ON (A, m, n)-ISOSYMMETRIC COMMUTING TUPLES OF OPERATORS ON A HILBERT SPACE

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ABSTRACT. In this paper, we consider a generalization of (A, m, n)-isosymmetric operators on a Hilbert space to the multivariable setting. For a d-tuple of commuting operators $\mathbf{T} = (T_1, \ldots, T_d)$, A a positive operator and m, n are two positive integers, we introduce the class of (A, m, n)-isosymmetric d-tuple of commuting operators and we explore some of their basic properties. Furthermore, we study the stability of an (A, m, n)-isosymmetric d-tuple of commuting operators $\mathbf{T}$ under the perturbation by a l-nilpotent d-tuple $\mathbf{Q} = (Q_1, \ldots, Q_d)$. Finally, we give some spectral properties of such family.

1. Introduction and Preliminary

Throughout this paper, we denote by H a complex Hilbert space with inner product $\langle ., . \rangle$, $\mathcal{B}(H)$ the algebra of all bounded linear operators on H and by $\mathcal{B}(H)^+$ the cone of positive (semi-definite) operators; that is,

$$\mathcal{B}(H)^+ := \{A \in \mathcal{B}(H) \ : \ \langle Ax, x\rangle \geq 0, \ \forall x \in H\}.$$

Note that any $A \in \mathcal{B}(H)^+$ defines a positive semi-definite sesquilinear form:

$$\begin{array}{cccc} \langle ., . \rangle_A & : & H \times H & \longrightarrow & \mathbb{C} \\ & & (u, v) & \longmapsto & \langle u, v\rangle_A = \langle Au, v\rangle. \end{array}$$

We denote by $\|.\|_A$ the semi-norm induced by $\langle ., . \rangle_A$; that is, $\|u\|_A = \langle u, u\rangle_A^{\frac{1}{2}}$. Furthermore, $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A. Hence, $\|.\|_A$ is a norm if and only if A is injective. We use the notations $\mathbb{N}$ and $\mathbb{C}$ the set of natural numbers and the set of complex numbers, respectively. For $d \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, let

$$\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(H)^d := \underbrace{\mathcal{B}(H) \times \cdots \times \mathcal{B}(H)}_{d-times},$$

and

$$|\alpha| := \sum_{i=1}^{d} \alpha_i, \quad \alpha! := \alpha_1! \ldots \alpha_d!, \quad \mathbf{T}^\alpha := T_1^{\alpha_1} \ldots T_d^{\alpha_d}, \quad \mathbf{T}^* := (T_1^*, \ldots, T_d^*),$$

where T^* is the adjoint of the operator T. Recall that a d-tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(H)^d$ is said to be commuting d-tuple of operators if $T_iT_j = T_jT_i$, for all $i, j \in \{1, \ldots, d\}$. Unless mentioned otherwise, we will denote by $\mathbf{T}$ a commuting d-tuple of bounded linear operators. For $T \in \mathcal{B}(H)$, $A \in \mathcal{B}(H)^+$ and n, $m \in \mathbb{N}$, we consider the following identities:

$$\begin{aligned} \mathcal{I}_m(T, A) &:= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} A T^k, \\ \mathcal{S}_n(T, A) &:= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} T^{*j} A T^{n-j}. \end{aligned}$$

In 1990, Agler [1] introduced the notion of m-isometric operators, which satisfies $\mathcal{I}_m(T, I) = 0$, where $I \in \mathcal{B}(H)$ is the identity operator. Alger and stakus in 1995, detailed the study of such

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family in a series of three papers [2, 3, 4]. This class of operators was naturally generalized on a Hilbert space by Sid Ahmed et al.[20], they introduced a family of operators in a semi-Hilbertian space called (A, m)-isometric operators, they showed many important results and proved that most of well-known properties related to m-isometries hold true for a such family. Similarly, J.W. Helton [14, 15] investigated the study of n-symmetries, which satisfy an identity of the form $\mathcal{S}_n(T, I) = 0$. Recently, Jeridi et al [16] has been introduced (A, n)-symmetries as a generalization of n-symmetries on a Hilbert space by considering an additional semi-inner product; that is, operators satisfying an identity of the form $\mathcal{S}_n(T, A) = 0$.
In his PhD Thesis, Stakus [22] introduced and study a class of operators called isosymmetries; that is, operators satisfying the following identity

$$T^{*2}T - T^*T^2 - T^* + T = 0.$$

This definition was extended to an higher order by Stakus in 2013 [21], he defines a class of operators includes m-isometries and n-symmetries called (m, n)-isosymmetries; that is, operators satisfy the following identity

$$\gamma_{m,n}(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}\mathcal{S}_n(T, I)T^j := \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}T^{*k}\mathcal{I}_m(T, I)T^{n-k} = 0$$

Many authors were involved in the development of this classes of operators. They considered certain types of operators like composition operators, elementary operators, shift operators, the dynamic of the orbits, etc. Furthermore, some results on sum, product, tensor product and the stability under powers or perturbation by a nilpotent operator are also obtained. For more information about these topics, we refer to see [5, 6, 7, 8, 10, 13, 19]
Some of these concepts were extended naturally from a single operator to a d-tuple of commuting operators. The m-isometric d-tuple of commuting operators was initiated by Gleason and Richter[11], they extended some properties that have appeared in the literature for the theory of m-isometries to the multi-variable setting. Continuing this work, Gu [12] obtained some results about sums, products and functions of m-isometries d-tuples of operators and they used to give more more examples of a such family. moreover, he constructed a class of m-isometric tuples of unilateral weighted shifts parametrized by polynomials.
Recall that [9] $\mathbf{T}$ is said to be (A, m)-isometric d-tuple of operators, if it satisfies the following equality:

$$\mathcal{I}_m(\mathbf{T}, A) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}A\mathbf{T}^{\alpha} = 0.$$

