,u_*)\in\mathcal{C}^+$, then either $u_*\in S^+$ or $u_*\in\partial S^+$. if $u_*\in\partial S^+$, then there exists $t_0\in[1,T]_Z$ such that

$$u_*(t_0)=0,\ \Delta u_*(t_0)=0\quad\text{or}\quad u_*(t_0)=0,\ \Delta u_*(t_0-1)\Delta u_*(t_0+1)\ge 0.$$

In view of the Lemma 3.1, it follows that $u_*\equiv 0$. Hence, there exists a sequence $\{(\lambda_n,u_n)\}\subset\mathbb{R}\times S^+$ such that (λ_n,u_n) is a solution of (5) corresponding to $\varepsilon=0$, and $(\lambda_n,u_n)\to(\lambda_*,0)$ in $\mathbb{R}\times E$. Lemma 3.2 implies that $\lambda_*\in I^0$. Thus $(\lambda_*,u_*)\in I^0\times\{0\}$.

If $u_* \in S^+$, we see that

$$\begin{cases} -\Delta[\varphi_p(\Delta u_*(t-1))] = \lambda_* a(t)\varphi_p(u_*(t)) + a(t)f(t,u_*(t),\lambda) + g(t,u_*(t),\lambda), & t\in[1,T]_Z, \\ u_*(0) = u_*(T+1) = 0, \end{cases}$$

This implies that $\mu = 1$ is a eigenvalue of the following problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \mu\Big(\lambda_* a(t) + \frac{a(t)f(t,u_*(t),\lambda_*)}{\varphi_p(u_*(t))} + \frac{g(t,u_*(t),\lambda_*)}{\varphi_p(u_*(t))}\Big)\varphi_p(u(t)), & t\in[1,T]_Z, \\ u(0) = u(T+1) = 0. \end{cases}$$

By $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we obtain that for any $\tau > 0$, there exists a positive constant $\gamma \le 1$ such that when $|s| \in [0,\gamma]$, there is

$$|g(t,s,\lambda)| \le \tau\varphi_p(|s|)$$

uniformly for $t \in [1,T]_Z$ and fixed λ. Hence, we deduce that

$$K(t,u_*,\lambda_*) \le |\lambda_*| \max_{t\in[1,T]_Z} a(t) + H + \tau + \max_{t\in[1,T]_Z} \max_{|s|\in[\gamma,\|u_*\|]} \left|\frac{f(t,s,\lambda_*) + g(t,s,\lambda_*)}{\varphi_p(s)}\right|,$$

$$H = \max_{t\in[1,T]_Z} a(t)\cdot \max\{|f_0|,|f^0|\},$$

$$K(t,u_*,\lambda_*) = \lambda_* a(t) + \frac{a(t)f(t,u_*(t),\lambda_*)}{\varphi_p(u_*(t))} + \frac{g(t,u_*(t),\lambda_*)}{\varphi_p(u_*(t))}.$$

Similar to the proof in [5, Proposition 3.2], it is easy to get that $u > 0$ holds on $[1,T]_Z$. Therefore, $\mathcal{C}^+ \subset (\mathbb{R}\times S^+)\cup(I^0\times\{0\})$. The same can be proved $\mathcal{C}^- \subset (\mathbb{R}\times S^-)\cup(I^0\times\{0\})$.

Step 2. We can prove that $\mathcal{C}^+$ and $\mathcal{C}^-$ are both unbounded in $\mathbb{R}\times E$ by using the method of Theorem 2.1 in [13]. □

Corollary 3.4. *If $f \equiv 0$, then there exist two unbounded continua $\mathcal{P}^+$ and $\mathcal{P}^-$ of solutions of (2), bifurcating from $(\lambda_1, 0)$ and $\mathcal{P}^+ \subset (\mathbb{R}\times S^+)\cup\{(\lambda_1,0)\}, \mathcal{P}^- \subset (\mathbb{R}\times S^-)\cup\{(\lambda_1,0)\}$.*

4. Interval bifurcation from infinity

As in the semilinear case in [19], Rabinowitz established the global bifurcation results from infinity. Inspired by the Theorem 1.6 of [19], next, we shall study the unilateral global bifurcation phenomena of problem (2) which bifurcates from infinity. We use $(\mathbf{H_3})$ and $(\mathbf{H_4})$ instead of $(\mathbf{H_1})$ and $(\mathbf{H_2})$, let $\mathscr{D}$ to denote the set of nontrivial solutions of (2) under assumptions $(\mathbf{H_3})$ and $(\mathbf{H_4})$. Our second main result is the following theorem.

