A new approach to N-soft topological structures

İzzettin DEMİR and Merve OKURER

Department of Mathematics, Duzce University, 81620, Duzce-Turkey
e-mails: izzettindemir@duzce.edu.tr, merveokurer34@gmail.com

Abstract. In the present paper, we study the topological structure of N-soft sets given by Riaz et al. [29]. Firstly, we redefine N-soft closed sets using the complement operation of N-soft sets proposed in [17] and investigate their basic properties. Then, we introduce the concept of an N-soft continuous mapping and also obtain the initial N-soft topology determined by a family of N-soft mappings. Furthermore, we establish a new concept of N-soft topological subspace and analyze some related properties of this concept. Finally, we present some examples to better understand the defined concepts.

Keywords: N-soft set, N-soft topology, N-soft continuity, N-soft closure, initial N-soft topology, N-soft subspace.

2010 AMS Classification: 06D72, 54A05, 54C05, 54B10, 54B05.

1 Introduction

Molodtsov [28] initiated soft set theory as an innovative approach to deal with uncertainties and to overcome incompatibility with the parametrization tools, which were not solved by existing theories such as fuzzy set theory, vague set theory, interval mathematics theory, and rough set theory. This leads to the rapid growth of soft set theory in a short period of time, as well as a wide range of real-world applications of soft sets including decision-making problems, medical science and computer sciences [8, 10, 15, 19, 33].

Maji et al. [26, 27] gave the first practical application of soft sets in decision making problems and then studied on the basic concepts of soft set theory. Chen et al. [14] presented a novel concept of parameterization reduction in soft sets. Also, Aktaş and Çağman [5] defined the soft group and compared soft sets to fuzzy set and rough set. Ali et al. [7] provided new kinds of algebraic operations in soft set theory. Kharal and Ahmad [23] initiated the notion of a mapping on soft classes. Later, Das and Samanta [16] introduced the notions of soft real sets and soft real numbers and studied their properties. Furthermore, as a mixture of soft sets and topology, Shabir and Naz [31] put forward the idea of a soft topology and discussed some properties of soft topology. In this line, Aygünoğlu and Aygün [12] brought out the notions of compactness and product topology in soft settings. In 2018, El-Shafei et al. [20] defined partial belong and total non belong relations and then utilized them to study partial soft separation

23 Oct 2022 13:16:53 PDT
220930-Demir Version 2 - Submitted to Rocky Mountain J. Math.

axioms. Also, Al-Shami [9] presented the first practical application of soft separation axioms. Moreover, Al-Shami et al. [11] studied the notions of compactness and connectedness via the family of soft somewhat open sets which is one of the generalizations of soft open sets. Following these works, many researchers have been achieved the further studies on soft topology [6, 13, 18, 22, 32]

Use of the soft set allows the binary evaluation of the objects. However, in daily life, we face with a lot of decision-making problems that involve non-binary and discrete structure data. This non-binary evaluation usually appear in rating or ranking cases which can be represented in multinary values such as number of stars, dots, grades or any generalized notation. Inspired by these observations, Fatimah et al. [21] put forward the idea of an N-soft set as an extended form of soft set to indicate the significance of grades in daily life. Since the appearance of this new method, researchers have done a lot of work on N-soft set. As the generalization of N-soft sets, Akram et al. [1, 2] presented the new concepts having applications called fuzzy N-soft sets and hesitant N-soft sets. Besides, Akram et al. [3, 4] proposed intuitionistic fuzzy N-soft rough sets as a new decision-making hybrid model and investigated parameter reductions in N-soft sets. Riaz et al. [29] established an N-soft topology and studied certain new operations on N-soft set. Moreover, Riaz et al. [30] presented the notion of a neutrosophic N-soft set and investigated its application to the decision making. Kamacı and Petchimuthu [24] gave the notion of a bipolar N-soft set and constructed some decision-making approaches. Liu et al. [25] introduced neutrosophic vague N-soft sets, generalizing both the neutrosophic vague sets and the N-soft sets.

In this paper, we continue investigating the topological structure of N-soft sets. Firstly, in the light of the results given in [17], we introduce N-soft closed sets and present some of their properties. Then, we give the concept of an N-soft continuous mapping and determine the relations between this mapping and N-soft closed sets. Also, we establish the existence of initial N-soft topology, and based on this fact, we characterize products of N-soft topological spaces. Furthermore, we explore a novel concept of N-soft topological subspace and study some useful properties of it. Finally, we demonstrate examples to investigate the usability of the obtained results.

2 Preliminaries

To better understand the results obtained in this paper, we recall some basic concepts related to N-soft sets which we need in the sequel. Throughout this paper, X be a universe of alternatives (objects) and P be a set of specified parameters (attributes) unless otherwise explicit.

Definition 2.1. ([28]) *Let X be a universe of alternatives (objects) and P be a set of specified parameters (attributes). A pair (S, P) is called a soft set over X if S is a mapping from the set P to the set of all subsets of X, $S : P \to 2^X$.*

Definition 2.2 ([21])**.** *Let X be a universe of alternatives (objects) and P be a set of specified parameters (attributes). Also, let $\mathcal{L} = \{0, 1, 2, ..., N-1\}$ be a collection of ordered grades, where $N = \{2, 3, ...\}$. Then, the triplet (F, P, N) is called an N-soft set on X if F is a mapping $F : P \to 2^{X\times\mathcal{L}}$ where for each $p \in P$,*

$$F(p) = \{(x, \ell^F_{px}) : x \in X \ and \ \ell^F_{px} \in \mathcal{L} \ is \ the \ grade \ of \ x\}$$

such that there is a unique grade ℓ^F_{px} for each $x \in X$.

In this case, we type $F(p)(x) = \ell^F_{px}$ to match the standard notation for soft sets, which are the case where $N = 2$.

Throughout this paper, the family of all N-soft sets over X is denoted by $\mathcal{S}^N(X, P)$.

Example 2.3. *Let $X = \{x_1, x_2, x_3\}$ be a set of mobile phones under consideration and $P = \{p_1, p_2, p_3, p_4, p_5\} = \{design,\ software,\ performance,\ batterylife,\ camera\}$ be a set of attributes. Let $\mathcal{L} = \{0, 1, 2, 3, 4\}$ be a set of ordered grades, where*

$$\begin{aligned}
0 &= \textit{not having the attribute},\\
1 &= \textit{low},\\
2 &= \textit{medium},\\
3 &= \textit{high},\\
4 &= \textit{full}.
\end{aligned}$$

The grades assigned by the experts to mobile phones according to the parameters are given in Table 1. Therefore, 5-soft set over X can be viewed as follows:

$$\begin{aligned}
(F, P, 5) = \Big\{&\langle p_1, \{(x_1, 1), (x_2, 3), (x_3, 2)\}\rangle, \langle p_2, \{(x_1, 4), (x_2, 2), (x_3, 2)\}\rangle,\\
&\langle p_3, \{(x_1, 0), (x_2, 1), (x_3, 4)\}\rangle, \langle p_4, \{(x_1, 2), (x_2, 3), (x_3, 1)\}\rangle,\\
&\langle p_5, \{(x_1, 3), (x_2, 2), (x_3, 3)\}\rangle\Big\}.
\end{aligned}$$

Table 1
The tabular form of $(F, P, 5)$

$(F, P, 5)$	p_1	p_2	p_3	p_4	p_5
x_1	1	4	0	2	3
x_2	3	2	1	3	2
x_3	2	2	4	1	3

Definition 2.4 ([29])**.** *Let X be the universal set and P be the set of parameters. Let $\mathcal{L} = \{0, 1, 2, ..., N-1\}$. Then,*

(i) The empty N-soft set on X is defined by (X_0, P, N), where $X_0 : P \to 2^{X \times \{0\}}$ with the property that for all $p \in P$ and all $x \in X$, $X_0(p) = (x, 0)$.

(ii) The universal N-soft set on X is denoted by (X_{N-1}, P, N), where $X_{N-1} : P \to 2^{X \times \{N-1\}}$ with the property that for all $p \in P$ and all $x \in X$, $X_{N-1}(p) = (x, N-1)$.

Definition 2.5 ([29])**.** *Let (F, P, N), $(G, P, N) \in \mathcal{S}^N(X, P)$. Then,*

(i) (F, P, N) is an N-soft subset of (G, P, N) if for all $p \in P$ and all $x \in X$,

$$\ell^F_{px} \leq \ell^G_{px}.$$

It is symbolized by $(F, P, N) \sqsubseteq (G, P, N)$.

(ii) (F, P, N) and (G, P, N) are equal N-soft set if for all $p \in P$ and all $x \in X$,

$$\ell^F_{px} = \ell^G_{px}.$$

It is symbolized by $(F,P,N) = (G,P,N)$.

