ON THE JACOBSON UNIT-LIKE RINGS

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Abstract. Let R be a commutative ring with identity element, $U(R)$ its group of units and $J(R)$ its Jacobson radical. Recall that a ring R is a JU-ring (resp. a weakly JU-ring) if $U(R) = 1 + J(R)$ (resp. $U(R) = \pm 1 + J(R)$. In this note we investigate the properties of these notions in different contexts of commutative rings. Precisely, we study the transfer of the notions of JU-rings, WJU-rings, UU-rings, WUU-rings, and more to trivial ring extensions, amalgamations of rings and pullbacks. Our aim is to provide new classes of commutative rings satisfying these properties. Examples illustrating the limits and scopes of our results are provided.

1. Introduction

Let R be commutative ring with identity element, $U(R)$ its (multiplicative) group of units, $J(R)$ its Jacobson radical, $Id(R)$ the set of all idempotent elements of R, and $Nil(R)$ the set of all nilpotent elements of R. The influence on the structure of rings of properties defined elementwise is intensively studied in the literature. For example, clean rings and their generalizations. Recall that R is called clean if $R = U(R) + Id(R)$, nil-clean if $R = Nil(R) + Id(R)$, and weakly nil-clean, provided that $R = Nil(R) \pm Id(R)$. Besides, a ring R is said to be exchange if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Clean rings are always exchange, while the converse is false. In [15], a ring R is called a UU-ring if all units are unipotents, that is, $U(R) = 1 + Nil(R)$. This is obviously equivalent to the equality $U(R) = Nil(R) - 1$. On the other hand, in [3], the author defined the so-called weakly UU-rings (WUU-rings for short) as rings R whose $U(R) = Nil(R) \pm 1$ and which are a common extension of UU-rings. Rings with special types of units, generalizations of commutative rings have been investigated in relation to various global ring properties. For instance, it is well known that $1 + J(R) \subseteq U(R)$.

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However, the inclusion could be strict, so that it is rather natural to ask whether the equality holds. In [14], Danchev introduced the notion of a ring with Jacobson units (JU-ring for short) as a ring R such that $U(R) = 1 + J(R)$. Later, he defined and completely explore the so-called weakly JU-rings (WJU for short) and showed that his class properly encompasses the class of JU-rings.
The main purpose of this paper is to study the pre-mentioned notions in the context of trivial ring extensions, amalgamations of algebras along ideals and pullbacks. Our aim is to provide new classes of commutative rings satisfying these properties. First, we prove that if A is a commutative ring with identity, E is an A-module and $R := A \ltimes E$ is the trivial ring extension of A by E, then R is a weakly semiboolean (resp. semiboolean, resp. WUU-ring, resp. UU-ring, resp. WJU-ring, resp. JU-ring) if and only if so is A. Next, we deal with amalgamations of rings and we prove that if A and B are commutative rings, $f : A \mapsto B$ is a ring homomorphism, J is a nonzero ideal of B such that that $J \subseteq J(B)$ and $R := A \bowtie^f J$, then R is a is a weakly JU-ring (resp. JU-ring, resp. semiboolean ring) if and only if so is A (Theorem 2.7). We also prove that R is a UU-ring if and only if A is a UU-ring and $J \subseteq Nil(B)$ (Theorem 2.12). We close the paper by studying the transfer of these notions to pullbacks. We prove that for the diagram of type (Δ), assume that M is a prime ideal of T and $J(T) \subseteq M$ (in particular, if M is a maximal ideal of T):
(1) If T is a weakly JU-ring (resp. a JU-ring, resp. a weakly semiboolean ring, resp. semiboolean), then so is R.
(2) Assume that T is local with maximal ideal M. Then:
(i) R is a weakly JU-ring (resp. a JU-ring, resp. a weakly semiboolean ring, resp. semiboolean) if and only if so is D.
(ii) R is a UU-ring (resp. weakly UU ring) if and only if $M = Nil(R) = J(R)$ and $U(D) = \{1\}$ (resp. $M = Nil(R) = J(R)$ and $U(D) = \{-1, 1\}$) (Theorem 2.14). Several examples illustrating the limits and scopes of our results are given.

2. Motivation and General Results

We start with the following definition collecting definitions of different clean-like notions.

Definition 2.1. Let R be a commutative ring with identity element, $U(R)$ the set of all units of R, $J(R)$ the Jacobson radical of R, $Id(R)$ the set of all idempotent of R and $Nil(R)$, the nilradical of R.
(1) A ring R is called semiboolean (or J-clean) if $R = J(R) + Id(R)$ ([35]).
(2) A ring R is called weakly semiboolean if $R = J(R) \pm Id(R)$ ([17]).