Theorem 4.1. *Assume* $(\mathbf{H_3})$-$(\mathbf{H_4})$ *hold. Let* $I^\infty = [\lambda_1 - f^\infty, \lambda_1 - f_\infty]$ *and* $\sigma = +$ *and* $-$. *There exists a component* $\mathcal{D}^\sigma$ *of* $\mathscr{D} \cup (I^\infty \times \{\infty\})$, *containing* $I^\infty \times \{\infty\}$. *Moreover, if* $\Lambda \subset \mathbb{R}$ *is an interval such that* $\Lambda \cap (I^\infty) = I^\infty$ *and* $\mathcal{M}$ *is a neighborhood of* $I^\infty \times \{\infty\}$ *whose projection on* $\mathbb{R}$ *lies in* Λ *and whose projection on* E *is bounded away from 0, then either*

1°. $\mathcal{D}^\sigma - \mathcal{M}$ *is bounded in* $\mathbb{R} \times E$ *and* $\mathcal{D}^\sigma - \mathcal{M}$ *meets* $\mathcal{R} = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ *or*

2°. $\mathcal{D}^\sigma - \mathcal{M}$ *is unbounded.*

Furthermore, if 2° *occurs and* $\mathcal{D}^\sigma - \mathcal{M}$ *has a bounded projection on* $\mathbb{R}$, *then* $\mathcal{D}^\sigma - \mathcal{M}$ *meets* $I_j^\infty \times \{\infty\}$ *for some* $j \neq 1$, *where* $I_j^\infty = [\lambda_j - f^\infty, \lambda_j - f_\infty]$.

Proof. If $(\lambda, u) \in \mathscr{D}$ and $\|u\| \neq 0$. Let $\omega = \frac{u}{\|u\|^2}$, dividing (2) by $\|u\|^{2(p-1)}$, we obtain

$$\begin{cases} -\Delta[\varphi_p(\Delta\omega(t-1))] = \lambda a(t)\varphi_p(\omega(t)) + a(t)\frac{f(t,u,\lambda)}{\|u\|^{2(p-1)}} + \frac{g(t,u,\lambda)}{\|u\|^{2(p-1)}}, \quad t \in [1, T]_Z, \\ \omega(0) = \omega(T+1) = 0. \end{cases} \tag{12}$$

Define

$$\widetilde{f}(t, \omega, \lambda) = \begin{cases} \|\omega\|^{2(p-1)} f(t, \frac{\omega}{\|\omega\|^2}, \lambda), & \omega \neq 0, \\ 0, & \omega = 0, \end{cases}$$

and

$$\widetilde{g}(t, \omega, \lambda) = \begin{cases} \|\omega\|^{2(p-1)} g(t, \frac{\omega}{\|\omega\|^2}, \lambda), & \omega \neq 0, \\ 0, & \omega = 0. \end{cases}$$

Obviously, (12) is equivalent to

$$\begin{cases} -\Delta[\varphi_p(\Delta\omega(t-1))] = \lambda a(t)\varphi_p(\omega(t)) + a(t)\widetilde{f}(t, \omega, \lambda) + \widetilde{g}(t, \omega, \lambda), \quad t \in [1, T]_Z, \\ \omega(0) = \omega(T+1) = 0. \end{cases} \tag{13}$$

It is easily can be seen that $(\mathbf{H_3})$ and $(\mathbf{H_4})$ imply

$$\liminf_{|\omega| \to 0^+} \frac{\widetilde{f}(t, \omega, \lambda)}{|\omega|^{p-1}} = f_\infty, \qquad \limsup_{|\omega| \to 0^+} \frac{\widetilde{f}(t, \omega, \lambda)}{|\omega|^{p-1}} = f^\infty$$

and $\widetilde{g}(r, \omega, \lambda) = o(|\omega|^{p-1})$, near $\omega = 0$, uniformly for $t \in [1, T]_Z$ and in every bounded interval of λ.