Similarly, M. Chō et al [17] investigated the study of (A, m)-symmetric d-tuple of commuting operators, namely d-tuple of operators $\mathbf{T}$ satisfying

$$\mathcal{S}_n(\mathbf{T}, A) := \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(T_1^* + \cdots + T_d^*)^k A(T_1 + \cdots + T_d)^{n-k} = 0.$$

It has been proven that in general, the properties of (A, m)-isometric and (A, n)-symmetric for a single operator hold true for (A, m)-isometric and (A, n)-symmetric d-tuple of commuting operators.
Recently, Rabaoui [18] investigated the study of (A, m, n)-isosymmetric operators; that is, operators satisfying the following identity

$$\gamma_{m,n}(T, A) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}\mathcal{S}_n(T, A)T^j = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}T^{*k}\mathcal{I}_m(T, A)T^{n-k} = 0,$$

This class of operators contains (A, m)-ismetric and (A, n)-symmetric operators. Inspired from the obvious works, we will extend the study of (A, m, n)-isosymmetries operators to the multi-variable case .
This paper is structured as follows. In Section 2, we introduce the class of (A, m, n)-isosymmetries for a d-tuple of commuting operators $\mathbf{T}$ and we study some various structural properties of such

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family. In particular, we prove that if $\mathbf{T}$ is (A,m,n)-isosymmetric d-tuple of operators, then the operator $(T_1+\cdots+T_d)^p$ is (A,m,n)-isosymmetric for any positive integer p. In Section 3, we treat the stability of an (A,m,n)-isosymmetric d-tuple of operators under the perturbation by an l-nilpotent d-tuple. In Section 4, we give some spectral properties. Specifically, we show that the approximate point spectrum of an (A,m,n)-isosymmetric d-tuple lies in the unit ball of $\mathbb{C}^d$ or in the set

$$\{\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{C}^d \;:\; Im(\sum_{i=1}^d\lambda_i)=0\}.$$

2. Structural properties of (A,m,n)-isosymmetric tuples of operators

In this section, we introduce the class of (A,m,n)-isosymmetric d-tuples of commuting operators and we give some of their basic properties.
For $A\in\mathcal{B}(H)^+$ and $m,n\in\mathbb{N}$, let

$$\gamma_{m,n}(\mathbf{T},A):=\begin{cases}\displaystyle\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha},\\ \displaystyle\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}(T_1^*+\cdots+T_d^*)^k\mathcal{I}_m(\mathbf{T},A)(T_1+\cdots+T_d)^{n-k},\end{cases}\tag{2.1}$$

Definition 2.1. We say that $\mathbf{T}$ is an (A,m,n)-isosymmetric d-tuples of commuting operators if

$$\gamma_{m,n}(\mathbf{T},A)=0.$$

Remark 2.2. We have the following particular cases.

1. When $d=1$, Definition 2.1 coincides with the definition of an (A,m,n)-isosymmetry for a single operator.
2. When $m=0$ (respectively $n=0$), Definition 2.1 coincides with the definitions of (A,n)-symmetric (respectively (A,m)-isometric) d-tuples of commuting operators.

Let use recall some symbols that well be used in the sequel. In [17], the authors defined a polynomial $\{(y-x)^n\}_a$ by

$$\{(y-x)^n\}_a=\Big\{\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}y^kx^{n-k}\Big\}_a=\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}y^kax^{n-k}.$$

For a d-tuple $\mathbf{T}$ and $A\in\mathcal{B}(H)^+$, we have

$$\begin{aligned}\Big(\Big\{\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}y^kx^{n-k}\Big\}_a\Big)(\mathbf{T},A)&:=\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}y^kax^{n-k}\Big|_{y=T_1^*+\cdots+T_d^*,x=T_1+\cdots+T_d,a=A}\\&=\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}(T_1^*+\cdots+T_d^*)^kA(T_1+\cdots+T_d)^{n-k}\\&=\mathcal{S}_n(\mathbf{T},A).\end{aligned}$$

Similarly, Rabaoui [18] defined two polynomials as follow.

$$\{(yx-1)^m\}_a=\Big\{\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}y^jx^j\Big\}_a=\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}y^jax^j.$$

$$\begin{aligned}\{(yx-1)^m\}\{(y-x)^n\}_a&=\Big\{\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}y^jx^j\Big\}\{(y-x)^n\}_a\\&=\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}y^j\{(y-x)^n\}_a\,x^j.\end{aligned}$$

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Then, for a d-tuple $\mathbf{T}$ and $A \in \mathcal{B}(H)^+$, we have

$$\begin{aligned}
\{(yx-1)^m\}_a(\mathbf{T},A) &:= \Big(\Big\{\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}y^jx^j\Big\}_a\Big)(\mathbf{T},A)\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}y^jax^j\Big|_{y=T_1^*+\cdots+T_d^*,x=T_1+\cdots+T_d,a=A}\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\mathbf{T}^{*\alpha}A\mathbf{T}^{\alpha}\\
&= \mathcal{I}_m(\mathbf{T},A).
\end{aligned}$$

and

$$\Big(\{(yx-1)^m\}\{(y-x)^n\}_a\Big)(\mathbf{T},A) := \Big(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}y^j\{(y-x)^n\}_a\,x^j\Big)(\mathbf{T},A)$$

$$\begin{aligned}
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}y^j\Big[\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^kax^{n-k}\Big]x^j\Big|_{y=T_1^*+\cdots+T_d^*,x=T_1+\cdots+T_d,a=A}\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(T_1^*+\cdots+T_d^*)^j\mathcal{S}_n(\mathbf{T},A)(T_1+\cdots+T_d)^j\\
&= \gamma_{m,n}(\mathbf{T},A).
\end{aligned}$$

In the following proposition, we prove a recursive formula for $\gamma_{m+1,n}(\mathbf{T},A)$ and $\gamma_{m,n+1}(\mathbf{T},A)$.