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(3) A ring R is called a UU-ring if $U(R) = 1 + Nil(R)$ ([9]).
(4) A ring R is called weakly UU-ring if $U(R) = \pm 1 + Nil(R)$ ([15]).
(5) A ring R is called JU-ring if $U(R) = 1 + J(R)$ ([14]).
(6) A ring R is called weakly JU-ring if $U(R) = \pm 1 + J(R)$ ([20]).

The following diagram puts each of the pre-mentioned notions in perspective. Notice that reversible implications are not true in general.

$$\begin{array}{ccccccc}
\textit{semiboolean} & \Longrightarrow & \text{weakly semiboolean} & \Longrightarrow & \textit{clean} & \Longrightarrow & \text{weakly clean} \\
 & & \Downarrow & & & & \\
\text{JU ring} & \Longrightarrow & \text{weakly JU ring} & & & & \\
\Uparrow & & \Uparrow & & & & \\
\text{UU ring} & \Longrightarrow & \text{weakly UU ring} & & & &
\end{array}$$

The next proposition deals with local domains and power series rings that are weakly JU-rings.

Proposition 2.2. *Let R be a commutative ring with identity element.*
(1) *Assume that R is a weakly JU-ring. If $2 \in U(R)$, then $3 \in J(R)$.*
(2) *Assume that R is a local domain with maximal ideal M. If R is a weakly JU-ring, then either $R/M \cong \mathbb{Z}_2$ or $R/M \cong \mathbb{Z}_3$.*
(3) *The power series ring $R[[X]]$ is a weakly JU-ring if and only if so is R.*

Proof. (1) Assume that R is a weakly JU-ring and $2 \in U(R)$. Then $2 = \pm 1 + a$ for some $a \in J(R)$. If $2 = 1 + a$, then $1 = a$ which is absurd. Thus $2 = -1 + a$ and therefore $3 = a \in J(R)$ as desired.
(2) Assume that R is a local weakly JU-ring with maximal ideal M. Then for every $x \in R \setminus M$, $x \in U(R)$ and so $x = \pm 1 + a$ for some $a \in M$. Thus $\overline{x} = \overline{\pm 1}$ where $\overline{y}$ denotes the class of y modulo M. If $\overline{1} = \overline{-1}$ (equivalently $2 \in M$), then $R/M \cong \mathbb{Z}_2$ (and in this case R is a JU-ring); and if $\overline{1} \neq \overline{-1}$, then $R/M \cong \mathbb{Z}_3$ as desired.
(3) Follows immediately from the fact that $J(R[[X]]) = J(R) + XR[[X]]$. □

The next example provides a weakly JU-ring which is not a JU-ring.

Example 2.3. Let $\mathbb{Z}_3$ be the field of integers modulo 3, X an indeterminate over $\mathbb{Z}_3$ and set $R = \mathbb{Z}_3[[X]]$. Clearly R is local with maximal ideal $M = X\mathbb{Z}_3[[X]]$ and $U(R) = \{a + Xg(X) | 0 \neq a \in \mathbb{Z}_3, g(X) \in \mathbb{Z}_3[[X]]\} = \{1 + Xg(X) | g(X) \in \mathbb{Z}_3[[X]]\} \cup \{-1 + Xg(x) | g(X) \in \mathbb{Z}_3[[X]]\}$ (since $2 = -1$ in $\mathbb{Z}_3$). Thus R is a weakly JU-ring which is not a JU-ring.

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Theorem 2.4. *Let $\{R_i\}_{i\in I}$ be a family of commutative rings with identity elements. Then $R := \prod_{i\in I} R_i$ is a weakly JU-ring if and only if each R_i is a weakly JU-ring and $2 \notin J(R_i)$ for at most one $i \in I$, that is, either $2 \in J(R_i)$ for all $i \in I$, or $2 \notin J(R_j)$ for some $j \in I$ and $2 \in J(R_i)$ for every $j \neq i \in I$.*