(iii) The addition of (F,P,N) *and* (G,P,N) *is defined by N-soft set* $(F,P,N) \oplus (G,P,N)$=$(F \oplus G,P,N)$*, where* $(F \oplus G)(p) = \{(x, \ell^{F\oplus G}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F\oplus G}_{px} = \begin{cases} \ell^F_{px} + \ell^G_{px}, & \text{if } 0 \leq \ell^F_{px} + \ell^G_{px} < N-1; \\ N-1, & \text{if } \ell^F_{px} + \ell^G_{px} \geq N-1. \end{cases}$$

(iv) The subtraction of (F,P,N) *and* (G,P,N) *is defined by N-soft set* $(F,P,N) \ominus (G,P,N)$=$(F \ominus G,P,N)$*, where* $(F \ominus G)(p) = \{(x, \ell^{F\ominus G}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F\ominus G}_{px} = \begin{cases} \ell^F_{px} - \ell^G_{px}, & \text{if } \ell^F_{px} > \ell^G_{px}; \\ 0, & \text{if otherwise.} \end{cases}$$

Definition 2.6 ([21])**.** *Let* $(F,P,N_1) \in \mathcal{S}^{N_1}(X,P)$ *and* $(G,P,N_2) \in \mathcal{S}^{N_2}(X,P)$*. Then,*

(i) The union of (F,P,N_1) *and* (G,P,N_2) *is defined by N-soft set* $(F,P,N_1) \sqcup (G,P,N_2)$=$(F \sqcup G, P, max\{N_1,N_2\})$*, where* $(F \sqcup G)(p) = \{(x, \ell^{F\sqcup G}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F\sqcup G}_{px} = max\{\ell^F_{px}, \ell^G_{px}\}.$$

(ii) The intersection of (F,P,N_1) *and* (G,P,N_2) *is defined by N-soft set* $(F,P,N_1) \sqcap (G,P,N_2)$=$(F \sqcap G, P, min\{N_1,N_2\})$*, where* $(F \sqcap G)(p) = \{(x, \ell^{F\sqcap G}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F\sqcap G}_{px} = min\{\ell^F_{px}, \ell^G_{px}\}.$$

Definition 2.7. ([21]) *A bottom weak complement of N-soft set* (F,P,N) *is defined by N-soft set* $(F,P,N)^b = (F^b,P,N)$*, where* $F^b(p) = \{(x, \ell^{F^b}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F^b}_{px} = \begin{cases} 0, & \text{if } \ell^F_{px} > 0; \\ N-1, & \text{if } \ell^F_{px} = 0. \end{cases}$$

Definition 2.8 ([17])**.** *The relative complement (or briefly complement) of an N-soft set* (F,P,N) *is defined by N-soft set* $(F,P,N)^r = (F^r,P,N)$*, where* $F^r(p) = \{(x, \ell^{F^r}_{px}) : x \in X\}$ *for all* $p \in P$ *such that*

$$\ell^{F^r}_{px} = N - 1 - \ell^F_{px}.$$

Proposition 2.9 ([17])**.** *Let* (F,P,N)*,* (F_1,P,N) *and* (F_2,P,N) *be the N-soft sets over X. Then, the following statements are satisfied:*

(i) $((F,P,N)^r)^r = (F,P,N)$.

(ii) $((F_1,P,N) \sqcap (F_2,P,N))^r = (F_1,P,N)^r \sqcup (F_2,P,N)^r$.

(iii) $((F_1,P,N) \sqcup (F_2,P,N))^r = (F_1,P,N)^r \sqcap (F_2,P,N)^r$.

(iv) If $(F_1,P,N) \sqsubseteq (F_2,P,N)$*, then* $(F_2,P,N)^r \sqsubseteq (F_1,P,N)^r$.

Definition 2.10 ([17])**.** *Let* $\mathcal{S}^N(X,P)$ *and* $\mathcal{S}^N(Y,K)$ *be the families of all N-soft sets over X and Y with attributes from P and K, respectively. Let* $\mathfrak{u} : X \to Y$ *and* $\mathfrak{a} : P \to K$ *be two mappings. Then, the mapping* $\mathfrak{u}_\mathfrak{a}$ *is called an N-soft mapping from X to Y, denoted by* $\mathfrak{u}_\mathfrak{a} : \mathcal{S}^N(X,P) \to \mathcal{S}^N(Y,K)$.

(i) Let $(F,P,N) \in \mathcal{S}^N(X,P)$*. Then,* $\mathfrak{u}_\mathfrak{a}(F,P,N) = (\mathfrak{u}_\mathfrak{a}(F),K,N)$ *is the N-soft set over Y defined as follows: for all* $k \in K$,

$$\mathfrak{u}_\mathfrak{a}(F)(k) = \{(y, \ell^{\mathfrak{u}_\mathfrak{a}(F)}_{ky}) : y \in Y\},$$

where

$$\ell_{ky}^{\mathfrak{u}_{\mathfrak{a}}(F)} = \begin{cases} max\{\ell_{px}^{F} : p \in \mathfrak{a}^{-1}(k), x \in \mathfrak{u}^{-1}(y)\}, & if\ \mathfrak{a}^{-1}(k) \neq \emptyset,\ \mathfrak{u}^{-1}(y) \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

$\mathfrak{u}_{\mathfrak{a}}(F, P, N)$ *is called an N-soft image of an N-soft set* (F, P, N).

(ii) Let $(G, K, N) \in \mathcal{S}^N(Y, K)$. *Then,* $\mathfrak{u}_{\mathfrak{a}}^{-1}(G, K, N) = (\mathfrak{u}_{\mathfrak{a}}^{-1}(G), P, N)$ *is the N-soft set over X defined as follows: for all* $p \in P$,

$$\mathfrak{u}_{\mathfrak{a}}^{-1}(G)(p) = \{(x, \ell_{px}^{\mathfrak{u}_{\mathfrak{a}}^{-1}(G)}) : x \in X\},$$

where

$$\ell_{px}^{\mathfrak{u}_{\mathfrak{a}}^{-1}(G)} = \ell_{\mathfrak{a}(p)\mathfrak{u}(x)}^{G}.$$

$\mathfrak{u}_{\mathfrak{a}}^{-1}(G, K, N)$ *is called an N-soft inverse image of an N-soft set* (G, K, N).

Theorem 2.11 ([17]). *Consider an N-soft mapping* $\mathfrak{u}_{\mathfrak{a}} : \mathcal{S}^N(X, P) \to \mathcal{S}^N(Y, K)$. *Then, for* (F_1, P, N), (F_2, P, N) *and* $(F_i, P, N) \in \mathcal{S}^N(X, P)$ *with* $i \in J$, *where J is an index set, the following properties are satisfied.*

(i) $\mathfrak{u}_{\mathfrak{a}}(X_0, P, N) = (Y_0, K, N)$.

(ii) $\mathfrak{u}_{\mathfrak{a}}(X_{N-1}, P, N) \sqsubseteq (Y_{N-1}, K, N)$.

(iii) If $(F_1, P, N) \sqsubseteq (F_2, P, N)$, *then* $\mathfrak{u}_{\mathfrak{a}}(F_1, P, N) \sqsubseteq \mathfrak{u}_{\mathfrak{a}}(F_2, P, N)$.

(iv) $\mathfrak{u}_{\mathfrak{a}}((F_1, P, N) \sqcup (F_2, P, N)) = \mathfrak{u}_{\mathfrak{a}}(F_1, P, N) \sqcup \mathfrak{u}_{\mathfrak{a}}(F_2, P, N)$.

In general, we have $\mathfrak{u}_{\mathfrak{a}}(\bigsqcup_{i \in J}(F_i, P, N)) = \bigsqcup_{i \in J} \mathfrak{u}_{\mathfrak{a}}(F_i, P, N)$.

(v) $\mathfrak{u}_{\mathfrak{a}}((F_1, P, N) \sqcap (F_2, P, N)) \sqsubseteq \mathfrak{u}_{\mathfrak{a}}(F_1, P, N) \sqcap \mathfrak{u}_{\mathfrak{a}}(F_2, P, N)$.

In general, we have $\mathfrak{u}_{\mathfrak{a}}(\bigsqcap_{i \in J}(F_i, P, N)) \sqsubseteq \bigsqcap_{i \in J} \mathfrak{u}_{\mathfrak{a}}(F_i, P, N)$.

Theorem 2.12 ([17]). *Consider an N-soft mapping* $\mathfrak{u}_{\mathfrak{a}} : \mathcal{S}^N(X, P) \to \mathcal{S}^N(Y, K)$. *Then, for* (G_1, K, N), (G_2, K, N) *and* $(G_i, K, N) \in \mathcal{S}^N(Y, K)$ *with* $i \in J$, *where J is an index set, the following properties are satisfied.*

(i) $\mathfrak{u}_{\mathfrak{a}}^{-1}(Y_0, K, N) = (X_0, P, N)$.

(ii) $\mathfrak{u}_{\mathfrak{a}}^{-1}(Y_{N-1}, K, N) = (X_{N-1}, P, N)$.

(iii) If $(G_1, K, N) \sqsubseteq (G_2, K, N)$, *then* $\mathfrak{u}_{\mathfrak{a}}^{-1}(G_1, K, N) \sqsubseteq \mathfrak{u}_{\mathfrak{a}}^{-1}(G_2, K, N)$.

(iv) $\mathfrak{u}_{\mathfrak{a}}^{-1}((G_1, K, N) \sqcup (G_2, K, N)) = \mathfrak{u}_{\mathfrak{a}}^{-1}(G_1, K, N) \sqcup \mathfrak{u}_{\mathfrak{a}}^{-1}(G_2, K, N)$.

In general, we have $\mathfrak{u}_{\mathfrak{a}}^{-1}(\bigsqcup_{i \in J}(G_i, K, N)) = \bigsqcup_{i \in J} \mathfrak{u}_{\mathfrak{a}}^{-1}(G_i, K, N)$.

