Upper and lower powerspaces of directed spaces☆

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Abstract

In domain theory, powerdomains play an important role in modeling the semantics of nondeterministic functional programming languages. In this paper, we extend the notion of powerdomains to the cartesian closed category of directed spaces. We define the notion of upper and lower powerspace of a directed space by the way of free algebras. We show that the upper and lower powerspaces of any directed space exist and give their concrete structures. Moreover, the upper and lower functors preserve the continuity of directed spaces. The upper and lower powerspaces of c-spaces (resp., FS-spaces) are c-spaces (resp., FS-spaces).

Keywords: powerdomain, observationally-induced lower powerspace, directed space, c-space

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1. Introduction

The notion of powerdomains is one of the most important parts of domain theory. Powerdomains are used to provide mathematical models for the semantics of nondeterministic functional programming languages. There are three classical power structures in domain theory: the lower powerdomain, the upper powerdomain, and the convex powerdomain. In recent years, a lot of generalizations were made on these power structures [8, 19, 20, 21]. Generally speaking, these power structures are free algebras generated by domains with respect to some binary operations. In 2015, I. Battenfeld and M. Schöder [2] introduced a kind of power structures on general topological spaces, where the Boolean algebra **2** is endowed with the Sierpinski topology as an observable structure. Then they defined the upper power structure (observationally-induced upper powerspace) and lower power structure (observationally-induced lower powerspace).

In an invited presentation at the 6th International Symposium in Domain Theory, Lawson gave evidence from recent development in domain theory to highlight the intimate relationship between domains and T_0 spaces and called to develop the domain theory in T_0 spaces. Directed spaces were introduced by Yu and Kou [18], which are equivalent to T_0 monotone determined spaces defined by Erné in [6]. Directed spaces with continuous maps form a cartesian closed category, denoted by **Dtop** [18], and also contain many objects in domain theory such as dcpos endowed with the Scott topology, posets endowed with the Alexandroff topology, c-spaces and locally hypercompact spaces. In particular, the exponential objects and categorical products in the category of dcpos coincide with those in **Dtop** by viewing dcpos as topological spaces endowed with the Scott topology.

A c-space can be viewed as a continuous directed space just like domains as continuous dcpos [3, 16]. Recently, the notion of FS-domains, which form one of the most important cartesian closed categories in domain theory, is extended to FS-spaces [16]. FS-spaces form a cartesian closed subcategory of directed spaces and are contained in c-spaces. Besides, many other classical results about dcpos can be extended

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to directed spaces as well. For example, a directed space X is core compact iff for any directed space Y, the topological product of X and Y is equal to the categorical product in **DTop** of X and Y (see [3]). Based on these facts, we believe that directed spaces are very suitable topological extensions of dcpos. It is meaningful to investigate the power structures on directed spaces.

Like the category of dcpos and the category of topological spaces, the existence of the free algebras over a directed space in the context of **Dtop** can be proved by the Adjoint Functor Theorem (see [1, 4]). We will define the concept of upper and lower powerspaces of directed spaces through free algebras, and give the concrete topological representations of the directed upper powerspace and the directed lower powerspace over an arbitrary directed space. Moreover, we show that the upper and lower powerspaces of c-spaces (resp., FS-spaces) are c-spaces (resp., FS-spaces) as well.

2. Preliminaries

We assume some basic knowledges in domain theory, topology, and category theory as in [1, 9, 10, 14]. Let P be a poset. Given any $A \subseteq P$, we denote $\downarrow A = \{x \in P : \exists a \in A, \ \ x \leq a\}$, $\uparrow A = \{x \in P : \exists a \in A, \ a \leq x\}$. We say that A is a lower set (upper set) if $A = \downarrow A$ ($A = \uparrow A$). We denote by $\sigma(P)$ the Scott topology on P. Let $P,\ E$ be two posets, a map $f : P \longrightarrow E$ is Scott continuous iff it is continuous with respect to Scott topology $\sigma(P)$ and $\sigma(E)$. The upper topology on P is denoted by $\upsilon(P)$.