Proposition 2.3. *Let $A \in \mathcal{B}(H)^+$. The following identities hold.*

$$\gamma_{m+1,n}(\mathbf{T},A) = (T_1^*+\cdots+T_d^*)\gamma_{m,n}(\mathbf{T},A)(T_1+\cdots+T_d) - \gamma_{m,n}(\mathbf{T},A),$$

and

$$\gamma_{m,n+1}(\mathbf{T},A) = (T_1^*+\cdots+T_d^*)\gamma_{m,n}(\mathbf{T},A) - \gamma_{m,n}(\mathbf{T},A)(T_1+\cdots+T_d).$$

In particular, if $\mathbf{T}$ is an (A,m,n)-isosymmetric d-tuple of operators, then $\mathbf{T}$ is an (A,m',n')-isosymmetric d-tuple of operators of all $m \geq m'$ and $n \geq n'$.

Proof. Note that

$$\begin{aligned}
\mathcal{I}_m(\mathbf{T},A) &= \big\{(yx-1)^{m+1}\big\}_{\mathbf{a}})(\mathbf{T},A) = \big(y\big\{(yx-1)^m\big\}_{\mathbf{a}}x - \big\{(yx-1)^m\big\}_{\mathbf{a}}\big)(\mathbf{T},A)\\
&= (T_1^*+\ldots,T_d^*)\mathcal{I}_m(\mathbf{T},A)(T_1+\ldots,T_d) - \mathcal{I}_m(\mathbf{T},A),\\
\mathcal{S}_{n+1}(\mathbf{T},A) &= \big\{(y-x)^{n+1}\big\}_{\mathbf{a}} = \big(y\big\{(y-x)^n\big\}_{\mathbf{a}} - \big\{(y-x)^n\big\}_{\mathbf{a}}x\big)(\mathbf{T},A)\\
&= (T_1^*+\ldots,T_d^*)\mathcal{S}_A^n(T) - \mathcal{S}_n(\mathbf{T},A)(T_1+\ldots,T_d),
\end{aligned}$$

then by using (2.1) we get the two equalities desired, which enables us to conclude. □

Recall that two d-tuples of commuting operators $\mathbf{S}$ and $\mathbf{T}$ are unitary equivalent if there exists some unitary operator $U \in \mathcal{B}(H)$ such that $\mathbf{S} = (S_1,\ldots,S_d) = (U^*T_1U,\ldots,U^*T_dU)$.

Proposition 2.4. *If $\mathbf{S}$ and $\mathbf{T}$ are unitary equivalent, then $\mathbf{S}$ is an (A,m,n)-isosymmetric d-tuple of operators if and only if $\mathbf{T}$ is an (UAU^*,m,n)-isosymmetric d-tuple of operators.*

Proof. Since the following identity holds.

$$\mathcal{S}_n(\mathbf{S},A) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}U^*(T_1^*+\cdots+T_d^*)^kUAU^*(T_1+\cdots+T_d)^{n-k}U = U^*\mathcal{S}_n(\mathbf{T},UAU^*)U.$$

We obtain

$$\begin{aligned}
\gamma_{m,n}(\mathbf{S},A) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}U^*\sum_{|\alpha|=j}\frac{j!}{\alpha!}\mathbf{T}^{*\alpha}UU^*\mathcal{S}_n(T,UAU^*)UU^*\mathbf{T}^{\alpha}U\\
&= U^*\gamma_{m,n}(\mathbf{T},UAU^*)U.
\end{aligned}$$

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As a result, we obtain the requested equivalence. $\square$

Theorem 2.5. *If $\mathbf{T}$ is an (A,m,n)-isosymmetric d-tuple of operators, then $(T_1+\cdots+T_d)^k$ is an (A,m,n)-isosymmetric operator for any positive integer k.*

Proof. Let k be a positive integer, then the following equation holds:

$$\begin{aligned}
& (y^kx^k-1)^m(y^k-x^k)^n \\
= & \Big((yx-1)(y^{k-1}x^{k-1}+y^{k-2}x^{k-2}+\ldots+1)\Big)^m\Big((y-x)(y^{k-1}+y^{k-2}x+\ldots+x^{k-1})\Big)^n \\
= & \sum_{l=0}^{m(k-1)}\sum_{j=0}^{n(k-1)}\lambda_l\mu_j y^{m(k-1)-l}y^{n(k-1)-j}(yx-1)^m(y-x)^n x^j x^{m(k-1)-l},
\end{aligned}$$

where λ_l and μ_j are some constants. It follows that

$$\{(y^kx^k-1)^m\}\{(y^k-x^k)^n\}_a=\sum_{l=0}^{m(k-1)}\sum_{j=0}^{n(k-1)}\lambda_l\mu_j y^{m(k-1)-l}y^{n(k-1)-j}\{(yx-1)^m\}\{(y-x)^n\}_a x^j x^{m(k-1)-l}.$$

Hence

$$\begin{aligned}
& \gamma_{m,n}((T_1+\cdots+T_d)^k,A)=\{(y^kx^k-1)^m\}\{(y^k-x^k)^n\}_a(\mathbf{T},A) \\
= & \sum_{l=0}^{m(k-1)}\sum_{j=0}^{n(k-1)}\lambda_l\mu_j(T_1^*+\cdots+T_d^*)^{m(k-1)-l+n(k-1)-j}\gamma_{m,n}(\mathbf{T},A)(T_1+\cdots+T_d)^{j+m(k-1)-l}.
\end{aligned}$$

Consequently, we obtain the desired result. $\square$

The Euclidean operator norm of $\mathbf{T}$ is denoted by $\|\mathbf{T}\|$ and defined as $\|\mathbf{T}\|^2:=\sum_{l=1}^{d}\|T_l\|^2$. In the following Theorem, we prove that the class of (A,m,n)-isosymmetric d-tuple of operators is norm closed.