We using Theorem 3.3 to the problem (13), which implies that there exists connected component D^σ of $\mathscr{C}^\sigma \cup (I^\infty \times \{0\})$, containing $I^\infty \times \{0\}$ is unbounded and

$$\mathrm{D}^\sigma \subset (\mathbb{R} \times S^\sigma \cup (I^\infty \times \{0\})).$$

In view of $\omega \to \frac{\omega}{\|\omega\|^2} = u$, it follows that $\mathrm{D}^\sigma \to \mathcal{D}^\sigma$. Furthermore, the conclusions in the theorem can be obtained. □

Combining the facts of Theorem 3.3 with the proof of Theorem 4.1, we can obtain the following result.

Theorem 4.2. *There exists a neighborhood $\mathcal{N} \subset \mathcal{M}$ of $I^\infty \times \{\infty\}$ such that $(\mathcal{D}^\sigma \cap \mathcal{N}) \subset (\mathbb{R} \times S^\sigma \cup (I^\infty \times \{\infty\}))$ for $\sigma = +$ and $\sigma = -$.*

Remark 4.3. *Note that we can only construct the connected components of positive and negative solutions, but cannot construct connected components of change-sign solutions. The main reason is that we do not know whether other eigenvalues of the problem (3) are simple. If we can overcome this difficulty in the future, the conclusions of these two sections can be naturally extended.*

5. Existence of one-sign solutions for half-quasilinear discrete problem

Based on the global interval bifurcation conclusions of the previous sections, we first study the spectrum of the following half-quasilinear eigenvalue problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + \alpha(t)\varphi_p(u^+(t)) + \beta(t)\varphi_p(u^-(t)), & t \in [1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases} \tag{14}$$

where $\lambda \in \mathbb{R}$ is a parameter, $u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\}, a : [1,T]_Z \to (0, +\infty), \alpha, \beta : [1,T]_Z \to \mathbb{R}$.

We say λ is a half-eigenvalue of (14) if there exists a nontrivial solution (λ, u). λ is said to be simple if all solutions (λ, ω) of (14), with $\omega = ku$ ($k > 0$ is a constant). A half-eigenvalue is called a principal half-eigenvalue if its corresponding eigenfunction is positive or negative.

Theorem 5.1. *There exist two simple half-eigenvalues λ_+ and λ_- for problem (14). The corresponding half-quasilinear solutions are in $\{\lambda_+\} \times S^+$ and $\{\lambda_-\} \times S^-$. Furthermore, aside from λ_+ and λ_-, the problem (14) has no other half-eigenvalue with positive or negative eigenfunction.*

Proof. According to Lemma 3.2, for $\sigma = +$ or $\sigma = -$, the problem (14) has at least one solution $(\lambda_\sigma, u_\sigma) \in \mathbb{R} \times S^\sigma$. Note that $\{(\lambda_\sigma, ku_\sigma), k > 0\}$ are half-quasilinear solutions in $\{\lambda_\sigma\} \times S^\sigma$. We only prove the case of $\sigma = -$, since the case of $\sigma = +$ is similar.

We claim that for any solution (λ, u) of problem (14) with $u \in S^-$, we have $\lambda = \lambda_-$ and $u = ku_-$ for some constant $k > 0$.

Similar to the arguments of the Lemma 3.2, we have that

$$\begin{aligned}&\sum_{t=1}^{T}\Delta\Big\{u^-(t-1)\varphi_p(\Delta u^-(t-1))-\frac{|u^-(t-1)|^p}{\varphi_p(u(t-1))}\varphi_p(\Delta u(t-1))\Big\}\\&=\sum_{t=1}^{T}\big(\lambda_--\lambda\big)a(t)|u^-(t)|^p\\&+\sum_{t=1}^{T}\Big[|\Delta u^-(t-1)|^p+\frac{\varphi_p(\Delta u(t-1))}{\varphi_p(u(t-1))}|u^-(t-1)|^p-\frac{\varphi_p(\Delta u(t-1))}{\varphi_p(u(t))}|u^-(t)|^p\Big].\end{aligned}$$

It is easy to see that

$$\sum_{t=1}^{T}\Delta\Big\{u^-(t-1)\varphi_p(\Delta u^-(t-1))-\frac{|u^-(t-1)|^p}{\varphi_p(u(t-1))}\varphi_p(\Delta u(t-1))\Big\}=0$$

$$\sum_{t=1}^{T}\Big[|\Delta u^-(t-1)|^p+\frac{\varphi_p(\Delta u(t-1))}{\varphi_p(u(t-1))}|u^-(t-1)|^p-\frac{\varphi_p(\Delta u(t-1))}{\varphi_p(u(t))}|u^-(t)|^p\geq 0,$$

and the equality holds if and only if $\frac{\Delta u^-(t-1)}{u^-(t-1)}=\frac{\Delta u(t-1)}{u(t-1)}$.