(v) $\mathfrak{u}_{\mathfrak{a}}^{-1}((G_1, K, N) \sqcap (G_2, K, N)) = \mathfrak{u}_{\mathfrak{a}}^{-1}(G_1, K, N) \sqcap \mathfrak{u}_{\mathfrak{a}}^{-1}(G_2, K, N)$.

In general, we have $\mathfrak{u}_{\mathfrak{a}}^{-1}(\bigsqcap_{i \in J}(G_i, K, N)) = \bigsqcap_{i \in J} \mathfrak{u}_{\mathfrak{a}}^{-1}(G_i, K, N)$.

Theorem 2.13 ([17]). *Let* $\mathfrak{u}_{\mathfrak{a}} : \mathcal{S}^N(X, P) \to \mathcal{S}^N(Y, K)$ *be an N-soft mapping. Then, for* $(F, P, N) \in \mathcal{S}^N(X, P)$ *and* $(G, K, N) \in \mathcal{S}^N(Y, K)$, *the following properties hold.*

(i) $(F, P, N) \sqsubseteq \mathfrak{u}_{\mathfrak{a}}^{-1}(\mathfrak{u}_{\mathfrak{a}}(F, P, N))$.

(ii) $\mathfrak{u}_{\mathfrak{a}}(\mathfrak{u}_{\mathfrak{a}}^{-1}(G, K, N)) \sqsubseteq (G, K, N)$.

Definition 2.14 ([29]). *Let* τ *be the collection of N-soft sets over X, then* τ *is said to be an N-soft topology on X if*

(Nst_1) (X_0, P, N), (X_{N-1}, P, N) *belong to* τ.

(Nst_2) *the union of any number of N-soft sets in* τ *belongs to* τ.

(Nst_3) *the intersection of any two N-soft sets in* τ *belongs to* τ.

23 Oct 2022 13:16:53 PDT
220930-Demir Version 2 - Submitted to Rocky Mountain J. Math.

(X, P, N, τ) is called an N-soft topological space. The members of τ are called N-soft open sets in X and their bottom weak complements as defined by Definition 2.7 are called N-soft closed sets.

Definition 2.15 ([29]). *Let (X, P, N, τ) be an N-soft topological space and $(F, P, N) \in \mathcal{S}^N(X, P)$. The N-soft interior of (F, P, N), denoted by $(F, P, N)^o$, is the union of all N-soft open subsets of (F, P, N).*
Clearly, $(F, P, N)^o$ is the largest N-soft open set contained in (F, P, N).

Definition 2.16 ([29]). *Let (X, P, N, τ) be an N-soft topological space and $\mathcal{B}$ be subcollection of τ. If each element of τ can be express as a union of members of $\mathcal{B}$, then $\mathcal{B}$ is called an N-soft base for the N-soft topology τ.*

3 N-soft closed sets and N-soft continuity

Riaz et al. [29] defined N-soft closed sets by taking bottom weak complements of N-soft open sets. However, since this complement operation is not suitable for double complementation law in N-soft sets, some classical properties of closure operators were missing in the previous approach. So, in this section, on the basis of the notion of complement of an N-soft set given by Demir [17], we redefine N-soft closed sets, in a different way than in Definition 2.14, and investigate its related properties. Besides, we present the notions of an N-soft continuous mapping and an initial N-soft topology.

Definition 3.1. *Let (X, P, N, τ) be an N-soft topological space. An N-soft set (F, P, N) over X is called an N-soft closed in X if $(F, P, N)^r \in \tau$.*

Theorem 3.2. *Let (X, P, N, τ) be an N-soft topological space. Then, the following properties satisfied.*

(Nsc_1) (X_0, P, N), (X_{N-1}, P, N) are N-soft closed sets.
(Nsc_2) the union of any two N-soft closed sets is an N-soft closed set.
(Nsc_3) the intersection of any family of N-soft closed sets is an N-soft closed set.

Proof. These properties can be easily checked from De Morgan's laws and properties $(Nst_1) - (Nst_3)$ of N-soft open sets. □

Definition 3.3. *Let (X, P, N, τ) be an N-soft topological space and $(F, P, N) \in \mathcal{S}^N(X, P)$. The N-soft closure of (F, P, N), denoted by $\overline{(F, P, N)}$, is the intersection of all N-soft closed containing (F, P, N). It is clear that $\overline{(F, P, N)}$ is the smallest N-soft closed set over X which contains (F, P, N).*

Theorem 3.4. *Let (X, P, N, τ) be an N-soft topological space and $(F, P, N), (G, P, N) \in \mathcal{S}^N(X, P)$. Then, the following properties hold:*

(i) $\overline{(X_0, P, N)} = (X_0, P, N)$ *and* $\overline{(X_{N-1}, P, N)} = (X_{N-1}, P, N)$.
(ii) $(F, P, N) \sqsubseteq \overline{(F, P, N)}$.
(iii) (F, P, N) *is an N-soft closed set if and only if* $\overline{(F, P, N)} = (F, P, N)$.
(iv) $\overline{\overline{(F, P, N)}} = \overline{(F, P, N)}$.
(v) If $(F, P, N) \sqsubseteq (G, P, N)$, *then* $\overline{(F, P, N)} \sqsubseteq \overline{(G, P, N)}$.
(vi) $\overline{(F, P, N) \sqcup (G, P, N)} = \overline{(F, P, N)} \sqcup \overline{(G, P, N)}$.
(vii) $\overline{(F, P, N) \sqcap (G, P, N)} \sqsubseteq \overline{(F, P, N)} \sqcap \overline{(G, P, N)}$.

Proof. Properties (i), (ii) and (v) follow directly from the definition of N-soft closure.

(iii) Let (F,P,N) be an N-soft closed set. Since $\overline{(F,P,N)}$ is the smallest N-soft closed set over X which contains (F,P,N), we have $\overline{(F,P,N)} \sqsubseteq (F,P,N)$. Thus, from (ii) it follows that $\overline{(F,P,N)} = (F,P,N)$. Conversely, let $\overline{(F,P,N)} = (F,P,N)$. Since $\overline{(F,P,N)}$ is an N-soft closed set, then (F,P,N) is an N-soft closed set.

(iv) It follows from the fact that $\overline{(F,P,N)}$ is an N-soft closed set.

(vi) By (v), we have $\overline{(F,P,N)} \sqsubseteq \overline{(F,P,N) \sqcup (G,P,N)}$ and $\overline{(G,P,N)} \sqsubseteq \overline{(F,P,N) \sqcup (G,P,N)}$. Then, we obtain $\overline{(F,P,N)} \sqcup \overline{(G,P,N)} \sqsubseteq \overline{(F,P,N) \sqcup (G,P,N)}$. Conversely, since $\overline{(F,P,N) \sqcup (G,P,N)}$ is the smallest N-soft closed set which contains $(F,P,N) \sqcup (G,P,N)$ and from $(F,P,N) \sqcup (G,P,N) \sqsubseteq \overline{(F,P,N)} \sqcup \overline{(G,P,N)}$ it follows that $\overline{(F,P,N) \sqcup (G,P,N)} \sqsubseteq \overline{(F,P,N)} \sqcup \overline{(G,P,N)}$. Thus, the equality (vi) holds.

(vii) It is clear by the property (v) and the definition of N-soft closure. □

Theorem 3.5. *Let us consider an operator associating with each N-soft set $(F,P,N) \in \mathcal{S}^N(X,P)$ another N-soft set $\overline{(F,P,N)} \in \mathcal{S}^N(X,P)$ such that the following properties hold:*

$(Nsco_1)$ $(F,P,N) \sqsubseteq \overline{(F,P,N)}$,
$(Nsco_2)$ $\overline{\overline{(F,P,N)}} = \overline{(F,P,N)}$,
$(Nsco_3)$ $\overline{(F,P,N) \sqcup (G,P,N)} = \overline{(F,P,N)} \sqcup \overline{(G,P,N)}$,
$(Nsco_4)$ $\overline{(X_0,P,N)} = (X_0,P,N)$.

Then, the family

$$\tau = \{ (F,P,N) \in \mathcal{S}^N(X,P) : \overline{(F,P,N)^r} = (F,P,N)^r \}$$

defines an N-soft topology on X and for every $(F,P,N) \in \mathcal{S}^N(X,P)$, the N-soft set $\overline{(F,P,N)}$ is the N-soft closure of (F,P,N) in the N-soft topological space (X,P,N,τ).

This operator is called the N-soft closure operator.

Proof. To prove the first part of the theorem it suffices to show that the family $C = \{(F,P,N) : \overline{(F,P,N)} = (F,P,N)\}$ of N-soft closed sets has properties $(Nsc_1) - (Nsc_3)$. Since $\overline{(F,P,N)} \sqsubseteq (X_{N-1},P,N)$ for every $(F,P,N) \sqsubseteq (X_{N-1},P,N)$ we have $\overline{(X_{N-1},P,N)} \sqsubseteq (X_{N-1},P,N)$, and this together with $(Nsco_1)$ show that $\overline{(X_{N-1},P,N)} = (X_{N-1},P,N)$. By $(Nsco_4)$, we have $\overline{(X_0,P,N)} = (X_0,P,N)$ so that the family C has property (Nsc_1).

Let us take $(F,P,N),(G,P,N) \in C$. By $(Nsco_3)$,

$$\overline{(F,P,N) \sqcup (G,P,N)} = \overline{(F,P,N)} \sqcup \overline{(G,P,N)} = (F,P,N) \sqcup (G,P,N)$$

and this implies that $(F,P,N) \sqcup (G,P,N) \in C$. Therefore, the family C has property (Nsc_2).