Proof. Assume that R is a weakly JU-ring and suppose that $2 \notin J(R_j)$ for some $j \in I$. Let $u = (u_k)_{k\in I}$ where $u_k = 1$ if $k \neq j$ and $u_j = -1$. Since $u \in U(R)$, $u = \pm 1_R + a$ where $1_R = (1)_{k\in I}$ and $a = (a_k)_{k\in I} \in J(R) = \prod_{k\in I} J(R_k)$. If $u = 1_R + a$, then $-1 = 1 + a_j$ and so $2 = -a_j \in J(R_j)$ which is absurd. Hence $u = -1_R + a$. Thus, for every $j \neq i \in I$, $1 = -1 + a_i$ and so $2 = a_i \in J(R_i)$, as desired.
Conversely, let $u = (u_i)_{i\in I} \in U(R)$ where $u_i \in U(R_i)$ for every $i \in I$. Since R_i is a weakly JU-ring, $u_i = \pm 1 + a_i$ where $a_i \in J(R_i)$. Let $\Omega_+ = \{i \in I | u_i = 1 + a_i\}$ and $\Omega_- = \{i \in I | u_i = -1 + a_i\}$. Notice that if $2 \in J(R_i)$, then $1 + a_i = -1 + (2 + a_i) = -1 + b_i$ where $b_i = 2 + a_i \in J(R_i)$ and $-1 + a_i = 1 + (a_i - 2) = 1 + c_i$ where $c_i = a_i - 2 \in J(R_i)$. Thus u_i can always expressed as $u_i = 1 + a_i = -1 + b_i$ or $u_i = -1 + a_i = 1 + c_i$. Thus, if $2 \in J(R_i)$ for all $i \in I$, then $u = 1_R + (d_i)$ where $d_i = a_i$ if $i \in \Omega_+$ and $d_i = c_i$ if $i \in \Omega_-$. (Similarly, we can express $u = -1_R + (d_i)$ where $d_i = a_i$ if $i \in \Omega_-$ and $d_i = b_i$ if $i \in \Omega_+$). Now assume that $2 \notin J(R_j)$ for a unique $j \in I$. Then $2 \in J(R_i)$ for every $i \neq j$. If $u_j = 1 + a_j$ (that is, $j \in \Omega_+$), we express u as $u = 1_R + (d_i)$ where $d_i = a_i$ if $i \in \Omega_+$ and $d_i = c_i$ if $i \in \Omega_-$. If $u_j = -1 + a_j$ (that is, $j \in \Omega_-$), we express u as $u = -1_R + (d_i)$ where $d_i = a_i$ if $i \in \Omega_-$ and $d_i = b_i$ if $i \in \Omega_+$. Hence $u = \pm 1_R + (d_i)_{i\in I}$ with $(d_i)_{i\in I} \in \prod_{i\in I} J(R_i) = J(R)$ and therefore R is a weakly JU-ring. □

Let A be a ring and E an $A-$module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a,x)(b,y) = (ab, ay + bx)$. An element $(a,x) \in R$ is a unit if and only if so is a. Considerable work, part of it summarized in Glaz's book [30] and Huckaba's book [32], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [23, 25, 26]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations)(cf. [33, page 2]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([24, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it ([24, Example 2.7 and Remark 2.8]).

Our next two results deal with the transfer of the notions of JU-rings and weakly JU-rings to trivial extensions of rings by modules and amalgamations of algebras along ideals. Before stating our first result, we need the following lemma collecting information about units, idempotents and nilpotents of a trivial ring extension. Tough the results are well-known and their proofs are easy, we include them for the convenience of the reader.

Lemma 2.5. *Let A be a commutative ring with identity, E an A-module and $R := A \ltimes E$ be the trivial ring extension of A by E. Then:*
(1) $J(R) = J(A) \ltimes E$;
(2) $U(R) = U(A) \ltimes E$;
(3) $Id(R) = Id(A) \ltimes 0$;
(4) $Nil(R) = Nil(A) \ltimes E$.

Proof. (1) Follows from the fact that the maximal ideals of R are of the form $M \ltimes E$ where M is a maximal ideal of A, see [2, Theorem 3.2].
(2) Clearly $U(R) \subseteq U(A) \ltimes E$. Conversely, let $a \in U(A)$ and $e \in E$. Then $(a,e)(a^{-1}, -a^{-2}e) = (1,0)$ and so $(a,e) \in U(R)$, as desired.
(3) Clearly $Id(A) \ltimes 0 \subseteq Id(R)$. Conversely, let $(a,e) \in Id(R)$. Then $(a^2, 2ae) = (a,e)$. So $a = a^2$ and $2ae = e$. Then $a \in Id(A)$. Now, $2ae = e$ and multiplying by a, we obtain $2ae = 2a^2e = ae$. Thus $ae = 0$ and so $e = 2ae = 0$. Thus $(a,e) = (a,0) \in Id(A) \ltimes 0$.
(4) Follows from [2, Theorem 3.2]. In fact it is easy to check that if $b \in Nil(A)$ and $b^n = 0$, then for every $e \in E$, $(b,e)^{n+1} = (b^{n+1}, (n+1)b^n e) = (0,0)$. □

Proposition 2.6. *Let A be a commutative ring with identity, E an A-module and $R := A \ltimes E$ be the trivial ring extension of A by E. Then:*
(1) *R is a weakly semiboolean (resp. semiboolean) ring if and only if so is A.*
(2) *R is a WUU-ring (resp. UU-ring) if and only if so is A.*
(3) *R is a WJU-ring (resp. JU-ring) if and only if so is A.*

Proof. Straightforward by Lemma 2.5. □

Recall that if $R := A \bowtie^f J$ is the amalgamation of A and B along J with respect to f, then the maximal ideals of R are of two types: the

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maximal ideals of the form $P \bowtie^f J$ where P is a maximal ideal of A and the maximal ideals of the form $\overline{Q}^f := \{(a, f(a)+i) | f(a)+i \in Q | Q$ is a maximal ideal of B with $J \nsubseteq Q\}$. In particular, if $J \subseteq J(B)$, then $Max(R) = \{P \bowtie^f J | P \in Max(A)\}$.