Hence,

$$\sum_{t=1}^{T}\big(\lambda_--\lambda\big)a(t)|u^-(t)|^p\leq 0,$$

which implies $\lambda\geq\lambda_-$. Similarly,

$$\sum_{t=1}^{T}\big(\lambda_--\lambda\big)a(t)|u(t)|^p\geq 0,$$

it follows that $\lambda\leq\lambda_-$. Thus there exists constant $k>0$ such that $u=ku_-$ and $\lambda=\lambda_-$. $\square$

Remark 5.2. $\min\{\lambda_-,\lambda_+\}$ *is the smallest half-eigenvalue of (14).*

Remark 5.3. *By some simple computations, we obtain that if $\beta\equiv 0$, then $\lambda_-=\lambda_1$; if $\alpha\equiv 0$, then $\lambda_+=\lambda_1$; if $\alpha\equiv\beta\equiv 0$, then $\lambda_-=\lambda_+=\lambda_1$.*

Next, we discuss the bifurcation phenomena of the problem

$$\begin{cases}-\Delta[\varphi_p(\Delta u(t-1))]=\lambda a(t)\varphi_p(u(t))+\alpha(t)\varphi_p(u^+(t))+\beta(t)\varphi_p(u^-(t))+h(t,u(t),\lambda),\\ \qquad t\in[1,T]_Z,\\ u(0)=u(T+1)=0,\end{cases}\tag{15}$$

where $h(t,s,\lambda)=o(|s|^{p-1})$ near $s=0$, uniformly for $t\in[1,T]_Z$ and λ on bounded sets.

Theorem 5.4. *For $\sigma = +$ or $-$, $(\lambda_\sigma, 0)$ is a bifurcation point for problem (15). Moreover, there exists an unbounded continuum C^σ of solutions to the problem (15) such that $C^\sigma \subset \big((\mathbb{R}\times S^\sigma) \cup \{(\lambda_\sigma, 0)\}\big)$.*

Proof. Let $\overline{\alpha} = \max\limits_{t\in[1,T]_Z} |\alpha(t)|, \overline{\beta} = \max\limits_{t\in[1,T]_Z} |\beta(t)|$ and

$$I = [\lambda_1 - \frac{\overline{\alpha}+\overline{\beta}}{a_0}, \lambda_1 + \frac{\overline{\alpha}+\overline{\beta}}{a_0}],$$

where $a_0 = \min\limits_{t\in[1,T]_Z} a(t)$. It follows from Theorem 3.3 that from $I\times\{0\}$ it bifurcates two unbounded continua $\mathbf{C}^+$ and $\mathbf{C}^-$ in $\mathbb{R}\times E$ of the solutions to the problem (15). Further,

$$\mathbf{C}^\sigma \subset (\mathbb{R}\times S^\sigma) \cup (I\times\{0\}),$$

where $\sigma = +$ or $-$. We claim that $\mathbf{C}^\sigma \cap (\mathbb{R}\times\{0\}) = (\lambda_\sigma, 0)$. In fact, let $(\lambda_n, u_n) \in C^\sigma, u_n \not\equiv 0$ and $(\lambda_n, u_n) \to (\lambda, 0)$. Set $\omega_n = \frac{u_n}{\|u_n\|}$, thus ω_n satisfies

$$-\Delta[\varphi_p(\Delta\omega_n(t-1))] = \lambda_n a(t)\varphi_p(\omega_n(t)) + \alpha(t)\varphi_p(\omega_n^+(t)) + \beta(t)\varphi_p(\omega_n^-(t)) + \frac{h(t,u_n(t),\lambda_n)}{\|u_n\|^{p-1}}.$$

According to the proof of the (8) and the compactness of G_p, we deduce that $\omega_n \to \omega_0$ in E as $n\to\infty$. Hence, ω_0 satisfies

$$-\Delta[\varphi_p(\Delta\omega_0(t-1))] = \lambda a(t)\varphi_p(\omega_0(t)) + \alpha(t)\varphi_p(\omega_0^+(t)) + \beta(t)\varphi_p(\omega_0^-(t))$$

and $\|\omega_0\| = 1$. This implies that $\lambda = \lambda_\sigma$.