All topological spaces in this paper are supposed to be T_0. A net of a topological space X is a map $\xi : J \longrightarrow X$ and is denoted by $(x_j)_{j\in J}$ or (x_j), where J is a directed set. Given $x \in X$, we say that (x_j) converges to x, denote by $(x_j) \to x$ or $x \equiv \lim x_j$, if (x_j) is eventually in every open neighborhood of x, that is, for any open neighborhood U of x, there exists a $j_0 \in J$ such that for every $j \in J$, $j \geq j_0 \Rightarrow x_j \in U$.

Let X be a T_0 topological space. Its topology is denoted by $\mathcal{O}(X)$, the specialization order $\sqsubseteq$ on X is defined by $x \sqsubseteq y \Leftrightarrow x \in \overline{\{y\}}$, where $\overline{\{y\}}$ means the closure of $\{y\}$. From now on, the order of a T_0 topological space always indicates the specialization order $\sqsubseteq$.

For any topological space X, a directed subset $D \subseteq X$ can be regarded as a monotone net $(d)_{d\in D}$. We use $D \to x$ or $x \equiv \lim D$ to denote that D converges to x. Define

$$D(X) = \{(D, x) : x \in X,\ D \text{ is a directed subset of } X \text{ and } D \to x\}.$$

It is easy to verify that, for each $x, y \in X$, $x \sqsubseteq y \Leftrightarrow \{y\} \to x$. Therefore, if $x \sqsubseteq y$ then $(\{y\}, x) \in D(X)$. Next, we introduce the notion of a directed space, which is equivalent to the notion of a T_0 monotone determined space [6]. A subset U of X is called directed open if $\forall (D, x) \in D(X),\ x \in U \Rightarrow D \cap U \neq \emptyset$. All directed open subsets from a topology, called the directed topology of X, denoted by $d(X)$. Obviously, all open subsets of X are directed open, and all directed open sets are upper sets.

Definition 2.1. [18] X is called a directed space if each directed open subset of X is an open subset, i.e., $d(X) = \mathcal{O}(X)$.

The following are some basic properties of directed spaces.

Proposition 2.2. [18] Let X be a T_0 topological space. Then

(1) X equipped with $d(X)$ is a T_0 topological space such that $\sqsubseteq_d = \sqsubseteq$, where $\sqsubseteq_d$ is the specialization order relative to $d(X)$.
(2) For a directed subset D of X, $D \to x$ iff $D \to_d x$ for all $x \in X$, where $D \to_d x$ means that D converges to x with respect to the topology $d(X)$.
(3) $(X, d(X))$ is a directed space.

Directed spaces contain many important structures in domains theory such as dcpo endowed with the Scott topology, c-spaces, and posets endowed with the Alexandroff topology (we refer to [3, 16, 18] for directed spaces). In general, a dcpo with the upper topology is not a directed space.

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Example 2.3. Let $\mathbb{N}$ be the set of natural numbers. Denote by $\mathbb{N}^\top$ the flat domain, i.e., the poset with carrier set $\mathbb{N} \cup \{\top\}$ and $x \leq y$ iff $y = \top$ or $x = y$. It is easily seen that $\mathbb{N}^\top$ is a dcpo. Given any upper set U of $\mathbb{N}^\top$, since any directed subset D of $\mathbb{N}^\top$ must have a largest element, $D \to x$ for some $x \in U$ iff $D \cap U \neq \emptyset$. Thus, U is a directed open subset of $(\mathbb{N}^\top, \upsilon(\mathbb{N}^\top))$. Let $U = \{2n : n \in \mathbb{N}\} \cap \{\top\}$. Then, U is an directed open set, but not an open subset of $\upsilon(\mathbb{N}^\top))$. Thus, $(\mathbb{N}^\top, \upsilon(\mathbb{N}^\top))$ is not a directed space. The directed topology of $(\mathbb{N}^\top, \upsilon(\mathbb{N}^\top))$ is just the Scott topology.