Theorem 2.7. *Let A and B be commutative rings, $f : A \mapsto B$ a ring homomorphism, J a nonzero ideal of B and $R := A \bowtie^f J$.*
(1) *Assume that $J \subseteq J(B)$ (in particular, if B is a local ring). Then R is a weakly JU-ring (resp. JU-ring, resp. semiboolean ring) if and only if so is A.*
(2) *Assume that $J \nsubseteq J(B)$ and set $\overline{J}(J) = \cap\{Q | Q \in Max(B), J \nsubseteq Q\}$. Then R is a JU-ring if and only if A is a JU-ring, $U(A) \subseteq 1 + f^{-1}(\overline{J}(J))$ and for every $u = (a, f(a)+i) \in U(R)$, $i \in \overline{J}(J)$.*
(3) *Assume that $Id(A) = \{0,1\}$ and $Id(B) \cap J = \{0\}$ (in particular, if A and B are integral domains). Then R is semiboolean if and only if A is semiboolean and $J \subseteq J(B)$.*

Proof. (1) Assume that A is a weakly JU-ring and let $u = (a, f(a)+i) \in U(R)$. Then $a \in U(A)$ and so $a = \pm 1 + b$ where $b \in J(A)$. Thus $U = (a, f(a)+i) = \pm(1,1) + (b, f(b)+i)$, and clearly $(b, f(b)+i) \in J(R)\}$.
Assume that A is semiboolean and let $x = (a, f(a)+i) \in R$. Write $a = b+e$ where $b \in J(A)$ and $e \in Id(A)$. Then $x = (a, f(a)+i) = (b, f(b)+i) + (e, f(e))$ and clearly $(b, f(b)+i) \in J(A) \bowtie^f J = J(R)$ and $(e, f(e)) \in Id(R)$.
(2) Assume that R is a JU-ring and let $a \in U(A)$. Then $(a, f(a)) \in U(R)$ and so $(a, f(a)) = (1,1) + (b, f(b)+i)$ where $(b, f(b)+i) \in J(R)$. Then $a = 1+b$, $i = 0$ and clearly $b \in J(A)$. Hence A is a JU-ring. Let Q be a maximal ideal of B such that $J \nsubseteq Q$. Since $(b, f(b)) \in J(R) \subseteq \overline{Q}^f$, $f(a-1) = f(b) \in Q$ and so $f(a-1) \in \overline{J}(J)$. Hence $a-1 \in f^{-1}(\overline{J}(J))$ and so $a \in 1 + f^{-1}(\overline{J}(J))$. Thus $U(A) \subseteq 1 + f^{-1}(\overline{J}(J))$. Now, let $u = (a, f(a)+i) \in U(R)$. Then $(a, f(a)+i) = (1,1) + (b, f(b)+j)$ where $(b, f(b)+j) \in J(R)$. Thus $a = 1+b$ and $i = j$. Since $a \in U(A)$, by the first part, $a = 1+c$ for some $c \in f^{-1}(\overline{J}(J))$. Thus $b = c$ and so $b \in f^{-1}(\overline{J}(J))$. Therefore $f(b) \in \overline{J}(J))$. But since $(b, f(b)+i) = (b, f(b)+j) \in J(R)$, $f(b)+i \in \overline{J}(J))$. Hence $i = (f(b)+i) - f(b) \in \overline{J}(J))$, as desired.
Conversely, let $u = (a, f(a)+i) \in U(R)$. Then $a \in U(A)$ and so $a = 1+b$ where $b \in J(A)$. So $(a, f(a)+i) = (1,1) + (b, f(b)+i)$. Let $P \in Max(A)$. Then $b \in J(A) \subseteq P$, and so $(b, f(b)+i) \in P \bowtie^f J$. On the other hand, let Q be a maximal ideal of B do not containing J. By hypothesis, $i \in \overline{J}(J) \subseteq Q$. Since $b+1 = a \in U(A) \subseteq 1 + f^{-1}(\overline{J}(J))$, $b \in f^{-1}(\overline{J}(J))$ and so $f(b) \in \overline{J}(J)) \subseteq Q$. Thus $f(b)+i \in Q$ and therefore $(b, f(b)+i) \in \overline{Q}^f$. Hence