Let us note that $(Nsco_3)$ implies that

$$If\ (F,P,N) \sqsubseteq (G,P,N),\ then\ \overline{(F,P,N)} \sqsubseteq \overline{(G,P,N)}. \quad (1)$$

Indeed, if $(F,P,N) \sqsubseteq (G,P,N)$, then $(F,P,N) \sqcup (G,P,N) = (G,P,N)$ and $\overline{(F,P,N)} \sqcup \overline{(G,P,N)} = \overline{(F,P,N) \sqcup (G,P,N)} = \overline{(G,P,N)}$. The last equality gives $\overline{(F,P,N)} \sqsubseteq \overline{(G,P,N)}$. Let us take now a family $\{(F_i,P,N)\}_{i\in I}$ of members of C, where $I \subseteq \mathbb{N}$. i.e., let $\overline{(F_i,P,N)} = (F_i,P,N)$ for $i \in I$. As $\bigsqcap_{i\in I}(F_i,P,N) \sqsubseteq (F_i,P,N)$, by (1) we have $\overline{\bigsqcap_{i\in I}(F_i,P,N)} \sqsubseteq \overline{(F_i,P,N)} = (F_i,P,N)$ and this implies that $\overline{\bigsqcap_{i\in I}(F_i,P,N)} \sqsubseteq \bigsqcap_{i\in I}(F_i,P,N)$. The last inclusion together with $(Nsco_1)$ show that $\overline{\bigsqcap_{i\in I}(F_i,P,N)} = \bigsqcap_{i\in I}(F_i,P,N)$ Thus, the family C has property (Nsc_3).

23 Oct 2022 13:16:53 PDT
220930-Demir Version 2 - Submitted to Rocky Mountain J. Math.

Let the symbol $c((F,P,N))$ denote the closure of the N-soft set (F,P,N) in the N-soft topological space $((X_{N-1}P,N),\tau)$. We have to show that $c((F,P,N)) = \overline{(F,P,N)}$ for every $(F,P,N) \in \mathcal{S}^N(X,P)$. By $(Nsco_2)$, for every $(F,P,N) \in \mathcal{S}^N(X,P)$, we have $\overline{(F,P,N)} \in \mathcal{C}$, therefore $c((F,P,N)) \sqsubseteq \overline{(F,P,N)}$. For every N-soft closed subset (H,P,N) on X that contains (F,P,N), we obtain $\overline{(F,P,N)} \sqsubseteq \overline{(H,P,N)} = (H,P,N)$ by virtue of (1). Hence,

$$\begin{aligned}\overline{(F,P,N)} &\sqsubseteq \bigsqcap\{(H,P,N) : (H,P,N) \; is \; N-soft \; closed \; and \; (F,P,N) \sqsubseteq (H,P,N)\} \\ &= c((F,P,N))\end{aligned}$$

and this proves that $c((F,P,N)) = \overline{(F,P,N)}$. □

Theorem 3.6. *Let (X,P,N,τ) be an N-soft topological space and $(F,P,N) \in \mathcal{S}^N(X,P)$. Then,*
(i) $((F,P,N)^o)^r = \overline{(F,P,N)^r}$.
(ii) $\overline{((F,P,N))}^r = ((F,P,N)^r)^o$.

Proof. It is clear from Proposition 2.9 (iv) and Theorem 3.4 (v). □

Definition 3.7. *Let (X,P,N,τ_1) and (Y,K,N,τ_2) be two N-soft topological spaces. An N-soft mapping $\mathfrak{u}_\mathfrak{a} : (X,P,N,\tau_1) \to (Y,K,N,\tau_2)$ is called N-soft continuous if $\mathfrak{u}_\mathfrak{a}^{-1}(G,K,N) \in \tau_1$ for every $(G,K,N) \in \tau_2$.*

Example 3.8. *Let $X = P = \{0,1,2\}$. Let us consider the following 3-soft sets $(F_1,P,3)$ and $(G_1,P,3)$ on X with the set P of parameters: for all $p \in P$ and all $x \in X$,*

$$\ell_{px}^{F_1} = \begin{cases} 0, & if \; 0 \le x \le p; \\ x-p, & if \; p \le x \le 2, \end{cases} \qquad \ell_{px}^{G_1} = \begin{cases} p-x, & if \; 0 \le x \le p; \\ 0, & if \; p \le x \le 2. \end{cases}$$

Then, $\tau_1 = \{(X_0,P,3),(X_2,P,3),(F_1,P,3),(G_1,P,3),(F_1,P,3) \sqcup (G_1,P,3)\}$ is a 3-soft topology on X. On the other hand, let $Y = K = \{a,b,c\}$ and take the 3-soft sets $(F_2,K,3)$ and $(G_2,K,3)$ on Y with the set K of parameters such that

$$(F_2,K,3) = \Big\{\langle a,\{(a,0),(b,1),(c,2)\}\rangle, \langle b,\{(a,0),(b,0),(c,1)\}\rangle, \langle c,\{(a,0),(b,0),(c,0)\}\rangle\Big\},$$

$$(G_2,K,3) = \Big\{\langle a,\{(a,0),(b,1),(c,2)\}\rangle, \langle b,\{(a,1),(b,0),(c,1)\}\rangle, \langle c,\{(a,2),(b,1),(c,0)\}\rangle\Big\}.$$

Therefore, $\tau_2 = \{(Y_0,K,3),(Y_2,K,3),(F_2,K,3),(G_2,K,3)\}$ is a 3-soft topology on Y. Now, we define a 3-soft mapping $\mathfrak{u}_\mathfrak{a} : (X,P,3,\tau_1) \to (Y,K,3,\tau_2)$ by

$$\begin{aligned} &\mathfrak{u}(0) = a, \qquad \mathfrak{u}(1) = b, \qquad \mathfrak{u}(2) = c, \\ &\mathfrak{a}(0) = a, \qquad \mathfrak{a}(1) = b, \qquad \mathfrak{a}(2) = c. \end{aligned}$$

Thus, the 3-soft mapping $\mathfrak{u}_\mathfrak{a}$ is 3-soft continuous because $\mathfrak{u}_\mathfrak{a}^{-1}(F_2,K,3) = (F_1,P,3) \in \tau_1$ and $\mathfrak{u}_\mathfrak{a}^{-1}(G_2,K,3) = (F_1,P,3) \sqcup (G_1,P,3) \in \tau_1$.

Proposition 3.9. *If $(\mathfrak{u}_1)_{\mathfrak{a}_1} : (X,P,N,\tau_1) \to (Y,K,N,\tau_2)$ is an N-soft continuous mapping and $(\mathfrak{u}_2)_{\mathfrak{a}_2} : (Y,K,N,\tau_2) \to (Z,E,N,\tau_3)$ is an N-soft continuous mapping, then the composition $(\mathfrak{u}_2)_{\mathfrak{a}_2} \circ (\mathfrak{u}_1)_{\mathfrak{a}_1} = (\mathfrak{u}_2 \circ \mathfrak{u}_1)_{(\mathfrak{a}_2 \circ \mathfrak{a}_1)} : (X,P,N,\tau_1) \to (Z,E,N,\tau_3)$ is also an N-soft continuous mapping, where $\mathfrak{u}_2 \circ \mathfrak{u}_1 : X \to Z$ and $\mathfrak{a}_2 \circ \mathfrak{a}_1 : P \to E$.*

23 Oct 2022 13:16:53 PDT
220930-Demir Version 2 - Submitted to Rocky Mountain J. Math.

Proof. It will suffice to prove that $(\mathfrak{u}_2 \circ \mathfrak{u}_1)^{-1}_{(\mathfrak{a}_2 \circ \mathfrak{a}_1)}(H, E, N) = (\mathfrak{u}_1)^{-1}_{\mathfrak{a}_1}((\mathfrak{u}_2)^{-1}_{\mathfrak{a}_2}(H, E, N))$ for every $(H, E, N) \in \mathcal{S}^N(Z, E)$. Then, since for all $p \in P$ and all $x \in X$,

$$\ell_{px}^{(\mathfrak{u}_2 \circ \mathfrak{u}_1)^{-1}_{(\mathfrak{a}_2 \circ \mathfrak{a}_1)}(H)} = \ell^H_{\mathfrak{a}_2(\mathfrak{a}_1(p))\mathfrak{u}_2(\mathfrak{u}_1(x))} \quad and \quad \ell_{px}^{(\mathfrak{u}_1)^{-1}_{\mathfrak{a}_1}\left((\mathfrak{u}_2)^{-1}_{\mathfrak{a}_2}(H)\right)} = \ell^{(\mathfrak{u}_2)^{-1}_{\mathfrak{a}_2}(H)}_{\mathfrak{a}_1(p)\mathfrak{u}_1(x)} = \ell^H_{\mathfrak{a}_2(\mathfrak{a}_1(p))\mathfrak{u}_2(\mathfrak{u}_1(x))}$$

the required equality is satisfied. □

Proposition 3.10. *Consider an N-soft mapping $\mathfrak{u}_\mathfrak{a} : \mathcal{S}^N(X, P) \to \mathcal{S}^N(Y, K)$ and let $(F, P, N) \in \mathcal{S}^N(X, P)$ and $(G, K, N) \in \mathcal{S}^N(Y, K)$. Then, the following results hold.*

(i) $(\mathfrak{u}_\mathfrak{a}(F, P, N))^r \sqsubseteq \mathfrak{u}_\mathfrak{a}(F, P, N)^r$.