Definition 2.4. Let X, Y be two T_0 spaces. A map $f : X \longrightarrow Y$ is called directed continuous if it is monotone and preserves all limits of directed set of X, i.e., $(D, x) \in D(X)$ implies $(f(D), f(x)) \in D(Y)$.

Proposition 2.5. [18] Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ be a map between X and Y.

(1) f is directed continuous if and only if $\forall U \in d(Y),\ f^{-1}(U) \in d(X)$.
(2) If X, Y are directed spaces, then f is continuous if and only if it is directed continuous.

We now introduce the categorical product in **Dtop**, the category of all directed spaces together with continuous maps as morphisms.

Suppose that X, Y are two directed spaces. Let $X \times Y$ denote the cartesian product of X and Y, then we have a natural partial order on it, called the pointwise order: $\forall (x_1,\ y_1),\ (x_2,\ y_2) \in X \times Y$,

$$(x_1,\ y_1) \leq (x_2,\ y_2) \iff x_1 \sqsubseteq x_2,\ y_1 \sqsubseteq y_2.$$

Now, we define a topological space $X \otimes Y$ as follows:

1. The underlying set of $X \otimes Y$ is $X \times Y$;
2. The topology on $X \times Y$ is generated as follows: for each given $\leq$-directed set $D \subseteq X \times Y$ and $(x, y) \in X \times Y$,
$$D \to (x, y) \in X \otimes Y \iff \pi_1 D \to x \in X, \pi_2 D \to y \in Y.$$
That is, a subset $U \subseteq X \times Y$ is open if and only if for every directed subset $D \to (x, y)$, $(x, y) \in U \Rightarrow U \cap D \neq \emptyset$.

Theorem 2.6. [18] *Let X, Y, Z be directed spaces.*

(1) *The topological space $X \otimes Y$ defined as above is a directed space and satisfies the following properties: the specialization order of $X \otimes Y$ is equal to the pointwise order on $X \times Y$, i.e., $\sqsubseteq = \leq$.*
(2) *A map $f : X \otimes Y \longrightarrow Z$ is continuous if and only if it is continuous in each variable separately.*
(3) *$X \otimes Y$ is the categorical product of X and Y in* **Dtop**

Dtop is cartesian closed and contains all dcpos endowed with the Scott topology [18]. Directed spaces are a very natural topological extended framework for dcpos in domain theory (see [3, 16, 18]).

3. The directed upper powerspaces of directed spaces

In the category of dcpos and the category of topological spaces, the existence of the free algebras can be shown by the Adjoint Functor Theorem (see [1]). Recently, the free algebras over a directed space in the context of **Dtop** were defined in a similar way as the free algebras over dpcos, and the existence of the free algebras over any directed space was proved by the Adjoint Functor Theorem (see [4]). However, the proof is not constructive. It is meaningful to construct the representation of the free algebra over a directed space with respect to a concrete signature Σ and a set of inequalities $\mathcal{E}$.

In classical domain theory, the concrete representation of the upper powerdomain over a dcpo is still not known. In this section, we will give the concrete representation of the directed upper powerspace of a directed space, which is a free algebra generated by the same signature and inequalities of the upper powerdomains.

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Definition 3.1. Let X be a directed space.

(1) A binary operation $\oplus : X \otimes X \to X$ on X is called a deflationary operation if it is continuous and satisfies the following four conditions: $\forall x, y, z \in X$,
  (a) $x \oplus x = x$,
  (b) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
  (c) $x \oplus y = y \oplus x$,
  (d) $x \oplus y \leq x$.

(2) If $\oplus$ is a deflationary operation on X, then $(X, \oplus)$ is called a directed deflationary semilattice, that is, directed deflationary semilattices are those directed spaces with deflationary operations.

By Theorem 2.6(2), the operation $\oplus$ on a directed space X is continuous if and only if it is monotone and for any $x, y \in X$ and directed set $D \subseteq X$, $D \to x$ implies $(D \oplus y) \to x \oplus y$. Here, $D \oplus y = \{d \oplus y : d \in D\}$.