$(b, f(b)+i) \in J(R)$, as desired.
(3) Assume that $Id(A) = \{0,1\}$ and $Id(B) \cap J = \{0\}$ and suppose that R is semiboolean. By way of contradiction, suppose that $J \nsubseteq J(B)$. Let $Q \in Max(B)$ and let $j \in J \setminus Q$. Write $(0, j) = (b, f(b)+i)+(e, f(e)+h)$ where $(b, f(b)+i) \in J(R)$ and $(e, f(e)+h) \in Id(R)$. Then $f(b)+i \in Q$ and $b+e = 0$. Since $b \in J(A)$, $b = e = 0$. Since $h = f(e)+h \in Id(B)$ and $h \in J$, $h = 0$. Thus $j = i+h = i = f(b)+i \in Q$, which is absurd. Hence $J \subseteq J(B)$, as desired. Conversely, let $(a, f(a)+i) \in R$. Since A is semiboolean, $a = b+e$ where $b \in J(A)$ and $e \in Id(A)$. So $(a, f(a) = i) = (b, f(b)+i)+(e, f(e))$, and clearly $(e, f(e)) \in Id(R)$ and $(b, f(b)+i) \in J(A) \bowtie^f J = J(R)$ since $J \subseteq J(B)$. □

Example 2.8. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and X an indeterminate over $\mathbb{Z}_2$. Set $A = \mathbb{Z}_2$, $B = \mathbb{Z}_2[[X]]$, $J = X^2B$, $f : A \mapsto B$ the canonical injection and $R = A \bowtie^f J$. By Theorem 2.7(1), R is a JU-ring.

Example 2.9. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and X an indeterminate over $\mathbb{Z}_2$. Set $A = \mathbb{Z}_2$, $B = \mathbb{Z}_2[X]$, $J = XB$, $f : A \mapsto B$ the canonical injection and $R = A \bowtie^f J$. Clearly $\overline{J}(J) = (0)$, $U(A) = \{1\}$. Let $u = (a, f(a)+i) \in U(R)$ and $v = (b, f(b)+j) \in R$ such that $uv = (1,1)$. Necessarily $a = b = 1$ and so $(1+i)(1+j) = 1$. Write $i = Xg$ and $j = Xh$ for some $g,h \in \mathbb{Z}_2[X]$. If $g \neq 0$ or $h \neq 0$, $deg(i) = deg(1+Xg) \geq 1$ or $deg(j) = deg(1+Xh) \geq 1$ which is absurd. Thus $g = h = 0$ and so $i = j = 0 \in \overline{J}(J)$. By Theorem 2.7(2), R is a JU-ring.

Example 2.10. Let $\mathbb{Z}_4$ be the ring of integers modulo 4, X an indeterminate over the field $\mathbb{Z}_2$ and $A = \mathbb{Z}_4$. Clearly A is a semiboolean ring.
(1) Set $B = \mathbb{Z}_2[[X]]$, $J = XB$ and let $f : A \mapsto B$ defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Since $Id(A) = \{0,1\}$, B is an integral local domain, $R = A \bowtie^f J$ is a semiboolean ring.
(2) Set $B = \mathbb{Z}_2[X]$, $J = XB$ and let $f : A \mapsto B$ defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Since $Id(A) = \{0,1\}$, B is an integral domain and $J \nsubseteq J(B)$, $R = A \bowtie^f J$ is not a semiboolean ring.

Example 2.11. Let $\mathbb{Z}_4$ be the ring of integers modulo 4 and set $A = \mathbb{Z}_4$, $B = \mathbb{Z}_2 \times \mathbb{Z}_2$, $J = 0 \times \mathbb{Z}_2$ and let $f : A \mapsto B$ defined by $f(0) = f(2) = (0,0)$ and $f(1) = f(3) = (1,1)$. Since $Id(A) = \{0,1\}$, $Id(B) \cap J = J$. However, $R = A \bowtie^f J$ is a semiboolean ring. Indeed, it is easy to check that $R = \{(0,(0,0)),(0,(0,1)),(1,(1,1)),(1,(1,0)),(2,(0,0)),(2,(0,1)),(3,(1,1)),$ $(3,(1,0))\}$, $Id(R) = \{(0,(0,0)),(0,(0,1)),(1,(1,1)),(1,(1,0))\}$, $J(R) = \{(0,(0,0)),$ $(2,(0,0))\}$. Thus $R = J(R)+Id(R)$ as desired.