Let $\mathbb{C}^\sigma$ denote the closure in $\mathbb{R}\times E$ of the solutions set $\{(\lambda,\omega) : \omega\in S^\sigma\}$ of (15), we have

$$\mathbb{C}^\sigma \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_\sigma, 0)\}.$$

If $\mathbf{C}^\sigma$ is the component given by Theorem 3.3, we define $C^\sigma = \mathbf{C}^\sigma \cap \mathbb{C}^\sigma$. It is easy to verify that C^σ is an unbounded continuum of $\mathbb{C}^\sigma$ and

$$(\lambda_\sigma, 0) \in C^\sigma \subset \big((\mathbb{R}\times S^\sigma) \cup \{(\lambda_\sigma, 0)\}\big).$$

□

According to the above spectral results and bifurcation conclusions, we discuss the existence of one-sign solutions for the p-Laplacian discrete problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \mu a(t)F(u(t)) + \alpha(t)\varphi_p(u^+(t)) + \beta(t)\varphi_p(u^-(t)), \\ \quad t\in[1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases} \tag{16}$$

where $\mu \neq 0$ is a parameter, we assume that $F \in C(\mathbb{R}, \mathbb{R})$ satisfies

($\mathbf{H_5}$) $sF(s) > 0$ for all $s \neq 0$;

($\mathbf{H_6}$) there exist $F_0, F_\infty \in (0, +\infty)$ such that

$$F_0 = \lim_{|s| \to 0^+} \frac{F(s)}{\varphi_p(s)}, \qquad F_\infty = \lim_{|s| \to +\infty} \frac{F(s)}{\varphi_p(s)}.$$

Under the above assumptions, we can obtain the following results for the existence of one-sign solutions.

Theorem 5.5. *If $\mu \in \big(\min\{\frac{\lambda_\sigma}{F_0}, \frac{\lambda_\sigma}{F_\infty}\}, \max\{\frac{\lambda_\sigma}{F_\infty}, \frac{\lambda_\sigma}{F_0}\}\big)$, then (16) has at least one solution u_σ such that $\sigma u_\sigma > 0$ in $[1, T]_Z$, where $\sigma = +$ or $-$.*

Proof. By ($\mathbf{H_6}$), it is easy to see that there exists $\iota \in C(\mathbb{R}, \mathbb{R})$ such that

$$F(s) = F_0 \varphi_p(s) + \iota(s).$$

Clearly,

$$\lim_{|s| \to 0^+} \frac{\iota(s)}{\varphi_p(s)} = 0.$$

Let us consider

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \mu a(t) F_0 \varphi_p(u(t)) + \alpha(t)\varphi_p(u^+(t)) + \beta(t)\varphi_p(u^-(t)) + \mu a(t)\iota(u(t)), \\ \qquad t \in [1, T]_Z, \\ u(0) = u(T+1) = 0 \end{cases} \tag{17}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. Applying the Theorem 5.4 to (17), we know that there is a nontrivial solution branch $\mathcal{C}$ of (17) bifurcating from $(\frac{\lambda_\sigma}{F_0}, 0)$ such that $\mathcal{C} \subset \big((\mathbb{R} \times S^\sigma) \cup \{(\frac{\lambda_\sigma}{F_0}, 0)\}\big)$, and it joins $(\frac{\lambda_\sigma}{F_0}, 0)$ to infinity.

Let us prove that $\mathcal{C}$ joins $(\frac{\lambda_\sigma}{F_0}, 0)$ to $(\frac{\lambda_\sigma}{F_\infty}, +\infty)$. Set $(\lambda_n, u_n) \in \mathcal{C}, u_n \not\equiv 0$, we know that

$$\lambda_n + \|u_n\| \to +\infty.$$

We claim that if there exists positive constant K such that $|\lambda_n| \in [0, K]$ holds for any $n \in \mathbb{N}$, then $\mathcal{C}$ joins $(\frac{\lambda_\sigma}{F_0}, 0)$ to $(\frac{\lambda_\sigma}{F_\infty}, +\infty)$.