Theorem 2.6. *Let $A\in\mathcal{B}(H)^+$ and $(\mathbf{T}_p=(T_{1p},\ldots,T_{dp}))_p\in\mathcal{B}(H)^d$ be a sequence of (A,m,n)-isosymmetric d-tuple of operators such that $T_{jp}\longrightarrow T_j$ for each $j=1,\ldots,d$ as $p\to+\infty$ in the strong topology of $\mathcal{B}(H)$. Then $\mathbf{T}$ is an (A,m,n)-isosymmetric d-tuple of operators.*

Proof.

$$\begin{aligned}
& \|\gamma_{m,n}(\mathbf{T}_p,A)-\gamma_{m,n}(\mathbf{T},A)\|\leq\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\|\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T}_p,A)\mathbf{T}_p^{\alpha}-\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\| \\
\leq & \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\|\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T}_p,A)\mathbf{T}_p^{\alpha}-\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T}_p,A)\mathbf{T}^{\alpha}+\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T}_p,A)\mathbf{T}^{\alpha} \\
& \qquad -\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}+\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}-\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\| \\
\leq & \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\|\mathbf{T}_p^{*\alpha}\mathcal{S}_n(\mathbf{T}_p,A)(\mathbf{T}_p^{\alpha}-\mathbf{T}^{\alpha})+\mathbf{T}_p^{*\alpha}(\mathcal{S}_n(\mathbf{T}_p,A)-\mathcal{S}_n(\mathbf{T},A))\mathbf{T}^{\alpha} \\
& \qquad +(\mathbf{T}_p^{*\alpha}-\mathbf{T}^{*\alpha})\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\| \\
\leq & \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\{\|\mathbf{T}_p^{*\alpha}\|\|\mathcal{S}_n(\mathbf{T}_p,A)\|\|\mathbf{T}_p^{\alpha}-\mathbf{T}^{\alpha}\|+\|\mathbf{T}_p^{*\alpha}\|\|\mathcal{S}_n(\mathbf{T}_p,A)-\mathcal{S}_n(\mathbf{T},A)\|\|\mathbf{T}^{\alpha}\| \\
& \qquad +\|\mathbf{T}_p^{*\alpha}-\mathbf{T}^{*\alpha}\|\|\mathcal{S}_n(\mathbf{T},A)\|\|\mathbf{T}^{\alpha}\|\}.
\end{aligned}$$

Since $\|\mathbf{T}_p-\mathbf{T}\|\longrightarrow 0$ as $p\to+\infty$, we get that $\sum_{l=1}^{d}\|T_{lp}-T_l\|^2\longrightarrow 0$ as $p\to+\infty$. Therefore, $\|T_{lp}-T_l\|\longrightarrow 0$ as $p\to+\infty$ for $l=1,\ldots,d$. Thus $\|T_{lp}^{\alpha_l}-T_l^{\alpha_l}\|\longrightarrow 0$ $(p\to+\infty)$ for every $l=1,\ldots,d$. Which implies that $\|\mathbf{T}_p^{\alpha}-\mathbf{T}^{\alpha}\|\longrightarrow 0$ as $p\to+\infty$. On the other hand, set $R_p=T_{1p}+\cdots+T_{dp}$ and $R=T_1+\cdots+T_d$. Since adjoint operation is continuous and multiplication is jointly continuous, it follow that $R_p\longrightarrow R$, $R_p^*\longrightarrow R^*$ in $\mathcal{B}(H)$ and $R_p^h\longrightarrow R^h$, $R_p^{*h}\longrightarrow R^{*h}$ in $\mathcal{B}(H)$. Then

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$$\begin{aligned}
\|\mathcal{S}_n(\mathbf{T}_p,A)-\mathcal{S}_n(\mathbf{T},A)\| &= \|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\{(T_{1p}^*+\cdots+T_{dp}^*)^{n-k}A(T_{1p}+\cdots+T_{dp})^k \\
&\quad -(T_1^*+\cdots+T_d^*)^{n-k}A(T_1+\cdots+T_d)^k\}\| \\
&= \|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\{R_p^{*n-k}AR_p^k-R^{*n-k}AR^k\| \\
&= \sum_{k=0}^{n}\binom{n}{k}\|R_p^{*n-k}AR_p^k-R_p^{*n-k}AR^k+R_p^{*n-k}AR^k-R^{*n-k}AR^k\| \\
&\leq \sum_{k=0}^{n}\binom{n}{k}\{\|R_p^{*n-k}A(R_p^k-R^k)\|+\|(R_p^{*n-k}-R^{*n-k})AR^k\|\} \\
&\leq \sum_{k=0}^{n}\binom{n}{k}\{\|R_p^{*n-k}A\|\|R_p^k-R^k\|+\|R_p^{*n-k}-R^{*n-k}\|\|AR^k\|\}
\end{aligned}$$

Since $(\mathbf{T}_p)_p$ is an (A,m,n)-isosymmetric d-tuple of operators, we get

$$\gamma_{m,n}(\mathbf{T},A)=0.$$

□

Proposition 2.7. *Let $\mathbf{T}$ be an (A,m,n)-isosymmetric d-tuple of operators. Then the following statements hold.*

(1) *If $p\geq m$, then*

$$\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^i\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}=0,\quad i=0,1,\cdots,p-m. \tag{2.2}$$

(2) *If $p\geq n$, then*

$$\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^i(T_1^*+\cdots+T_d^*)^k\mathcal{I}_m(\mathbf{T},A)(T_1+\cdots+T_d)^{p-k}=0,\quad i=0,1,\cdots,p-n. \tag{2.3}$$