Theorem 2.12. *Let A and B be commutative rings, $f : A \mapsto B$ a ring homomorphism, J a nonzero ideal of B and $R := A \bowtie^f J$. Assume that $J \subseteq J(B)$. Then R is a UU-ring if and only if A is a UU-ring and $J \subseteq Nil(B)$.*

Proof. Assume that R is a UU-ring and let $j \in J$. Clearly A is a UU-ring. Since $J \subseteq J(B)$, $1+j \in U(B)$ and so $(1,1+j) \in U(R)$. (Indeed, let $b \in B$ such that $(1+j)b = 1$. Then $b = 1-bj$, and since $bj \in J$, $(1,b) = (1,1-bj) \in R$ and $(1,1+j)(1,1-bj) = (1,1+j)(1,b) = (1,1)$). Since R is a UU-ring, $(1,1+j) = (1,1)+(c,f(c)+i)$ for some $(c,f(c)+i) \in Nil(R)$. Then $c = 0$ and $j = i$. Thus $(0,j) = (0,i) = (c,f(c)+i) \in Nil(R)$ and so $j \in Nil(B)$. Thus $J \subseteq Nil(B)$, as desired.
Conversely, let $x = (a,f(a)+i) \in U(R)$. Then $a \in U(A)$ and so $a = 1+b$ for some $b \in Nil(A)$. Thus $x = (a,f(a)+i) = (1,1)+(b,f(b)+i)$. Since $i \in Nil(B)$ and $b \in Nil(A)$, $(b,f(b)+i) \in Nil(R)$ and therefore R is a UU-ring. □

Example 2.13. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and X an indeterminate over $\mathbb{Z}_2$. Set $A = \mathbb{Z}_2$, $B = \mathbb{Z}_2[X]/(X^2)$, $J = xB$, where x is the class of X mod X^2, $f : A \mapsto B$ the canonical injection and $R = A \bowtie^f J$. Since B is local with maximal ideal J and $J^2 = (0)$, by Theorem 2.12, R is a UU-ring.

Let T be a domain, M an ideal of T (M is not necessarily maximal), $\phi : T \longrightarrow T/M$ the canonical surjection and D a proper subring of T/M. Let R be the pullback issued from the following diagram of canonical homomorphisms:

$$(\square) \qquad \begin{array}{ccc} R := \phi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T/M. \end{array}$$

Clearly, $M = (R : T)$ and $D \cong R/M$. For ample details on the ideal structure of R and its ring-theoretic properties, we refer the reader to [3, 5, 8, 27, 28, 29].

Theorem 2.14. *For the diagram of type* (Δ) *assume that M is a prime ideal of T and $J(T) \subseteq M$ (in particular, if M is a maximal ideal of T):*
(1) *If T is a weakly JU-ring (resp. JU-ring, resp. weakly semiboolean ring, resp. semiboolean), then so is R.*
(2) *Assume that T is local with maximal ideal M. Then:*
(i) *R is a weakly JU-ring (resp. JU-ring, resp. weakly semiboolean ring, resp. semiboolean) if and only if so is D.*
(ii) *R is a UU-ring (resp. weakly UU ring) if and only if $M = Nil(R) = J(R)$ and $U(D) = \{1\}$ (resp. $M = Nil(R) = J(R)$ and $U(D) = \{-1,1\}$).*

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Proof. (1) First we need to show that $J(T)\cap R\subseteq J(R)$ and $Id(T)=Id(R)$. Clearly $Id(R)\subseteq Id(T)$, and conversely if $e\in Id(T)$, then $e=e^2$. So $e(1-e)=0\in M$ and thus $e\in M$ or $1-e\in M$ (since M is a prime ideal of T) and so $e\in R$. Hence $Id(R)=Id(T)$. Next, let $x\in J(T)\cap R$ and let Q be a maximal ideal of R. If $M\subseteq Q$, then $x\in J(T)\subseteq M\subseteq Q$. If $M\nsubseteq Q$, then there is a unique maximal ideal N of T such that $N\cap R=Q$ (see [27, Theorem 1.4]). But then $x\in J(T)\cap R\subseteq N\cap R=Q$ and thus $x\in J(R)$.
Assume that T is a weakly JU-ring (resp. a JU-ring) and let $u\in U(R)\subseteq U(T)$. Then $u=\pm 1+a$ (resp. $u=1+a$) where $a\in J(T)$. Then $a=u\mp 1$ (resp. $a=u-1$) is in $J(T)\cap R\subseteq J(R)$, and the conclusion follows. Now assume that T is weakly semiboolean (resp. semiboolean) and let $x\in R\subseteq T$. Write $x=a\pm e$ (resp. $x=a+e$) where $a\in J(T)$ and $e\in Id(T)=Id(R)$. Thus $a=x\pm e$ (resp. $a=x-e$) is in $J(T)\cap R\subseteq J(R)$, as desired.