Note that

$$\|u_n\| \to +\infty, \qquad \text{as } n \to +\infty.$$

We can see that there exists $\kappa \in C(\mathbb{R}, \mathbb{R})$ such that

$$F(s) = F_\infty \varphi_p(s) + \kappa(s).$$

Obviously,

$$\lim_{|s|\to+\infty}\frac{\kappa(s)}{\varphi_p(s)}=0.$$

Consider the problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))]=\mu a(t)F_\infty\varphi_p(u(t))+\alpha(t)\varphi_p(u^+(t))+\beta(t)\varphi_p(u^-(t))+\mu a(t)\kappa(u(t)),\\ \qquad t\in[1,T]_Z,\\ u(0)=u(T+1)=0.\end{cases}\tag{18}$$

Set $v_n=\frac{u_n}{\|u_n\|^2}$, dividing (18) by $\|u_n\|^{2(p-1)}$, we obtain

$$\begin{cases} -\Delta[\varphi_p(\Delta v_n(t-1))]=\lambda_n a(t)F_\infty\varphi_p(v_n(t))+\alpha(t)\varphi_p(v_n^+(t))+\beta(t)\varphi_p(v_n^-(t))+\\ \ \lambda_n a(t)\frac{\kappa(u_n)}{\|u_n\|^{2(p-1)}},\quad t\in[1,T]_Z,\\ v_n(0)=v_n(T+1)=0.\end{cases}\tag{19}$$

Noting that

$$\lim_{n\to+\infty}\frac{\kappa(u_n(t))}{\|u_n\|^{2(p-1)}}=0.$$

The boundedness of v_n ensures that $v_n\to v$ and $v\in E$, thus

$$-\Delta[\varphi_p(\Delta v(t-1))]=\lambda a(t)F_\infty\varphi_p(v(t))+\alpha(t)\varphi_p(v^+(t))+\beta(t)\varphi_p(v^-(t)),$$

where $\lambda=\lim\limits_{n\to+\infty}\lambda_n$. By virtue of Theorem 5.1 and $v\to\frac{v}{\|v\|^2}=u$, it follows that $\lambda F_\infty=\lambda_\sigma$. Hence, $\lambda=\frac{\lambda_\sigma}{F_\infty}$.

Consequently, $\mathcal{C}$ joins $(\frac{\lambda_\sigma}{F_0},0)$ to $(\frac{\lambda_\sigma}{F_\infty},\infty)$.

In fact, we show that there exists a constant K such that $|\lambda_n|\in[0,K]$, for any $n\in\mathbb{N}$. Suppose there is no such K, choosing a subsequence and relabelling if necessary, it follows that $\lim\limits_{n\to+\infty}|\lambda_n|=+\infty$. Since $(\lambda_n,u_n)\in\mathcal{C}$, thus

$$-\Delta[\varphi_p(\Delta u_n(t-1))]=\lambda_n a(t)\widehat{F_n}(t)\varphi_p(u_n(t))+\alpha(t)\varphi_p(u_n^+(t))+\beta(t)\varphi_p(u_n^-(t)),$$

where

$$\widehat{F_n}(t)=\begin{cases}\frac{F(u_n(t))}{\varphi_p(u_n(t))}, & u_n(t)\neq 0,\\ F_0, & u_n(t)=0.\end{cases}$$

We see from $(\mathbf{H_5})$-$(\mathbf{H_6})$ that there exists constant $\chi>0$ such that $\frac{F(s)}{\varphi_p(s)}\geq\chi,\forall s\neq 0$. Therefore,

$$\lim_{n\to+\infty}\lambda_n\widehat{F_n}(t)=\pm\infty.$$

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Suppose that ϕ_σ is the eigenfunction corresponding to λ_σ. If $\lim\limits_{n\to+\infty}\lambda_n\widehat{F_n}(t)=-\infty$. Applying the Theorem 3 of [16] to ϕ_σ and u_n, then ϕ_σ must change sign for n large enough, it is conflicted.. Similarly, we can prove that $\lim\limits_{n\to+\infty}\lambda_n\widehat{F_n}(t)=+\infty$ is impossible.

$\square$

Finally, we present an example to illustrate the result of Theorem 5.5.