Proof. (1) We well prove (2.2) by induction on p. Suppose that $\mathbf{T}$ is (A,m,n)-isosymmetric d-tuple of operators, then Proposition 2.3 implies that $\mathbf{T}$ is (A,l,n)-isosymmetric d-tuple of operators for each $l\geq m$. Therefore, for $i=0$ the proof of (2.2) is immediate. Furthermore, the result is true for $p=m$. Suppose that (2.2) is true for $i\in\{1,2,\cdots,p-m\}$ and prove it for $i\in\{1,2,\cdots,p+1-m\}$. Then by using the induction hypothesis, we get

$$
\begin{aligned}
&\sum_{k=0}^{p+1}(-1)^{p+1-k}\binom{p+1}{k}k^i \sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\\
=\;& \sum_{k=1}^{p+1}(-1)^{p+1-k}\binom{p+1}{k}k^i \sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\\
=\;& \sum_{k=0}^{p}(-1)^{p-k}\binom{p+1}{k+1}(k+1)^i \sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\\
=\;& (p+1)\Big(T_1^*+\cdots+T_d^*\Big)\Big\{\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}(k+1)^{i-1}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\Big\}\\
&\Big(T_1^*+\cdots+T_d^*\Big)\\
=\;& (p+1)\Big(T_1^*+\cdots+T_d^*\Big)\Big\{\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}\Big(\sum_{j=0}^{i-1}\binom{i-1}{j}k^j\Big)\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\Big\}\\
&\Big(T_1+\cdots+T_d\Big)\\
=\;& (p+1)\Big(T_1^*+\cdots+T_d^*\Big)\Big\{\sum_{j=0}^{i-1}\binom{i-1}{j}\underbrace{\Big(\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^j\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}\Big)}_{=0}\Big\}\\
&\Big(T_1+\cdots+T_d\Big)\\
=\;& 0.
\end{aligned}
$$

(2) The identity (2.3) is also holds by a similar argument.

□

Proposition 2.8. *Let $T_i \in \mathcal{B}(H)$ and $A_i \in \mathcal{B}(H)^+$, for $i = 1, 2$. If T_i is (A_i, m, n)-isosymmetric (i=1, 2), then $T_1 \oplus T_2$ is $(A_1 \oplus A_2, m, n)$-isosymmetric.*

Proof. Assume that T_i is (A_i, m, n)-isosymmetric (i=1, 2). By [16, Proposition 2.2], we have $\mathcal{S}_n(A_1 \oplus A_2, T_1 \oplus T_2) = \mathcal{S}_n(A_1, T_1) \oplus \mathcal{S}_n(A_2, T_2)$, then

$$
\begin{aligned}
\gamma_{m,n}(T_1\oplus T_2, A_1\oplus A_2) &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}\oplus T_2^{*m-j}\mathcal{S}_n(T_1\oplus T_2, A_1\oplus A_2)T_1^{m-j}\oplus T_2^{m-j}\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}\oplus T_2^{*m-j}\mathcal{S}_n(T_1, A_1)\oplus\mathcal{S}_n(T_1, A_1)T_1^{m-j}\oplus T_2^{m-j}\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}\mathcal{S}_n(T_1, A_1)T_1^{m-j}\oplus T_2^{*m-j}\mathcal{S}_n(T_2, A_2)T_2^{m-j}\\
&= \gamma_{m,n}(T_1, A_1)\oplus\gamma_{m,n}(T_2, A_2) = 0.
\end{aligned}
$$

Therefore, $T_1 \oplus T_2$ is $(A_1 \oplus A_2, m, n)$-isosymmetric.

□

Remark 2.9. As a consequence of Proposition 2.8, we can prove that if $T_i \in \mathcal{B}(H)$ is (A_i, m, n)-isosymmetric, with $A_i \in \mathcal{B}(H)^+$, $i = 1, \ldots, d$, then $\displaystyle\bigoplus_{i=1}^{d} T_i = (T_1, \ldots, T_d)$ is $(\displaystyle\bigoplus_{i=1}^{d} A_i, m, n)$-isosymmetric by applying an induction argument.

3. Nilpotent perturbations of (A,m,n)-isosymmetric tuples of operators

The aim objective of this section is to study the stability of an (A,m,n)-isosymmetric d-tuple of operators $\mathbf{T}$ under the perturbation by a l-nilpotent d-tuple $\mathbf{Q}$.
Let $A \in \mathcal{B}(H)^+$, $\mathbf{T} = (T_1,\dots,T_d) \in \mathcal{B}(H)^d$ and $\mathbf{Q} = (Q_1,\dots,Q_d) \in \mathcal{B}(H)^d$ be two d-tuples of commuting operators. Let use denote by $\mathbf{T}+\mathbf{Q} := (T_1+Q_1,\dots,T_d+T_d)$. The d-tuple $\mathbf{Q}$ is said to be l-nilpotent if $\mathbf{Q}^\alpha = Q_1^{\alpha_1}\dots Q_d^{\alpha_d} = 0$ for all $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ such that $|\alpha| := \sum_{i=1}^d \alpha_i = l$ and $\mathbf{Q}^\alpha \neq 0$ for some α with $|\alpha| = l-1$.