Here are two simple examples of directed deflationary semilattices.

Example 3.2. (1) Suppose that P is a poset endowed with the Scott topology, and for each $a, b \in P$, the infimum of a and b exists in P (denote by $a \wedge b$). Then $(P, \wedge)$ is a directed deflationary semilattice.

(2) Let $I = [0, 1]$ (the unit interval) and $\mathcal{T}$ be the topology generated by $\{[0, a] : a \in I\}$. It is easy to check that $(I, \mathcal{T})$ is a directed space, and (I, min) is a directed deflationary semilattice endowed with topology $\mathcal{T}$.

Definition 3.3. Suppose that $(X, \oplus)$ and $(Y, \uplus)$ are two directed deflationary semilattices. A map $f : (X, \oplus) \to (Y, \uplus)$ is called a deflationary homomorphism between X and Y, if f is continuous and $f(x \oplus y) = f(x) \uplus f(y)$ holds for all $x, y \in X$.

Denote the category of all directed deflationary semilattices with deflationary homomorphisms by **Ddsl**. Then **Ddsl** is a subcategory of **Dtop**.

Lemma 3.4. *Suppose that $(X, \oplus)$ is a directed deflationary semilattice. Then we have $\oplus = \wedge_{\sqsubseteq}$, where $x \wedge_{\sqsubseteq} y$ means the infimum of x and y with respect to the specialization order $\sqsubseteq$ on X (called the meet operation). Conversely, suppose that X is a directed space and for any $x, y \in X$, $x \wedge_{\sqsubseteq} y$ exists. Then the continuity of $\wedge_{\sqsubseteq}$ naturally implies that $(X, \wedge_{\sqsubseteq})$ is a directed deflationary semilattice.*

Proof. By Definition 3.1, for all $x, y \in X$, $x \oplus y \leq x, y$. Suppose that z is an another lower bound of $\{x, y\}$. By Theorem 2.6(1), the pointwise order equals to the specialization order of $X \otimes X$. Thus $(x, y) \sqsubseteq (z, z)$. By the continuity and idempotence of the deflationary operation, we have $z \oplus z = z \sqsubseteq x \oplus y$. Therefore, $x \oplus y$ is the infimum of $\{x, y\}$, i.e., $x \oplus y = x \wedge_{\sqsubseteq} y$. Conversely, a continuous meet operation naturally satisfies all conditions in Definition 3.1. □

The above result shows that a directed deflationary semilattice $(X, \oplus)$ is just a directed space with a continuous meet operation $\wedge_{\sqsubseteq}$ satisfying $\oplus = \wedge_{\sqsubseteq}$. We omit the subscript $\sqsubseteq$ if there is no confusion. Therefore, a directed deflationary semilattice can be represented by $(X, \wedge)$, where X is a directed space, and $\wedge$ represents the continuous meet operation on X.

Next, we give the definition of a directed upper powerspace.

Definition 3.5. Let X be a directed space. A directed space Z is called the directed upper powerspace over X if and only if the following two conditions are satisfied:

(1) Z is a directed deflationary semilattice, i.e., the meet operation $\wedge$ on Z exists and is continuous;

(2) There is a continuous map $i : X \longrightarrow Z$ satisfying: for any directed deflationary semilattice $(Y, \wedge)$ and any continuous map $f : X \longrightarrow Y$, there exists a unique deflationary homomorphism $\bar{f} : (Z, \wedge) \to (Y, \wedge)$ such that $f = \bar{f} \circ i$.

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If directed deflationary semilattices $(Z_1, \wedge)$ and $(Z_2, \wedge)$ are both directed upper powerspaces of X, then there exists a topological homomorphism which is also a deflationary homomorphism $g : Z_1 \to Z_2$. Therefore, up to isomorphism, the directed upper powerspace of a directed space is unique. Particularly, we denote the directed upper powerspace of each directed space X by $P_U(X)$.