Example 5.6. Consider the problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))]=\mu F(u(t))+\varphi_p(u^+(t))+\varphi_p(u^-(t)),\ t\in[1,6]_Z, \\ u(0)=u(7)=0, \end{cases} \tag{20}$$

where

$$F(s)=f_1(s)+f_2(s)\varphi_p(s),$$

$$f_1(s)=\begin{cases} 0, & s=0, \\ 2\varphi_p(s), & |s|\in(0,1], \\ 10\varphi_p(s)-\frac{1}{4}, & |s|\in(1,\infty) \end{cases}$$

and

$$f_2(s)=\frac{1}{5}\quad \text{for } |s|\in[0,\infty).$$

It is clear that

$$\lim_{|s|\to0^+}\frac{F(s)}{\varphi_p(s)}=F_0,\qquad \lim_{|s|\to+\infty}\frac{F(s)}{\varphi_p(s)}=F_\infty,$$

where

$$F_0=\frac{11}{5},\quad F_\infty=\frac{51}{5}.$$

Hence, if $\mu\in\big(\min\{\frac{5\lambda_\sigma}{11},\frac{5\lambda_\sigma}{51}\},\max\{\frac{5\lambda_\sigma}{51},\frac{5\lambda_\sigma}{11}\}\big)$, then problem (20) has at least a constant-sign solution u_σ satisfying $\sigma u_\sigma>0$ with $\sigma=+$ or $\sigma=-$, where λ_σ is the half-eigenvalue of problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))]=\lambda\varphi_p(u(t))+\varphi_p(u^+(t))+\varphi_p(u^-(t)),\quad t\in[1,6]_Z, \\ u(0)=u(7)=0. \end{cases}$$

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

The author read and approved the final manuscript.

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References

[1] G. D'Aguì, J. Mawhin, A. Sciammetta, Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian, J. Math. Anal. Appl., 447(1)(2017)383–397.

[2] D. Bai, Y. Chen, Global continuum of positive solutions for discrete p-Laplacian eigenvalue problems, Appl. Math., 60(4)(2015)343–353.

[3] H. Berestycki, On some nonlinear Sturm-Liouville problems, J. Differential Equations, 26(1997)375–390.

[4] M. Chehabi, O. Chakrone, Properties of the first eigenvalue with sign-changing weight of the discrete p-Laplacian and applications, Bol. Soc. Parana. Mat., 36(2)(2018)151–167.

[5] M. Cuesta, Eigenvalue problems for the p-Laplacian with indefinite weights, Electron. J. Differential Equations, 2001(33)(2001)1–9.

[6] Z. He, On the existence of positive solutions of p-Laplacian difference equations, J. Comput. Appl. Math., 161(2003)193–201.

[7] S. Heidarkhani, G.A. Afrouzi, G. Caristi, J. Henderson, S. Moradi, A variational approach to difference equations, J. Difference Equ. Appl., 22(2016)1761–1776.

[8] S. Heidarkhani, G.A. Afrouzi, J. Henderson, S. Moradi, G. Caristi, Variational approaches to p-Laplacian discrete problems of Kirchhoff-type, J. Difference Equ. Appl., 23(5)(2017)917–938.

[9] S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, Existence of multiple solutions for a perturbed discrete anisotropic equation, J. Difference Equ. Appl., 23(2017)1491–1507.

[10] D. Ji, W. Ge, Positive solutions for discrete boundary value problems with p-Laplacian, J. Appl. Anal., 16(1)(2010)107–119.

[11] W.G. Kelley, A.C. Peterson, Difference equations, An introduction with applications, Second edition, Harcourt/Academic Press, San Diego, 2001.

[12] M. Khaleghi Moghadam, S. Heidarkhani, J. Henderson, Infinitely many solutions for perturbed difference equations, J. Difference Equ. Appl., 20(7)(2014)1055–1068.

[13] R. Ma, G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, J. Funct. Anal., 265(8)(2013)1443–1459.

[14] Y. Tian, W. Ge, Two solutions for a discrete problem with a p-Laplacian, J. Appl. Math. Comput., 38(1-2)(2012)353–365.

[15] Q. Wang, D. Lu, Existence of three solutions to nonlinear discrete Dirichlet problem involving p-Laplacian, Ann. Differential Equations, 30(4)(2014)432–437.

[16] P. Řehák, Oscillatory properties of second order half-linear difference equations, Czechoslovak Math. J., 51(126)(2001)303–321.

[17] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, J. Math. Anal. Appl., 228(1)(1998)141–156.

[18] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal., 7(1971)487–513.

[19] P.H. Rabinowitz, On bifurcation from infinity, J. Differential Equations, 14(1973)462–475.

[20] P.H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, Rocky Mountain J. Math., 3(2)(1973)161–202.