Lemma 3.1. *If* $\mathbf{T}$ *and* $\mathbf{Q}$ *commute. Then*

$$\gamma_{m,n}(\mathbf{T}+\mathbf{Q},A) = \sum_{|\alpha|+|\beta|+k=m}\ \sum_{|\lambda|+|\mu|+h=n} (-1)^{|\mu|}\binom{m}{\alpha,\beta,k}\binom{n}{\lambda,\mu,h}(\mathbf{T}+\mathbf{Q})^{*\alpha}\mathbf{Q}^{*\beta+\lambda}\gamma_{k,h}(\mathbf{T},A)\mathbf{T}^\beta\mathbf{Q}^{\alpha+\mu}. \tag{3.1}$$

Proof. The multinomial theorem gives

$$\begin{aligned}
&\big((y_1+y_2)(x_1+x_2)-1\big)^m\big\{\big((y_1+y_2)-(x_1+x_2)\big)^n\big\}_{\mathbf{a}}\\
=\ &\big((y_1x_1-1)+(y_1+y_2)x_2+y_2x_1\big)^m\big\{\big((y_1-x_1)+(y_2-x_2)\big)^n\big\}_{\mathbf{a}}\\
=\ &\Big(\sum_{i+j+k=m}\binom{m}{i,j,k}(y_1+y_2)^i y_2^j(y_1x_1-1)^k x_1^j x_2^i\Big)\\
&\cdot\Big\{\sum_{s+l+h=n}\binom{n}{s,l,h}\, y_2^s(y_1-x_1)^h(-x_2)^l\Big\}_{\mathbf{a}}\\
=\ &\Big(\sum_{i+j+k=m}\binom{m}{i,j,k}(y_1+y_2)^i y_2^j(y_1x_1-1)^k x_1^j x_2^i\Big)\\
&\cdot\Big(\sum_{s+l+h=n}\binom{n}{s,l,h}\,(-1)^l y_2^s\Big\{(y_1-x_1)^h\Big\}_{\mathbf{a}} x_2^l\Big)\\
=\ &\sum_{i+j+k=m}\ \sum_{s+l+h=n}\binom{m}{i,j,k}\binom{n}{s,l,h}(-1)^l(y_1+y_2)^i y_2^j y_2^s(y_1x_1-1)^k\Big\{(y_1-x_1)^h\Big\}_{\mathbf{a}} x_1^j x_2^i x_2^l
\end{aligned}$$

Note that, we rearrange the terms so that the variables y_1 and y_2 are on the left. Then, by taking $x_1 = T_1+\dots+T_d$, $y_1 = T_1^*+\dots+T_d^*$, $x_2 = Q_1+\dots+Q_d$, $y_2 = Q_1^*+\dots+Q_d^*$ and by applying the multinomial theorem again we get

$$\gamma_{m,n}(\mathbf{T}+\mathbf{Q},A) = \sum_{|\alpha|+|\beta|+k=m}\ \sum_{|\lambda|+|\mu|+h=n} (-1)^{|\mu|}\binom{m}{\alpha,\beta,k}\binom{n}{\lambda,\mu,h}(\mathbf{T}+\mathbf{Q})^{*\alpha}\mathbf{Q}^{*\beta+\lambda}\gamma_{k,h}(\mathbf{T},A)\mathbf{T}^\beta\mathbf{Q}^{\alpha+\mu}.$$

The commuting condition of the d-tuples $\mathbf{T}$ and $\mathbf{Q}$ is needed to arrange the operators involved in the required order. □

Theorem 3.2. *Let* $A \in \mathcal{B}(H)^+$, $\mathbf{T}$ *be an* (A,m,n)*-isosymmetric* d*-tuple of operators and* $\mathbf{Q}$ *be an* l*-nilpotent* d*-tuple of operators such that* $\mathbf{T}$ *and* $\mathbf{Q}$ *commute. Then* $\mathbf{T}+\mathbf{Q}$ *is an* $(A,m+2l-2,n+2l-2)$*-isosymmetric* d*-tuple of operators.*

Proof. By Lemma 3.4, we have

$$\gamma_{m+2l-2,n+2l-2}(\mathbf{T}+\mathbf{Q},A) =$$
$$\sum_{|\lambda|+|\mu|+h=n+2l-2}\ \sum_{|\alpha|+|\beta|+k=m+2l-2}(-1)^{|\mu|}\binom{n}{\lambda,\mu,h}\binom{m}{\alpha,\beta,k}(\mathbf{T}+\mathbf{Q})^{*\alpha}\mathbf{Q}^{*\beta+\lambda}\gamma_{k,h}(\mathbf{T},A)\mathbf{T}^\beta\mathbf{Q}^{\alpha+\mu}.$$

Remark that :

- If $\max\{|\alpha|,|\beta|\} \geq l$ or $\max\{|\lambda|,|\mu|\} \geq l$, then $\mathbf{Q}^{*\beta+\lambda} = 0$ or $\mathbf{Q}^{\alpha+\mu} = 0$.
- If $\max\{|\alpha|,|\beta|\} \leq l-1$ and $\max\{|\lambda|,|\mu|\} \leq l-1$, then $k = m+2l-2-(|\alpha|+|\beta|) \geq m$ and $h = n+2l-2-(|\lambda|+|\mu|) \geq n$. Hence $\gamma_{k,h}(\mathbf{T},A) = 0$.

Consequently, $\gamma_{m+2l-2,n+2l-2}(\mathbf{T}+\mathbf{Q}, A) = 0$. Hence, $\mathbf{T}+\mathbf{Q}$ is an $(A, m+2l-2, n+2l-2)$-isosymmetric d-tuple of operators. □

In the following corollary, we give some immediate consequences of Theorem 3.2.