(2) Notice that since T is local with maximal ideal M, every ideal of R is comparable to M under the inclusion (see [27, 29]). So $M\subseteq J(R)$, and $J(R)=\phi^{-1}(J(D))$.
(i) Assume that R is a weakly JU-ring and let $d\in U(D)$. Let $x\in R$ such that $\phi(x)=d$. Since $d^{-1}\in D$, $d^{-1}=\phi(y)$ for some $y\in R$. Then $1=dd^{-1}=\phi(x)\phi(y)=\phi(xy)$ and so $1-xy=m\in M\subseteq J(R)$. Thus $xy\in U(R)$ and so $x\in U(R)$. Write $x=\pm 1+a$ for some $a\in J(R)$. Then $d=\phi(x)=\pm 1+\phi(a)$ and clearly $\phi(a)\in\phi(J(R))=J(D)$. Thus D is a weakly JU-ring.
Conversely, assume that D is a weakly JU-ring and let $x\in U(R)$. Then $\phi(x)\in U(D)$ and so $\phi(x)=\pm 1+d$ for some $d=\phi(y)\in J(D)$. Thus $x=\pm 1+y+m$ for some $m\in M$. But since $y\in J(R)$ and $M\subseteq J(R)$, $y+m\in J(R)$. Thus R is a weakly JU-ring, as desired. The proof of JU-ring is similar.
Assume that D is weakly semiboolean and let $x\in R$. If $x\in J(R)$, then $x=x+0$ and so we are done. Assume that $x\notin J(R)$. Then $\phi(x)=c\pm e$ where $c=\phi(y)\in J(D)$ and $e\in Id(D)$. Since D is an integral domain, either $e=0$ or $e=1$. If $e=0$, then $\phi(x)=c\in J(D)$ and so $x\in\phi^{-1}(J(D))=J(R)$, which is absurd. Thus $e=1$, and so $x=\pm 1+y+m$ for some $m\in M$ and clearly $y+m\in J(R)$ as desired.
Conversely, assume that R is a weakly semiboolean ring and let $d=\phi(x)$ for some $x\in R$. Write $x=a\pm e$ for some $a\in J(R)$ and $e\in Id(R)$. Then $d=\phi(a)\pm\phi(e)$ and clearly $\phi(a)\in\phi(J(R))=J(D)$ and $\phi(e)\in Id(D)$. Also it follows easily from the fact that $D=R/M$.
The proof of semiboolean case is similar.

(ii) Assume that R is a UU-ring and let $x \in J(R)$. Then $1+x \in U(R)$ and so $1+x = 1+a$ for some $a \in Nil(R)$. Thus $x = a \in Nil(R)$ and so $J(R) \subseteq Nil(R) \subseteq J(R)$. Therefore $Nil(R) = J(R)$. But since T is local with maximal ideal M, $J(R) = \phi^{-1}(J(D))$. Let $d \in J(D)$ and set $d = \phi(x)$ for some $x \in J(R) = Nil(R)$. Then $d^r = \phi(x^r) = 0$ for some positive integer r. Since D is an integral domain $d = 0$ and so $J(D) = (0)$. Hence $Nil(R) = J(R) = \phi^{-1}(J(D)) = M$. Now, let $d \in U(D)$ and set $d = \phi(x)$ and $d^{-1} = \phi(y)$ for some $x, y \in R$. Then $\phi(xy) = 1$ and so $1 - xy = m \in M = J(R)$. Thus $xy = 1 - m \in U(R)$ and so $x \in U(R)$. Hence $x = 1 + a$ for some $a \in Nil(R) = J(R) = M$. Thus $d = 1$ and so $U(D) = \{1\}$, as desired.
Conversely, let $x \in U(R)$. Then $\phi(x) \in U(D) = \{1\}$ and so $1 - x = m \in M = Nil(R)$. Thus $x = 1 + m$ as desired.
Assume now that R is a weakly UU-ring and let $a \in J(R)$. Since $1 + a \in U(R)$, $1 + a = \pm 1 + b$ for some $b \in Nil(R)$. If $1 + a = 1 + b$, then $a = b \in Nil(R)$. Assume that $1 + a = -1 + b$. Then $2 = b - a \in J(R)$ and so $3 \in U(R)$. Thus $3 = \pm 1 + c$ for some $c \in Nil(R)$. If $3 = 1 + c$, then $2 = c \in Nil(R)$ and so $a = b - 2 \in Nil(R)$. If $3 = -1 + c$, then $2^2 = 4 = c \in Nil(R)$ and so $2 \in Nil(R)$. Hence $a = b - 2 \in Nil(R)$ and so $J(R) \subseteq Nil(R)$. Thus $M \subseteq J(R) \subseteq Nil(R) \subseteq M$ and hence $Nil(R) = J(R) = M$. Finally, let $d \in U(D)$ and set $d = \phi(x)$ and $d^{-1} = \phi(y)$ for some $x, y \in R$. Then $\phi(xy) = 1$ and so $xy = 1 + m$ for some $m \in M = J(R)$. Thus $xy \in U(R)$ and so $x \in U(R)$. Hence $x = \pm 1 + a$ for some $a \in Nil(R) = J(R) = M$. Thus $d = \pm 1$ and so $U(D) = \{-1, 1\}$, as desired.
Conversely, Let $u \in U(R)$. Then $\phi(u) \in U(D)$ and so $\phi(u) = 1$ (resp. $\phi(u) = \pm 1$). So $x = 1 + m$ (resp. $x = \pm 1 + m$) for some $m \in M = Nil(R)$ and so R is a UU-ring (resp. weakly UU-ring). □

The next example shows that for the diagram of type (□), if R is a weakly JU-ring, then T is not necessarily a weakly JU-ring.