Next, we will give the concrete construction of the directed upper powerspace of a directed space X.

Let X be a directed space. Denote

$$UX = \{\uparrow F : F \subseteq_{fin} X\},$$

where $F \subseteq_{fin} X$ is an arbitrary nonempty finite subset of X. Define an order $\leq_U$ on UX as follows:

$$\uparrow F_1 \leq_U \uparrow F_2 \iff \uparrow F_2 \subseteq \uparrow F_1.$$

Let $\mathcal{F} \subseteq UX$ be a directed set (with respect to order $\leq_U$) and $\uparrow F \in UX$. Define $\Rightarrow_U$ convergence as follows: $\mathcal{F} \Rightarrow_U \uparrow F$ iff there exist finite directed sets $D_1, \ldots, D_n \subseteq X$ such that the following three conditions (called the $\Rightarrow_U$ convergence conditions) are satisfied:

(1) $F \cap \lim D_i \neq \emptyset, \ \forall 1 \leq i \leq n$;

(2) $F \subseteq \bigcup_{i=1}^{n} \lim D_i$;

(3) $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i$, there exists some $\uparrow F' \in \mathcal{F}$, such that $\uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i$.

A subset $\mathcal{U} \subseteq UX$ is called a $\Rightarrow_U$ convergence open set of UX if and only if for each directed subset $\mathcal{F}$ of UX and $\uparrow F \in UX$, $\mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U}$ implies $\mathcal{F} \cap \mathcal{U} \neq \emptyset$. Denote $O_{\Rightarrow_U}(UX)$ the set of all $\Rightarrow_U$ convergence open set of UX.

Proposition 3.6. *Let X be a directed space. Then the following statements hold*

(1) $(UX, O_{\Rightarrow_U}(UX))$ is a topological space, abbreviated as UX.
(2) The specialization order $\sqsubseteq$ of $(UX, \ O_{\Rightarrow_U}(UX))$ is equal to $\leq_U$.
(3) $(UX, O_{\Rightarrow_U}(UX))$ is a directed space.

Proof. (1) Obviously we have $\emptyset, \ UX \in O_{\Rightarrow_U}(UX)$. Suppose that $\mathcal{U} \in O_{\Rightarrow_U}(UX)$, $\uparrow F_1 \leq \uparrow F_2$, $\uparrow F_1 \in \mathcal{U}$, and $F_1 = \{a_1, \ldots, a_n\}$. Then $\{\uparrow F_2\} \Rightarrow_U \uparrow F_1$. We only need to take $D_i = \{a_i\}$, $i = 1, \ldots, n$. Then, $\{\uparrow F_2\} \cap \mathcal{U} \neq \emptyset$, which means $\uparrow F_2 \in \mathcal{U}$, and $\mathcal{U}$ is an upper set respect to the order $\leq_U$.

Let $\mathcal{U}_1, \mathcal{U}_2 \in O_{\Rightarrow_U}(UX)$, and $\mathcal{F} \subseteq UX$ be a directed set with $\mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U}_1 \cap \mathcal{U}_2$. Then, there exist some $\uparrow F_1 \in \mathcal{F} \cap \mathcal{U}_1$ and $\uparrow F_2 \in \mathcal{F} \cap \mathcal{U}_2$. Since $\mathcal{F}$ is directed, there exists some $\uparrow F_3 \in \mathcal{F}$ such that $\uparrow F_3 \subset \uparrow F_1, \uparrow F_2$. Then, $\uparrow F_3 \in \mathcal{F} \cap \mathcal{U}_1 \cap \mathcal{U}_2$. By the same way, we can prove that $O_{\Rightarrow_U}(UX)$ is closed under arbitrary unions. It follows that $O_{\Rightarrow_U}(UX)$ is a topology.

(2) Let $\uparrow F_1, \ \uparrow F_2 \in UX$. By (1), we know each $\Rightarrow_U$ convergence open set is an upper set with respect to $\leq_U$. If $\uparrow F_1 \leq_U \uparrow F_2$, then $\uparrow F_1 \in \overline{\{\uparrow F_2\}}$, that is, $\uparrow F_1 \sqsubseteq \uparrow F_2$.