(ii) $(\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^r = \mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)^r$.

Proof. It follows easily from Definition 2.8 and Definition 2.10. □

Theorem 3.11. *Let (X, P, N, τ_1) and (Y, K, N, τ_2) be two N-soft topological spaces and $\mathfrak{u}_\mathfrak{a} : (X, P, N, \tau_1) \to (Y, K, N, \tau_2)$ an N-soft mapping. Then, the following statements are equivalent:*

(i) $\mathfrak{u}_\mathfrak{a}$ is N-soft continuous.

(ii) For every N-soft closed (G, K, N) in Y, $\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)$ is N-soft closed in X.

(iii) For every $(F, P, N) \in \mathcal{S}^N(X, P)$, we have $\mathfrak{u}_\mathfrak{a}\overline{(F, P, N)} \sqsubseteq \overline{\mathfrak{u}_\mathfrak{a}(F, P, N)}$.

(iv) For every $(G, K, N) \in \mathcal{S}^N(Y, K)$, we have $\overline{\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)} \sqsubseteq \mathfrak{u}^{-1}_\mathfrak{a}\overline{(G, K, N)}$.

(v) For every $(G, K, N) \in \mathcal{S}^N(Y, K)$, we have $\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)^o \sqsubseteq (\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^o$.

Proof. The implication (i) ⇒ (ii) is obvious.

We shall prove that (ii) ⇒ (iii). Let us observe that $\mathfrak{u}^{-1}_\mathfrak{a}\overline{(\mathfrak{u}_\mathfrak{a}(F, P, N))}$ is an N-soft closed set containing $(F, P, N) \in \mathcal{S}^N(X, P)$, and so that $\overline{(F, P, N)} \sqsubseteq \mathfrak{u}^{-1}_\mathfrak{a}\overline{(\mathfrak{u}_\mathfrak{a}(F, P, N))}$, which gives

$$\mathfrak{u}_\mathfrak{a}\overline{(F, P, N)} \sqsubseteq \mathfrak{u}_\mathfrak{a}(\mathfrak{u}^{-1}_\mathfrak{a}\overline{(\mathfrak{u}_\mathfrak{a}(F, P, N))}) \sqsubseteq \overline{\mathfrak{u}_\mathfrak{a}(F, P, N)}.$$

To prove that (iii) ⇒ (iv) we apply (iii) to $(F, P, N) = \mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)$ and we obtain the inclusion

$$\mathfrak{u}_\mathfrak{a}\overline{(\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))} \sqsubseteq \overline{\mathfrak{u}_\mathfrak{a}(\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))} \sqsubseteq \overline{(G, K, N)},$$

which gives $\overline{\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)} \sqsubseteq \mathfrak{u}^{-1}_\mathfrak{a}\overline{(G, K, N)}$.

To prove that (iv) ⇒ (v) we apply (iv) to $(G, K, N)^r$ and we obtain the inclusion $\overline{\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)^r} \sqsubseteq \mathfrak{u}^{-1}_\mathfrak{a}\overline{(G, K, N)^r}$, which gives

$$\begin{aligned}
\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)^o &= \mathfrak{u}^{-1}_\mathfrak{a}(\overline{(G, K, N)^r})^r = (\mathfrak{u}^{-1}_\mathfrak{a}\overline{(G, K, N)^r})^r \\
&\sqsubseteq (\overline{\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)^r})^r = \left(\overline{(\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^r}\right)^r \\
&= (\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^o.
\end{aligned}$$

To complete the proof of the theorem it remains to show that (v) ⇒ (i). For every $(G, K, N) \in \tau_2$, we have $(G, K, N) = (G, K, N)^o$, and it follows from (v) that $\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N) \sqsubseteq (\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^o$. Thus, we obtain $(\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N))^o = \mathfrak{u}^{-1}_\mathfrak{a}(G, K, N)$, i.e., $\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N) \in \tau_1$. □

Theorem 3.12. *Let $\mathfrak{u}_\mathfrak{a} : (X, P, N, \tau_1) \to (Y, K, N, \tau_2)$ be an N-soft mapping and $\mathcal{B}$ be an N-soft base for τ_2. Then, $\mathfrak{u}_\mathfrak{a}$ is N-soft continuous if and only if $\mathfrak{u}^{-1}_\mathfrak{a}(G, K, N) \in \tau_1$ for every $(G, K, N) \in \mathcal{B}$.*

Proof. It follows immediately from the definition of N-soft base and Theorem 2.12 (iv). □

Definition 3.13. *Let (X, P, N, τ) be an N-soft topological space and $\mathcal{S}$ be subcollection of τ. If the family of all finite intersections of members of $\mathcal{S}$ forms an N-soft base for τ, then $\mathcal{S}$ is called an N-soft subbase for the N-soft topology τ.*

Theorem 3.14. *Let $\mathcal{S}$ be a family of N-soft sets over X such that $(X_0, P, N), (X_{N-1}, P, N) \in \mathcal{S}$. Then, $\mathcal{S}$ is an N-soft subbase for the N-soft topology τ, whose members are of the form $\bigsqcup_{i\in J} \left(\bigsqcap_{m\in\Lambda_i} (F_i^m, P, N)\right)$, where J is an arbitrary index set and for each $i \in J$, Λ_i is a finite index set with $(F_i^m, P, N) \in \mathcal{S}$ for $i \in J$ and $m \in \Lambda_i$.*

Proof. The prove is trivial. □

Definition 3.15. *Let $(F, P, N) \in \mathcal{S}^N(X, P)$ and $(G, K, N) \in \mathcal{S}^N(Y, K)$. The N-soft product $(F, P, N) \times (G, K, N)$ is defined by $(F \times G, P \times K, N)$, where*

$$\ell^{F\times G}_{(p,k)(x,y)} = min\{\ell^F_{px}, \ell^G_{ky}\}$$

for all $(p, k) \in P \times K$ and all $(x, y) \in X \times Y$.

Therefore, the N-soft set $(F, P, N)\times(G, K, N)$ is an N-soft set over $X\times Y$ with $P\times K$ a parameter set.

Example 3.16. *Let $X_1 = \{x_1, x_1'\}$, $X_2 = \{x_2, x_2'\}$, $P_1 = \{p_1, p_1'\}$ and $P_2 = \{p_2, p_2'\}$. Take two 5-soft sets over X_1 and X_2 with parameters from P_1 and P_2, respectively, as*

$$(F, P_1, 5) = \Big\{\langle p_1, \{(x_1, 2), (x_1', 4)\}\rangle, \langle p_1', \{(x_1, 3), (x_1', 3)\}\rangle\Big\}$$

and

$$(G, P_2, 5) = \Big\{\langle p_2, \{(x_2, 0), (x_2', 1)\}\rangle, \langle p_2', \{(x_2, 1), (x_2', 1)\}\rangle\Big\}.$$

Then, we obtain

$$\begin{aligned}(F \times G, P_1 \times P_2, 5) = \Big\{&\langle (p_1, p_2), \{((x_1, x_2), 0), ((x_1, x_2'), 1), ((x_1', x_2), 0), ((x_1', x_2'), 1)\}\rangle,\\ &\langle (p_1, p_2'), \{((x_1, x_2), 1), ((x_1, x_2'), 1), ((x_1', x_2), 1), ((x_1', x_2'), 1)\}\rangle,\\ &\langle (p_1', p_2), \{((x_1, x_2), 0), ((x_1, x_2'), 1), ((x_1', x_2), 0), ((x_1', x_2'), 1)\}\rangle,\\ &\langle (p_1', p_2'), \{((x_1, x_2), 1), ((x_1, x_2'), 1), ((x_1', x_2), 1), ((x_1', x_2'), 1)\}\rangle\Big\}.\end{aligned}$$

Remark 3.17. *Consider the projection mappings $\mathfrak{u}_i^p : X_1\times X_2 \to X_i$ and $\mathfrak{a}_i^p : P_1\times P_2 \to P_i$, where $i \in \{1, 2\}$. Let $(F, P_1, N) \in \mathcal{S}^N(X_1, P_1)$ and $(G, P_2, N) \in \mathcal{S}^N(X_2, P_2)$. In general, the N-soft sets $(\mathfrak{u}_1^p)_{\mathfrak{a}_1^p}(F \times G, P_1 \times P_2, N)$ and $(\mathfrak{u}_2^p)_{\mathfrak{a}_2^p}(F \times G, P_1 \times P_2, N)$ do not have to equal to the N-soft sets (F, P_1, N) and (G, P_2, N), respectively. Indeed, in Example 3.16, we see that*

$$(\mathfrak{u}_1^p)_{\mathfrak{a}_1^p}(F \times G, P_1 \times P_2, 5) \neq (F, P_1, 5) \textit{ since } \ell^{(\mathfrak{u}_1^p)_{\mathfrak{a}_1^p}(F\times G)}_{p_1 x_1} = 1 \neq 2 = \ell^F_{p_1 x_1}.$$

However, one can easily verify that

$$\begin{aligned}(\mathfrak{u}_1^p)^{-1}_{\mathfrak{a}_1^p}(F, P_1, N) &= (F \times (X_2)_{N-1}, P_1 \times P_2, N),\\ (\mathfrak{u}_2^p)^{-1}_{\mathfrak{a}_2^p}(G, P_2, N) &= ((X_1)_{N-1} \times G, P_1 \times P_2, N).\end{aligned}$$