Example 2.15. (1) Let $\mathbb{Z}_3$ be the field of integers modulo 3, X and Y indeterminates over $\mathbb{Z}_3$ and set $T = \mathbb{Z}_3((X))[[Y]] = \mathbb{Z}_3((X)) + M$ where $M = YT$ and $R = \mathbb{Z}_3[[X]] + M$. Since T is local with maximal ideal M and $\mathbb{Z}_3[[X]]$ is a weakly JU-ring (Proposition 2.2), R is a weakly JU-ring by Theorem 2.14(1), but T is not a weakly JU-ring since $T/M = \mathbb{Z}_3((X))$.

(2) Let $T = \mathbb{Z}_2(X)[[Y]] = \mathbb{Z}_2(X) + M$ and $R = \mathbb{Z}_2 + M$. By Theorem 2.14(1), R is a semiboolean ring, but T is not a semiboolean ring.

(3) Let $T = \mathbb{Z}_2(X)[Y]/(Y^2)$. Then T is local with maximal ideal $M = (Y)/(Y^2)$, $K = T/M = \mathbb{Z}_2(X)$ and $M^2 = (0)$. Set $R = \phi^{-1}(\mathbb{Z}_2)$. By Theorem 2.14, R is a UU-ring, however T is not.

The next example shows that if T is not local, the condition "$Nil(R) = J(R) = M$" is not a necessary condition.

Example 2.16. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2(X)[Y]/(Y^2)$, $M = \mathbb{Z}_2 \times (Y)/(Y^2)$ and $N = 0 \times \mathbb{Z}_2(X)[Y]/(Y^2)$. Then T is a semi-local ring with maximal ideals M and N and $T/M = \mathbb{Z}_2(X)$. Let R be the pullback of the diagram:

$$(\square) \quad \begin{array}{ccc} R := \phi^{-1}(\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \downarrow \\ T & \stackrel{\phi}{\longrightarrow} & T/M = \mathbb{Z}_2(X). \end{array}$$

Let y denotes the class of Y modulo Y^2. Then it is easy to see that $T = \{(a, a_0(X) + a_1(X)y) | a \in \mathbb{Z}_2, a_0(X), a_1(X) \in \mathbb{Z}_2(X)\}$, $R = \{(a, a_0 + a_1(X)y) | a, a_0 \in \mathbb{Z}_2, a_1(X) \in \mathbb{Z}_2(X)\}$, $U(R) = \{(1, 1 + a_1(X)y) | a_1(X) \in \mathbb{Z}_2(X)\}$ and $Nil(R) = J(R) = \{(0, a(X)y) | a(X) \in \mathbb{Z}_2(X)\} \subsetneq M$. But clearly $U(R) = 1 + Nil(R) = 1 + J(R)$ and so R is UU ring.

The next example illustrates Theorem 2.14(ii) and shows how to use pullbacks to construct weakly UU rings that are not UU rings.

Example 2.17. Let $\mathbb{Z}_3$ be the field of integers modulo 3, X and Y indeterminates over $\mathbb{Z}_3$ and set $T = \mathbb{Z}_3(X)[Y]/(Y^2)$. Then T is a local ring with maximal ideal $M = (Y)/(Y^2)$ and residue field $T/M = \mathbb{Z}_3(X)$. Let R be the pullback of the diagram:

$$(\square) \quad \begin{array}{ccc} R := \phi^{-1}(\mathbb{Z}_3) & \longrightarrow & \mathbb{Z}_3 \\ \downarrow & & \downarrow \\ T & \stackrel{\phi}{\longrightarrow} & T/M = \mathbb{Z}_3(X). \end{array}$$

Let y denotes the class of Y modulo Y^2. Then it is easy to see that $T = \{a_0(X) + a_1(X)y | a_0(X), a_1(X) \in \mathbb{Z}_3(X)\}$, $R = \{a_0 + a_1(X)y | a_0 \in \mathbb{Z}_3, a_1(X) \in \mathbb{Z}_3(X)\}$ and $Nil(R) = J(R) = M$. By Theorem 2.14, R is a weakly UU ring (since $U(\mathbb{Z}_3) = \{-1 = 2, 1\}$) which is not a UU ring.

We close this paper by listing some open questions, the first and second were listed in [20].

(1)-Find a criterion when the triangular matrix $n \times n$ ring $\mathbb{T}_n(R)$ is a JU ring or a weakly JU ring?

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(2)- Describe weakly exchange WUU rings. Are they weakly clean WUU?
(3)-Characterize the notion of JU, WJU, UU and WUU rings in bi-amalgamated of algebras along ideals.

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