For the converse, suppose that $\uparrow F_1 \sqsubseteq \uparrow F_2$. We need only to prove that $\{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\}$ is a closed subset of UX with respect to the topology $O_{\Rightarrow_U}(UX)$ and contains $\uparrow F_1$. Since $\{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\} = \{\uparrow F : \uparrow F \leq_U \uparrow F_2\} \subseteq \{\uparrow F : \uparrow F \sqsubseteq \uparrow F_2\} = \overline{\{\uparrow F_2\}}$ and $\{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\}$ is a closed set in UX with respect to $O_{\Rightarrow_U}(UX)$, then $\{\uparrow F \in UX : \uparrow F \subseteq \uparrow F_2\} = \overline{\{\uparrow F_2\}}$. Thus, $\uparrow F_1 \leq_U \uparrow F_2$.

Now, we prove that $\{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\}$ is closed with respect to $O_{\Rightarrow_U}(UX)$. Equivalently, $\mathcal{U} = UX \setminus \{\uparrow F \in UX : \uparrow F_2 \subseteq \uparrow F\}$ is a $\Rightarrow_U$ convergence open set in UX. By contradiction, suppose that $\mathcal{U}$ is not a $\Rightarrow_U$ convergence open set. Then there exists a directed set $\mathcal{F}$ of UX with $\mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{F} = \emptyset$. According to the definition of $\Rightarrow_U$ convergence, there exists finite directed sets $D_1, \ldots, \ D_n \subseteq X$ such that

1. $F \cap \lim D_i \neq \emptyset, \ i = 1, \ 2, \ldots, n$;

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2. $F \subseteq \bigcup_{i=1}^{n} \lim D_i$;

3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i$, there exists some $\uparrow F' \in \mathcal{F}$, such that $\uparrow F' \subseteq \bigcup_{i=1}^{n} \uparrow d_i$.

Since $\uparrow F \in \mathcal{U}$, then $\uparrow F_2 \nsubseteq \uparrow F$, and there exists some $a \in F_2$ with $a \notin \uparrow F$, then $F \subseteq X \backslash \downarrow a$. According to 1 and 2, for arbitrary $i \in \{1, 2, \ldots, n\}$, $D_i \cap (X \backslash \downarrow a) \neq \emptyset$. For each i, pick $d_i \in D_i \cap (X \backslash \downarrow a)$. Then $(d_1, d_2, \ldots, d_n) \in \prod_{i=1}^{n} D_i$ and $a \notin \bigcup_{i=1}^{n} \uparrow d_i$. Since $\mathcal{F} \cap \mathcal{U} = \emptyset$, then $\forall F' \in \mathcal{F}$, $\uparrow F_2 \subseteq \uparrow F'$, and $\uparrow F' \nsubseteq \bigcup_{i=1}^{n} \uparrow d_i$, which contradicts to 3. Therefore, $\mathcal{U}$ is a $\Rightarrow_U$ convergence open set in UX.

(3) For an arbitrary topological space X, $\mathcal{O}(X) \subseteq d(X)$ holds. Then $O_{\Rightarrow_U}(UX) \subseteq d(UX)$. On the other hand, according to the definition of $\Rightarrow_U$ convergence topology, if a directed set $\mathcal{F} \subseteq UX$ satisfies $\mathcal{F} \Rightarrow_U \uparrow F$, then $\mathcal{F}$ convergents to $\uparrow F$ respect to $O_{\Rightarrow_U}(UX)$. Thus, by the definition of directed open set, $\mathcal{F} \Rightarrow_F \uparrow F \in \mathcal{U} \in d(UX)$ will imply $\mathcal{U} \cap \mathcal{F} \neq \emptyset$. Then $\mathcal{U} \in O_{\Rightarrow_U}(UX)$. It follows that $O_{\Rightarrow_U}(UX) = d(UX)$, that is, $(UX, O_{\Rightarrow_U}(UX))$ is a directed space. □