23 Oct 2022 13:16:53 PDT
220930-Demir Version 2 - Submitted to Rocky Mountain J. Math.

Definition 3.18. *Let $\{(Y_i, K_i, N, \tau_i)\}_{i\in J}$ be a family of N-soft topological spaces and for each $i \in J$, $(\mathfrak{u}_i)_{\mathfrak{a}_i} : \mathcal{S}^N(X, P) \to (Y_i, K_i, N, \tau_i)$ be an N-soft mapping. Then, the N-soft topology τ generated from the N-soft subbase $\mathcal{S} = \{(\mathfrak{u}_i)^{-1}_{\mathfrak{a}_i}(G, K_i, N) : (G, K_i, N) \in \tau_i, i \in J\}$ is called the N-soft topology (or initial N-soft topology) on X induced by the family of N-soft mappings $\{(\mathfrak{u}_i)_{\mathfrak{a}_i}\}_{i\in J}$ and from the family of N-soft topological spaces $\{(Y_i, K_i, N, \tau_i)\}_{i\in J}$.*

Remark 3.19. *It evidently is that the initial N-soft topology on X induced by the family $\{(\mathfrak{u}_i)_{\mathfrak{a}_i} : \mathcal{S}^N(X, P) \to (Y_i, K_i, N, \tau_i)\}_{i\in J}$ is the coarsest N-soft topology making all the $(\mathfrak{u}_i)_{\mathfrak{a}_i}$'s N-soft continuous.*

Definition 3.20. *Let $\{(X_i, P_i, N, \tau_i)\}_{i\in J}$ be a family of N-soft topological spaces. Then, the initial N-soft topology on $X = \prod_{i\in J} X_i$ induced by the family $\{(\mathfrak{u}^p_i)_{\mathfrak{a}^p_i}\}_{i\in J}$, where $\mathfrak{u}^p_i : \prod_{i\in J} X_i \to X_i$ and $\mathfrak{a}^p_i : \prod_{i\in J} P_i \to P_i$ are projection mappings, is called the product N-soft topology on X.*

Theorem 3.21. *Let $\{(X_i, P_i, N, \tau_i)\}_{i\in J}$ be a family of N-soft topological spaces and τ^p be the product N-soft topology on $X = \prod_{i\in J} X_i$. Consider an N-soft topological space (Y, K, N, τ) and take an N-soft mapping $\mathfrak{u}_{\mathfrak{a}} : (Y, K, N, \tau) \to (X, P, N, \tau^p)$. Then, $\mathfrak{u}_{\mathfrak{a}}$ is N-soft continuous if and only if $(\mathfrak{u}^p_i)_{\mathfrak{a}^p_i} \circ \mathfrak{u}_{\mathfrak{a}} : (Y, K, N, \tau) \to (X_i, P_i, N, \tau_i)$ is N-soft continuous for all $i \in J$.*

Proof. It can be easily checked from Remark 3.19 and the definition of product N-soft topology. □

Theorem 3.22. *Let $\{(X_i, P_i, N, \tau_i)\}_{i\in J}$, $\{(Y_i, K_i, N, \tau'_i)\}_{i\in J}$ be two families of N-soft topological spaces and (X, P, N, τ^p_1), (Y, K, N, τ^p_2) be their product N-soft topological spaces, respectively. Let $(\mathfrak{u}_i)_{\mathfrak{a}_i} : (X_i, P_i, N, \tau_i) \to (Y_i, K_i, N, \tau'_i)$ be an N-soft mapping for all $i \in J$. Then, the product N-soft mapping $(\mathfrak{u}^p)_{\mathfrak{a}^p} = \prod_{i\in J}(\mathfrak{u}_i)_{\mathfrak{a}_i} : (X, P, N, \tau^p_1) \to (Y, K, N, \tau^p_2)$ is N-soft continuous if $(\mathfrak{u}_i)_{\mathfrak{a}_i}$ is N-soft continuous for all $i \in J$.*

Proof. The product N-soft mapping is defined as follows: for each $x \in X = \prod_{i\in J} X_i$, $(\mathfrak{u}^p(x))_i = \mathfrak{u}_i(\mathfrak{u}^p_i(x))$. That is, for each $x \in X$, $\mathfrak{u}^p(x)$ is the point in $Y = \prod_{i\in J} Y_i$ whose ith coordinate is $\mathfrak{u}_i(\mathfrak{u}^p_i(x))$ for each $i \in J$. Also, for each $s \in P = \prod_{i\in J} P_i$, $(\mathfrak{a}^p(s))_i = \mathfrak{a}_i(\mathfrak{a}^p_i(s))$. That is, for each $s \in P$, $\mathfrak{a}^p(s)$ is the point in $K = \prod_{i\in J} K_i$ whose ith coordinate is $\mathfrak{a}_i(\mathfrak{a}^p_i(s))$ for each $i \in J$. Thus, from Proposition 3.9 and Theorem 3.21 it follows that the product N-soft mapping is N-soft continuous. □

4 New N-soft topological subspace

In N-soft topological subspaces proposed by Riaz et al. [29], we can not obtain some basic results that hold for closed sets in ordinary subspaces. So, we bring out a new notion of N-soft topological subspace and study some fundamental properties of it with the help of examples.

Definition 4.1 ([29]). *Let (X, P, N, τ) be an N-soft topological space and $(F, P, N) \in \mathcal{S}^N(X, P)$. Then, the pair $((F, P, N), \tau_F)$ is called an N-soft topological subspace of (X, P, N, τ), where $\tau_F = \{(F, P, N) \sqcap (G, P, N) : (G, P, N) \in \tau\}$.*

Notice that τ_F is closed under finite intersections and arbitrary unions with $(X_0, P, N), (F, P, N) \in \tau_F$. So, the family τ_F is an N-soft topology and is called the subspace N-soft topology.

Definition 4.2. *Every member of an N-soft topological subspace* $((F,P,N),\tau_F)$ *is called an N-soft-open in* (F,P,N)*. If* $(T,P,N) \sqsubseteq (F,P,N)$ *and* $(F,P,N)\ominus(T,P,N) \in \tau_F$*, then* (T,P,N) *is called an N-soft closed set in* (F,P,N)*.*

Given an ordinary topological space (X,τ) and a crisp subset A of X, we have that a set in subspace topology τ_A is closed in A iff it equals the intersection of A with an closed set in X. However, this is not true, in general, in considering a similar problem for N-soft topological spaces as shown in the following example.

Example 4.3. *Let* $X = \{x_1, x_2\}$ *and* $P = \{p_1, p_2\}$*. Take* $(F_1, P, 5), (F_2, P, 5) \in \mathcal{S}^5(X,P)$*, where*

$$(F_1, P, 5) = \Big\{\langle p_1, \{(x_1, 1), (x_2, 3)\}\rangle, \langle p_2, \{(x_1, 3), (x_2, 2)\}\rangle\Big\},$$
$$(F_2, P, 5) = \Big\{\langle p_1, \{(x_1, 1), (x_2, 2)\}\rangle, \langle p_2, \{(x_1, 2), (x_2, 1)\}\rangle\Big\}.$$

Then, $\tau = \{(X_4, P, 5), (X_0, P, 5), (F_1, P, 5), (F_2, P, 5)\}$ *is a* 5*-soft topology on X. Consider a* 5*-soft subset* $(F, P, 5)$ *of* $(X_4, P, 5)$ *by*

$$(F, P, 5) = \Big\{\langle p_1, \{(x_1, 2), (x_2, 2)\}\rangle, \langle p_2, \{(x_1, 3), (x_2, 1)\}\rangle\Big\}.$$

Therefore, we obtain a 5*-soft topological subspace* $((F, P, 5), \tau_F)$*, where*

$$\tau_F = \Big\{(F, P, 5), (X_0, P, 5), \{\langle p_1, \{(x_1, 1), (x_2, 2)\}\rangle, \langle p_2, \{(x_1, 3), (x_2, 1)\}\rangle\}, (F_2, P, 5)\Big\}.$$

However, there is no 5*-soft closed set in X whose intersection with* $(F, P, 5)$ *is equal to* $(T, P, 5) = \Big\{\langle p_1, \{(x_1, 1), (x_2, 0)\}\rangle, \langle p_2, \{(x_1, 0), (x_2, 0)\}\rangle\Big\}$*, whereas* $(T, P, 5)$ *is a* 5*-soft closed set in* $(F, P, 5)$*.*

This prompts us to suggest a strengthening of the notion of N-soft topological subspaces, which remedies this defect.