Corollary 3.3. *Let $A \in \mathcal{B}(H)^+$, $\mathbf{T}$ and $\mathbf{Q}$ are two d-tuples of operators such that $\mathbf{T}$ and $\mathbf{Q}$ commute and $\mathbf{Q}$ be an l-nilpotent d-tuple of operators, then the following statements hold.*

1. *If $\mathbf{T}$ is an (A, m)-ismetric d-tuple of operators, then $\mathbf{T}+\mathbf{Q}$ is an $(A, m+2l-2)$-isometric d-tuple of operators.*
2. *If $\mathbf{T}$ is an (A, n)-symmetric d-tuple of operators, then $\mathbf{T}+\mathbf{Q}$ is an $(A, n+2l-2)$-symmetric d-tuple of operators.*

Let use denote by $H \otimes H$ the algebraic tensor product of H and its completion by $H \widehat{\otimes} H$, endowed with a reasonable uniform cross-norm. For given two non zero operators $R, S \in \mathcal{B}(H)$, the tensor product $R \otimes S$ of R and S on $H \widehat{\otimes} H$ is defined by

$$\langle (R \otimes S)(x \otimes y), z \otimes t \rangle = \langle Rx, z \rangle \langle Sy, t \rangle.$$

Lemma 3.4. *Let $A \in \mathcal{B}(H)^+$ and $\mathbf{T}$ be an (A, m, n)-isosymmetric d-tuple of operators. Then $\mathbf{T} \otimes \mathbf{I} \in \mathcal{B}(H \widehat{\otimes} H)^d$ is an $(A \otimes A, m, n)$-isosymmetric d-tuple of operators.*

Proof. Note that $(T_1 \otimes I + \cdots + T_d \otimes I)^p = (T_1 + \cdots + T_d)^p \otimes I$ and from [9, Lemma 3.2], we have $\mathcal{I}_m(\mathbf{T} \otimes \mathbf{I}, A \otimes A) = \mathcal{I}_m(\mathbf{T}, A) \otimes A$, then for all $u = x \otimes y \in H \otimes H$ and by using [9, Remark 3.1], we get

$$\begin{aligned}
& \langle \gamma_{m,n}(\mathbf{T} \otimes \mathbf{I}, A \otimes A)u, u \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle (T_1^* \otimes I + \cdots + T_d^* \otimes I)^k \mathcal{I}_m(\mathbf{T} \otimes \mathbf{I}, A \otimes A)(T_1 \otimes I + \cdots + T_d \otimes I)^{n-k} u, u \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle \Big((T_1^* + \cdots + T_d^*)^k \otimes I \Big) \Big(\mathcal{I}_m(\mathbf{T}, A) \otimes A \Big) \Big((T_1 + \cdots + T_d)^{n-k} \otimes I \Big) u, u \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle \Big((T_1^* + \cdots + T_d^*)^k \otimes I \Big) \Big(\mathcal{I}_m(\mathbf{T}, A)(T_1 + \cdots + T_d)^{n-k} \otimes A \Big) u, u \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle \Big((T_1^* + \cdots + T_d^*)^k \mathcal{I}_m(\mathbf{T}, A)(T_1 + \cdots + T_d)^{n-k} \otimes A \Big) u, u \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle (T_1^* + \cdots + T_d^*)^k \mathcal{I}_m(\mathbf{T}, A)(T_1 + \cdots + T_d)^{n-k} x, x \rangle \langle Ay, y \rangle \\
= & \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \langle (T_1^* + \cdots + T_d^*)^k \mathcal{I}_m(\mathbf{T}, A)(T_1 + \cdots + T_d)^{n-k} x, x \rangle \|y\|_A \\
= & \langle \gamma_{m,n}(\mathbf{T}, A)x, x \rangle \|y\|_A \\
= & 0.
\end{aligned}$$

□

Proposition 3.5. *Let $A \in \mathcal{B}(H)^+$ and $\mathbf{T}$ be an (A, m, n)-isosymmetric d-tuple of operators and $\mathbf{Q}$ be an l-nilpotent d-tuple of operators. Then $\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Q} := (T_1 \otimes I + I \otimes Q_1, \ldots, T_d \otimes I + I \otimes Q_d)$ is an $(A \otimes A, m+2l-2, n+2l-2)$-isosymmetric commuting d-tuple of operators.*

Proof. By the fact that $R \otimes S = (R \otimes I)(I \otimes S) = (I \otimes S)(R \otimes I)$, it follows that $\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Q}$ is a d-tuple of commuting operators. Furthermore, Lemma 3.4 implies that $\mathbf{T} \otimes \mathbf{I} \in \mathcal{B}(H \widehat{\otimes} H)^d$ is an $(A \otimes A, m, n)$-isosymmetric d-tuple of operators and $I \otimes \mathbf{Q} \in \mathcal{B}(H \widehat{\otimes} H)^d$ is an l-nilpotent d-tuple. Applying Theorem 3.2, we deduce that $\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Q}$ is an $(A \otimes A, m+2l-2, n+2l-2)$-isosymmetric d-tuple of operators. □

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4. Spectral Properties of (A, m, n)-isosymmetric commuting tuple of operators

The aim objective of this section is to investigate some spectral properties of (A, m, n)-isosymmetric d-tuple of commuting operators which extend known results for a single operators. Recall that a point $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ is called a joint point eigenvalue of $\mathbf{T}$ if there exists a nonzero vector $x \in H$ such that $(T_j - \lambda_j)x = 0$ for every $j = 1, 2, \ldots, d$, which equivalent to say that there exists a nonzero vector $x \in H$ such that $x \in \bigcap\limits_{1 \leq j \leq d} \mathcal{N}(T_j - \lambda_j)$. The set of all joint eigenvalues of $\mathbf{T}$ is called the joint point spectrum of $\mathbf{T}$, that is,

$$\sigma_p(\mathbf{T}) = \left\{ \lambda \in \mathbb{C}^d, \quad \bigcap_{1 \leq j \leq d} \mathcal{N}(T_j - \lambda_j) \neq \{0\} \right\}.$$

A point $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ is called a joint approximate point of $\mathbf{T}$ if and only if there exists a sequence $(x_n)_n \subset H$ such that $\|x_n\| = 1$ and $(T_j - \lambda_j)x_n \longrightarrow 0$ as $n \longrightarrow +\infty$, for every $j = 1, 2, \ldots, d$. The set of all joint approximate point of $\mathbf{T}$ denoted by $\sigma_{ap}(\mathbf{T})$ and its called the joint approximate point spectrum of $\mathbf{T}$. $\mathbb{B}(\mathbb{C}^d)$ and $\partial\mathbb{B}(\mathbb{C}^d)$ denoted the unit ball and its boundary, respectively, i.e.