Proposition 3.7. *Let X, Y be two directed spaces. Then a map $f : (UX, O_{\Rightarrow_U}(UX)) \longrightarrow Y$ is continuous if and only if for each directed set $\mathcal{F} \subseteq UX$ and $\uparrow F \in UX$, $\mathcal{F} \Rightarrow_U \uparrow F$ implies $f(\mathcal{F}) \to f(\uparrow F)$.*

Proof. The necessity is obvious. We only prove the sufficiency. Firstly, we check that f is monotone. If $\uparrow F_1, \uparrow F_2 \in UX$ and $\uparrow F_1 \leq_U \uparrow F_2$, then $\{\uparrow F_2\} \Rightarrow_U \uparrow F_1$. By the hypothesis, $\{f(\uparrow F_2)\} \to f(\uparrow F_1)$, thus $f(\uparrow F_2) \sqsubseteq f(\uparrow F_1)$. Suppose U is an open set of Y and the directed set $\mathcal{F} \Rightarrow_U \uparrow F \in f^{-1}(U)$, then $f(\mathcal{F})$ is a directed set of Y and $f(\mathcal{F}) \to f(\uparrow F) \in U$. Thus, there exists an $\uparrow F \in \mathcal{D}$ such that $f(\uparrow F) \in U$. That is, $\uparrow F \in \mathcal{F} \cap f^{-1}(U)$. According to the definition of $\Rightarrow_U$ convergence open set, $f^{-1}(U) \in O_{\Rightarrow_U}(UX)$, i.e., f is continuous. □

Define a binary operation $\cup$ on UX : $\forall \uparrow F_1, \uparrow F_2 \in UX$, $\uparrow F_1 \cup \uparrow F_2 = \uparrow (F_1 \cup F_2)$. Suppose $\uparrow F_1 = \uparrow F_2$, $\uparrow G_1 = \uparrow G_2$ with $F_1 \neq F_2$, $G_1 \neq G_2$. Then $F_1 \cup G_1 \subseteq \uparrow (F_2 \cup G_2)$ implies $\uparrow (F_1 \cup G_1) \subseteq \uparrow (F_2 \cup G_2)$. Similarly, the opposite containment holds. Thus, $\cup$ is well-defined.

Theorem 3.8. *Let X be a directed space. Then $(UX, O_{\Rightarrow_U}(UX))$ with the set union operation $\cup$ is a directed deflationary semilattice.*

Proof. By Proposition 3.6, $(UX, O_{\Rightarrow_U}(UX))$ is a directed space. We need only to prove that $\cup$ is a deflationary operation. For arbitrary $\uparrow F_1, \uparrow F_2 \in UX$, $\uparrow F_1 \cup \uparrow F_2 = \uparrow (F_1 \cup F_2) \in UX$. Obviously, $\cup$ satisfies the conditions (a), (b), (c), (d) in Definition 3.1, we now prove the continuity of $\cup$. The monotonicity of $\cup$ is obvious. By Theorem 2.6(2) and Proposition 3.7, we only need to prove that, for each directed set $\mathcal{F} \subseteq UX$ and $\uparrow F, \uparrow G \in UX$, $\mathcal{F} \Rightarrow_U \uparrow F$ will imply $G \cup \mathcal{F} \Rightarrow_U \uparrow G \cup \uparrow F = \uparrow (G \cup F)$. Here, $G \cup \mathcal{F} = \{\uparrow (G \cup F') : \uparrow F' \in \mathcal{F}\}$ is still a directed set. According to the definition of $\Rightarrow_U$ convergence, there exist finite directed sets $D_1, \ldots, D_k \subseteq X$ satisfying the conditions for $\mathcal{F} \Rightarrow_U \uparrow F$. Let $G = \{a_1, \ldots, a_n\}$, and $D_{k+1} = \{a_1\}$, $D_{k+2} = \{a_2\}$, $\ldots$, $D_{k+n} = \{a_n\}$. It is straightward to verify that, $D_1, D_2, \ldots, D_k, D_{k+1}, \ldots, D_{k+n}$ satisfy all the conditions for $G \cup \mathcal{F} \Rightarrow_U (\uparrow G \cup \uparrow F)$. It follows that $(UX, \cup)$ is a directed deflationary semilattice. □