Definition 4.4. *Let* (X, P, N, τ) *be an N-soft topological space and* $(F, P, N) \in \mathcal{S}^N(X, P)$*. The pair* $((F, P, N), \tau_F)$ *is called a strong N-soft topological subspace of* (X, P, N, τ) *if the family* $\tau_F = \{(F, P, N) \sqcap (G, P, N) : (G, P, N) \in \tau\}$ *satisfies the following properties:*

(s_1) *For every* $(H, P, N) \in \tau_F$*, there exists a* $(T_H, P, N) \in \mathcal{K}$ *such that* $(F, P, N) \ominus (H, P, N) = (F, P, N) \sqcap (T_H, P, N)$*, where* $\mathcal{K}$ *is a family of N-soft closed sets in X.*

(s_2) *For every* $(W, P, N) \in \mathcal{K}$*, there exists a* $(G_W, P, N) \in \tau$ *such that* $(F, P, N) \ominus ((F, P, N) \sqcap (W, P, N)) = (F, P, N) \sqcap (G_W, P, N)$*.*

Example 4.5. *Let* $(X, P, 5, \tau)$ *be a* 5*-soft topological space defined in Example 4.3. Define a* 5*-soft subset* $(F, P, 5)$ *of* $(X_4, P, 5)$ *as follows:*

$$(F, P, 5) = \Big\{\langle p_1, \{(x_1, 4), (x_2, 0)\}\rangle, \langle p_2, \{(x_1, 0), (x_2, 4)\}\rangle\Big\}.$$

Therefore, we obtain a strong 5*-soft topological subspace* $((F, P, 5), \tau_F)$ *of* $(X, P, 5, \tau)$ *such that*

$$\tau_F = \Big\{(F, P, 5), (X_0, P, 5), \{\langle p_1, \{(x_1, 1), (x_2, 0)\}\rangle, \langle p_2, \{(x_1, 0), (x_2, 2)\}\rangle\},$$
$$\{\langle p_1, \{(x_1, 1), (x_2, 0)\}\rangle, \langle p_2, \{(x_1, 0), (x_2, 1)\}\rangle\}\Big\}.$$

Example 4.6. *Let X be the universal set, P be the set of parameters and $N > 1$. Define an N-soft set (F^{α}, P, N) as follows:*

$$\ell_{px}^{F^{\alpha}} = \alpha$$

for all $p \in P$ and all $x \in X$ with $\alpha \in \{0, 1, ..., N-1\}$. Then,

$$\tau = \{(F^{\alpha}, P, N) : \alpha \in \{0, 1, ..., N-1\}\}$$

is an N-soft topology on X. Now, let us consider an N-soft set (G, P, N) such that

$$\ell_{px}^{G} = \begin{cases} 0, & \text{if } x \notin A; \\ t, & \text{if } x \in A, \end{cases}$$

for all $p \in P$ and all $x \in X$ with $t \in \{1, ..., N-1\}$, where A is a non-empty crisp subset of X. Therefore, $((G, P, N), \tau_G)$ is a strong N-soft topological subspace of (X, P, N, τ). Indeed, let $(H, P, N) \in \tau_G$. Then, for all $p \in P$ and all $x \in X$ with $h \in \{0, 1, ..., t\}$, we have

$$\ell_{px}^{H} = \begin{cases} 0, & \text{if } x \notin A; \\ h, & \text{if } x \in A. \end{cases}$$

Therefore, due to $(G, P, N) \ominus (H, P, N) = (F^{t-h}, P, N) \sqcap (G, P, N)$ and from the fact that $(F^{t-h}, P, N)^r = (F^{N-1-(t-h)}, P, N) \in \tau$ it follows that the property (s_1) is satisfied.
Now, let $(W, P, N) \in \mathcal{K}$. Then, we also get $(W, P, N) \in \tau$ and so that there exists an $\alpha \in \{0, 1, ..., N-1\}$ satisfying $(W, P, N) = (F^{\alpha}, P, N)$. Therefore, the N-soft set $((G, P, N) \ominus ((W, P, N) \sqcap (G, P, N)))$ is given by

$$\ell_{px}^{G \ominus (W \sqcap G)} = \begin{cases} 0, & \text{if } x \notin A; \\ t - min\{\alpha, t\}, & \text{if } x \in A, \end{cases}$$

for all $p \in P$ and all $x \in X$. If $min\{\alpha, t\} = t$, then there exists an $(F^0, P, N) \in \tau$ such that $((G, P, N) \ominus ((W, P, N) \sqcap (G, P, N))) = (G, P, N) \sqcap (F^0, P, N)$. If $min\{\alpha, t\} = \alpha$, then there is an $(F^{t-\alpha}, P, N) \in \tau$ such that $((G, P, N) \ominus ((W, P, N) \sqcap (G, P, N))) = (G, P, N) \sqcap (F^{t-\alpha}, P, N)$. Thus, the property (s_2) is satisfied.

Theorem 4.7. *Let $((F, P, N), \tau_F)$ be a strong N-soft topological subspace of (X, P, N, τ). The N-soft set $(T, P, N) \sqsubseteq (F, P, N)$ is N-soft closed in (F, P, N) if and only if there exists an N-soft closed set (W, P, N) in X such that $(T, P, N) = (F, P, N) \sqcap (W, P, N)$.*

Proof. Necessity and sufficiency conditions are easily demonstrated using the properties (s_1) and (s_2), respectively . □

Lemma 4.8. *Let $(F, P, N), (G, P, N), (H, P, N) \in \mathcal{S}^N(X, P)$. Then, we have*

$$\begin{aligned} &((F, P, N) \sqcap (G, P, N)) \ominus ((H, P, N) \sqcap ((F, P, N) \sqcap (G, P, N))) \\ &= ((F, P, N) \ominus ((H, P, N) \sqcap (F, P, N))) \sqcap ((G, P, N) \ominus ((H, P, N) \sqcap (G, P, N))). \end{aligned}$$

Proof. Let $p \in P$ and $x \in X$. Take $\ell_{px}^G \leq \ell_{px}^F$. Then, we consider the following three cases:

Case 1. Let $\ell_{px}^G \leq \ell_{px}^H \leq \ell_{px}^F$. Therefore, the left hand of the above equality is $\ell_{px}^G - \ell_{px}^G = 0$, and the right hand is $min\{(\ell_{px}^F - \ell_{px}^H), (\ell_{px}^G - \ell_{px}^G)\} = 0$.

Case 2. Let $\ell_{px}^H \leq \ell_{px}^G \leq \ell_{px}^F$. Then, the left hand of the above equality is $\ell_{px}^G - \ell_{px}^H$, and the right hand is $min\{(\ell_{px}^F - \ell_{px}^H), (\ell_{px}^G - \ell_{px}^H)\} = \ell_{px}^G - \ell_{px}^H$.

Case 3. Let $\ell^G_{px} \leq \ell^F_{px} \leq \ell^H_{px}$. Therefore, the left hand of the above equality is $\ell^G_{px} - \ell^G_{px} = 0$, and the right hand is $min\{(\ell^F_{px} - \ell^F_{px}), (\ell^G_{px} - \ell^G_{px})\} = 0$.
If $\ell^F_{px} \leq \ell^G_{px}$, then by repeating the same argument, we conclude the proof. □

Proposition 4.9. *Let $((F_1, P, N), \tau_{F_1})$ and $((F_2, P, N), \tau_{F_2})$ be two strong N-soft topological subspaces of (X, P, N, τ). Then, $((F_1 \sqcap F_2, P, N), \tau_{F_1 \sqcap F_2})$ is a strong N-soft topological subspace of (X, P, N, τ).*

Proof. We will show that the family

$$\tau_{F_1 \sqcap F_2} = \{((F_1 \sqcap F_2), P, N) \sqcap (G, P, N) : (G, P, N) \in \tau\}$$

has the properties (s_1) and (s_2).

(s_1) Let $(H, P, N) \in \tau_{F_1 \sqcap F_2}$. Then, there exists a $(G, P, N) \in \tau$ such that $(H, P, N) = ((F_1 \sqcap F_2), P, N) \sqcap (G, P, N)$. From Theorem 4.7 and Lemma 4.8 it follows that

$$\begin{aligned}
&((F_1 \sqcap F_2), P, N) \ominus (((F_1 \sqcap F_2), P, N) \sqcap (G, P, N)) \\
&= ((F_1, P, N) \ominus ((F_1, P, N) \sqcap (G, P, N))) \sqcap ((F_2, P, N) \ominus ((F_2, P, N) \sqcap (G, P, N))) \\
&= ((F_1, P, N) \sqcap (W_1, P, N)) \sqcap ((F_2, P, N) \sqcap (W_2, P, N)) \\
&= ((F_1, P, N) \sqcap (F_2, P, N)) \sqcap ((W_1, P, N) \sqcap (W_2, P, N)) \\
&= ((F_1 \sqcap F_2), P, N) \sqcap ((W_1 \sqcap W_2), P, N),
\end{aligned}$$

where (W_1, P, N) and (W_2, P, N) are N-soft closed sets in X. By the condition (Nsc_3), since $((W_1 \sqcap W_2), P, N)$ is an N-soft closed set in X, the property (s_1) holds.

(s_2) Let $(W, P, N) \in \mathcal{K}$. Using Theorem 4.7 and Lemma 4.8, we see that

$$\begin{aligned}
&((F_1 \sqcap F_2), P, N) \ominus (((F_1 \sqcap F_2), P, N) \sqcap (W, P, N)) \\
&= ((F_1, P, N) \ominus ((F_1, P, N) \sqcap (W, P, N))) \sqcap ((F_2, P, N) \ominus ((F_2, P, N) \sqcap (W, P, N))) \\
&= ((F_1, P, N) \sqcap (G_1, P, N)) \sqcap ((F_2, P, N) \sqcap (G_2, P, N)) \\
&= ((F_1, P, N) \sqcap (F_2, P, N)) \sqcap ((G_1, P, N) \sqcap (G_2, P, N)) \\
&= ((F_1 \sqcap F_2), P, N) \sqcap ((G_1 \sqcap G_2), P, N),
\end{aligned}$$

where (G_1, P, N) and (G_2, P, N) are N-soft open sets in X. From (Nst_3), we get $((G_1 \sqcap G_2), P, N) \in \tau$, completing the proof. □

The next example turns out that if $((F_1, P, N), \tau_{F_1})$ and $((F_2, P, N), \tau_{F_2})$ are two strong N-soft topological subspaces of (X, P, N, τ), then $((F_1 \sqcup F_2, P, N), \tau_{F_1 \sqcup F_2})$ is not a strong N-soft topological subspace of (X, P, N, τ), in general.