$$\mathbb{B}(\mathbb{C}^d) = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \ : \ \|\lambda\| = \left(\sum_{j=1}^{d} |\lambda_j|^2 \right)^{\frac{1}{2}} < 1\}$$

and

$$\partial\mathbb{B}(\mathbb{C}^d) = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \ : \ \|\lambda\| = \left(\sum_{j=1}^{d} |\lambda_j|^2 \right)^{\frac{1}{2}} = 1\}$$

Theorem 4.1. *Let $A \in \mathcal{B}(H)^+$. If $\mathbf{T}$ is (A, m, n)-isosymmetric d-tuple of operators and $0 \notin \sigma_{ap}(A)$, then*

(1) $\sigma_{ap}(\mathbf{T}) \subset \partial\mathbb{B}(\mathbb{C}^d) \bigcup \left\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \ : \ Im(\sum\limits_{i=1}^{d} \lambda_i) = 0 \right\}$

(2) *Eigenvectors of $\mathbf{T}$ corresponding to two joint eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ such that $\lambda.\overline{\mu} := \sum\limits_{i=1}^{d} \lambda_i.\overline{\mu_i} \neq 1$ and $\sum\limits_{i=1}^{d} (\overline{\mu_i} - \lambda_i) \neq 0$ are A-orthogonal*

(3) *If $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ are distinct approximate eigenvalues of $\mathbf{T}$ such that $\lambda.\overline{\mu} = \sum\limits_{i=1}^{d} \lambda_i.\overline{\mu_i} \neq 1$ and $\sum\limits_{i=1}^{d} (\overline{\mu_i} - \lambda_i) \neq 0$ and if $\{x_n\}_n$ and $\{y_n\}_n$ are two sequences of unit vectors such that*

$$(T_i - \lambda_i)x_n \longrightarrow 0 \quad \textit{and } (T_i - \mu_i)y_n \longrightarrow 0 \ (\textit{ as } n \longrightarrow +\infty) \ i \in \{1, \ldots, d\}.$$

Then

$$\langle Ax_n, y_n \rangle \longrightarrow 0 \ (\textit{ as } n \longrightarrow +\infty).$$

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Proof. (1). Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_n \subset H$ such that $\|x_n\| = 1$ and $(T_i - \lambda_i)x_n \longrightarrow 0$. Since $\mathbf{T}$ is (A,m,n)-isosymmetric d-tuple of operators, we get

$$\begin{aligned}
0 &= \langle \gamma_{m,n}(\mathbf{T},A)x_n, x_n\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}x_n, x_n\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}x_n, \mathbf{T}^{\alpha}x_n\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathcal{S}_n(\mathbf{T},A)\lambda^{\alpha}x_n, \lambda^{\alpha}x_n\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\lambda^{\alpha}\overline{\lambda}^{\alpha}\langle \mathcal{S}_n(\mathbf{T},A)x_n, x_n\rangle \\
&= \Big(\|\lambda\|^2 - 1\Big)^m \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle A(T_1+\cdots+T_d)^{n-k}x_n, (T_1+\cdots+T_d)^k x_n\rangle \\
&= \Big(\|\lambda\|^2 - 1\Big)^m \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda_1+\cdots+\lambda_d)^{n-k}(\overline{\lambda}_1+\cdots+\overline{\lambda}_d)^k\langle Ax_n, x_n\rangle \\
&= \Big(\|\lambda\|^2 - 1\Big)^m \Big((\overline{\lambda_1+\cdots+\lambda_d}) - (\lambda_1+\cdots+\lambda_d)\Big)^n\langle Ax_n, x_n\rangle .
\end{aligned}$$

Therefore, $\|\lambda\| = 1$ or $Im(\sum_{i=1}^{d}\lambda_i) = 0$.

(2). Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ be two distinct eigenvalues of $\mathbf{T}$. Assume that

$$T_i x = \lambda_i x \ \text{ and } T_i y = \mu_i y \ \text{ for all } \ i \in \{1, \ldots, d\}.$$

Since $\mathbf{T}$ is (A,m,n)-isosymmetric d-tuple of operators, we obtain

$$\begin{aligned}
0 &= \langle \gamma_{m,n}(\mathbf{T},A)x, y\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathbf{T}^{*\alpha}\mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}x, y\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathcal{S}_n(\mathbf{T},A)\mathbf{T}^{\alpha}x, \mathbf{T}^{\alpha}y\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\langle \mathcal{S}_n(\mathbf{T},A)\lambda^{\alpha}x, \mu^{\alpha}y\rangle \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\lambda^{\alpha}\overline{\mu}^{\alpha}\langle \mathcal{S}_n(\mathbf{T},A)x, y\rangle \\
&= \Big(1 - \sum_{i=1}^{d}\lambda_i\overline{\mu}_i\Big)^m \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle A(T_1+\cdots+T_d)^{n-k}x, (T_1+\cdots+T_d)^k y\rangle \\
&= \Big(1 - \sum_{i=1}^{d}\lambda_i\overline{\mu}_i\Big)^m \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda_1+\cdots+\lambda_d)^{n-k}(\overline{\mu_1+\cdots+\mu_d})^k\langle Ax, y\rangle \\
&= \Big(1 - \sum_{i=1}^{d}\lambda_i\overline{\mu}_i\Big)^m \Big(\sum_{i=1}^{d}(\overline{\mu_i} - \lambda_i)\Big)^n\langle Ax, y\rangle .
\end{aligned}$$

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Therefore, we deduce that $\langle Ax, y\rangle = 0$.
(3). By similar arguments as in the previous assertion, we get our desired result. $\square$

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