Example 4.10. *Let $X = \mathbb{Z}$, i.e. the set of integers, and P be any set of parameters. From Example 4.6 it follows that $(X, P, 6, \tau)$ is a 6-soft topological space, where $\tau = \{(F^\alpha, P, 6) : \alpha \in \{0, 1, 2, ..., 5\}\}$. Consider $(G_1, P, 6), (G_2, P, 6) \in \mathcal{S}^6(X, P)$ with the rule*

$$\ell^{G_1}_{px} = \begin{cases} 0, & if\ x \in \mathbb{Z}^-; \\ 3, & if\ x \in \mathbb{Z}^+ \cup \{0\}, \end{cases} \qquad \ell^{G_2}_{px} = \begin{cases} 4, & if\ x \in \mathbb{Z}^-; \\ 0, & if\ x \in \mathbb{Z}^+ \cup \{0\}, \end{cases}$$

for all $p \in P$ and all $x \in X$. Then, by Example 4.6, we know that $((G_1, P, 6), \tau_{G_1})$ and $((G_2, P, 6), \tau_{G_2})$ are two strong 6-soft topological subspaces of $(X, P, 6, \tau)$. However,

$((G_1 \sqcup G_2, P, 6), \tau_{G_1 \sqcup G_2})$ *is not a strong* 6*-soft topological subspace of* $(X, P, 6, \tau)$. *Indeed, let us take a* 6*-soft closed set* $(W, P, 6)$ *in* X *such that* $\ell_{px}^{W} = 1$ *for all* $p \in P$ *and all* $x \in X$. *Therefore, we obtain*

$$\ell_{px}^{(G_1 \sqcup G_2) \ominus (W \sqcap (G_1 \sqcup G_2))} = \ell_{px}^{(G_1 \sqcup G_2) \ominus W} = \begin{cases} 3, & \text{if } x \in \mathbb{Z}^-; \\ 2, & \text{if } x \in \mathbb{Z}^+ \cup \{0\}, \end{cases}$$

for all $p \in P$ *and all* $x \in X$. *In this case, we see that the property* (s_2) *is not satisfied.*

Theorem 4.11. *Let* $((F, P, N), \tau_F)$ *be a strong* N*-soft topological subspace of* (X, P, N, τ) *and* $(T, P, N) \sqsubseteq (F, P, N)$. *Then, the* N*-soft closure of* (T, P, N) *in* (F, P, N) *equals* $\overline{(T, P, N)} \sqcap (F, P, N)$.

Proof. This follows easily from Theorem 4.7. □

5 Conclusion

Closed sets play a very important role in the general topology since these sets are essential matters on a set that carries topology. Also, the closed sets make significant contributions to the study of many topological properties such as separation axioms, compactness and connectedness. Besides, the continuous mappings are an important subject of investigation not only in general topology but also in some other branches of mathematics. In particular, many topologists are interested in which the topological properties are preserved by these mappings. So, the stronger and weaker forms of both continuity and closed sets have been brought out and analyzed by many mathematicians.

In this research article, using the definition of complement in [17], we introduce a new concept of N-soft closed set. We establish some of its fundamental results and also, we give the notion of an N-soft continuous mapping. Furthermore, we characterize the connection of this mapping with N-soft closed sets. Then, we obtain initial N-soft topology. Later, we put forward a novel notion of N-soft topological subspace because, by using Definition 4.1, we can not obtain some basic results that hold for closed sets in ordinary subspaces. Finally, we demonstrate the utility of the proposed results by the examples. Therefore, it is expected that these theoretical studies will pave the way to further investigation of novel approaches for N-soft topology as well as in many areas of application. In the future, one can investigate further topological concepts via the N-soft continuous mappings and the family of N-soft closed sets such as separation axioms, compactness, connectedness and paracompactness. Also, one can extend our work to other N-soft models like fuzzy N-soft sets, bipolar N-soft sets, hesitant N-soft sets, neutrosophic vague N-soft sets and complex fuzzy N-soft sets to study topological structures and to develop decision making methods on these structures.

Conflicts of Interest

This work does not have any conflicts of interest.

References

[1] M. Akram, A. Adeel and J.C.R. Alcantud, Fuzzy N-soft sets: A novel model with applications, J. Intell. Fuzzy Syst. 35.4 (2018):4757-4771.

[2] M. Akram, A. Adeel and J.C.R. Alcantud, Group decision-making methods based on hesitant N-soft sets, Expert Syst. Appl. 115 (2019):95-105.

[3] M. Akram, G. Ali and J.C.R. Alcantud, New decision-making hybrid model: intuitionistic fuzzy N-soft rough sets, Soft Comput. 23.20 (2019):9853-9868.

[4] M. Akram, G. Ali, J.C.R. Alcantud and F. Fatimah, Parameter reductions in N-soft sets and their applications in decision-making, Expert Syst. 38 (2021):e12601.

[5] H. Aktaş H and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007):2726-2735.

[6] S. Al Ghour, Weaker forms of soft regular and soft T_2 soft topological spaces, Mathematics 9.17 (2021):2153.

[7] M.I. Ali, F. Feng, X. Liu, W.K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009):1547-1553.

[8] T.M. Al-Shami, Compactness on soft topological ordered spaces and its application on the information system, J. Math. 2021 (2021):12 pages.

[9] T.M. Al-Shami, On soft separation axioms and their applications on decision-making problem, Math. Probl. Eng. 2021 (2021):8876978.

[10] T.M. Al-Shami, Soft somewhat open sets: soft separation axioms and medical application to nutrition, Comput. Appl. Math. 41.5 (2022):1-22.

[11] T.M. Al-Shami, A. Mhemdi, R. Abu-Gdairi and M.E. El-Shafei, Compactness and connectedness via the class of soft somewhat open sets, AIMS Math. 8.1 (2023):815-840.

[12] A. Aygünoğlu and H. Aygün, Some notes on soft topological spaces, Neural Comput. Appl. 21 (2012): 113-119.

[13] A.A. Azzam, Z.A. Ameen, T.M. Al-Shami and M.E. El-Shafei, Generating soft topologies via soft set operators, Symmetry 14.5 (2022): 914.

[14] D. Chen, E.C.C. Tsang, D.S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005):757-763.

[15] O. Dalkılıç, A novel approach to soft set theory in decision-making under uncertainty, Int. J. Comput. Math. 98.10 (2021):1935-1945.

[16] S. Das and S.K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20.3 (2012):551-576.

[17] İ. Demir, N-soft mappings with application in medical diagnosis, Math. Methods Appl. Sci. 44.8 (2021):7343-7358.

[18] N. Demirtaş and O. Dalkılıç, Decompositions of Soft α-continuity and Soft A-continuity, J. New Theory 31 (2020):86-94.

[19] M.K. El-Bably and A.A.E. Atik, Soft β-rough sets and its application to determine COVID-19, Turk. J. Math. 45 (2021):1133-1148.

[20] M.E. El-Shafei, M. Abo-Elhamayel and T.M. Al-Shami, Partial soft separation axioms and soft compact spaces, Filomat 32.13 (2018):4755-4771.

[21] F. Fatimah, D. Rosadi, R.B.F. Hakim and J.C.R. Alcantud, N-soft sets and their decision-making algoritms, Soft Comput. 22 (2018):3829-3842.

[22] A.Ç. Güler and G. Kale, Regularity and normality on soft ideal topological spaces, Ann. Fuzzy Math. Inform. 9.3 (2015):373-383.

[23] A. Kharal and B. Ahmad, Mappings on soft classes, New Math. Nat. Comput. 7.3 (2011):471-481.

[24] H. Kamacı and S. Petchimuthu, Bipolar N-soft set theory with applications, Soft Comput. 24 (2020):16727-16743.

[25] J. Liu, Y. Chen, Z. Chen and Y. Zhang, Multi-attribute decision making method based on neutrosophic vague N-soft sets, Symmetry 12.5 (2020):853.

[26] P.K. Maji, A.R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002):1077-1083.

[27] P.K. Maji, R. Biswas and A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003):555-562.

[28] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999):19-31.

[29] M. Riaz, N. Çağman, I. Zareef and M. Aslam, N-soft topology and its applications to multi-criteria group decision making, J. Intell. Fuzzy Syst. 36.6 (2019):6521-6536.

[30] M. Riaz, K. Naeem, I. Zareef and D. Afzal, Neutrosophic N-soft sets with TOPSIS method for multiple attribute decision making, Neutrosophic Sets Syst. 32 (2020):146-170.

[31] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011):1786-1799.

[32] E. Turanlı, İ. Demir and O.B. Özbakır, Some types of soft paracompactness via soft ideals, Int. J. Nonlinear Anal. Appl. 10.2 (2019):197-211.

[33] J. Yang and Y. Yao, Semantics of soft sets and three-way decision with soft sets, Knowl. Based Syst. 194 (2020):105538.