Theorem 3.9. *Let X be a directed space. Then $(UX, O_{\Rightarrow_U}(UX))$ with the set union operation $\cup$ is the directed upper powerspace over X.*

Proof. Define map $i : X \to UX$ with $\forall x \in X$, $i(x) = \uparrow x$. We prove the continuity of i. It is evident that i is monotone. Suppose that there exist a directed set $D \subseteq X$ and an $x \in X$ with $D \to x$. Let $\mathcal{D} = \{\uparrow d : d \in D\}$, then $\mathcal{D}$ is a directed set in UX and $\mathcal{D} \Rightarrow_U \uparrow x$. Since $i(D) = \mathcal{D}$, then $i(D) \Rightarrow_U \uparrow x = i(x)$. By Proposition 2.5, i is continuous.

Let $(Y, \wedge)$ be an arbitrary directed deflationary semilattice and $f : X \to Y$ be a continuous map. Define $\bar{f} : UX \to Y$ as follows: $\forall \uparrow F \in UX$ (suppose that $F = \{a_1, a_2, \ldots, a_n\}$),

$$\bar{f}(\uparrow F) = f(a_1) \wedge f(a_2) \wedge \cdots \wedge f(a_n) = \bigwedge_{a \in F} f(a).$$

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Particularly, we write $\bar{f}(\uparrow F)$ as $\wedge f(F)$. If $\uparrow F = \uparrow G$ with $F \neq G$, then $f(F) \subseteq \uparrow f(G)$ implies $\wedge f(G) \leq \wedge f(F)$, that is $\bar{f}(\uparrow G) \leq \bar{f}(\uparrow F)$. Similarly, we have $\bar{f}(\uparrow F) \leq \bar{f}(\uparrow G)$. Therefore, $\bar{f}$ is well-defined.

(1) $f = \bar{f} \circ i$.

For arbitrary $x \in X$, $(\bar{f} \circ i)(x) = \bar{f}(i(x)) = \bar{f}(\uparrow x) = f(x)$.

(2) $\bar{f}$ is a deflationary homomorphism, that is, $\bar{f}$ is continuous and for arbitrary $\uparrow F_1, \uparrow F_2 \in UX$, $\bar{f}(\uparrow F_1 \cup \uparrow F_2) = \bar{f}(\uparrow F_1) \wedge \bar{f}(\uparrow F_2)$.

First, we prove that $\bar{f}$ preserves the union operation. Suppose $\uparrow F_1, \uparrow F_2 \in UX$. Then $\bar{f}(\uparrow F_1 \cup \uparrow F_2) = \bar{f}(\uparrow(F_1 \cup F_2)) = \wedge f(F_1 \cup F_2) = (\wedge f(F_1) \wedge (\wedge f(F_2)) = \bar{f}(\uparrow F_1) \wedge \bar{f}(\uparrow F_2)$. Next, we prove the continuity of $\bar{f}$. Since $\wedge$ is the meet operation, $\bar{f}$ is evidently monotone. Suppose that $\mathcal{F} \subseteq UX$ is a directed set and $\mathcal{F} \Rightarrow_U \uparrow F \in UX$. By the definition of $\Rightarrow_U$, there exist finite directed sets $D_1, \ldots, D_n \subseteq X$ such that

1. $F \cap \lim D_i \neq \emptyset,\ i = 1, 2, \ldots, n$;
2. $F \subseteq \bigcup\limits_{i=1}^{n} \lim D_i$;
3. $\forall (d_1, \ldots, d_n) \in \prod\limits_{i=1}^{n} D_i$, there exists some $\uparrow F' \in \mathcal{F}$, such that $\uparrow F' \subseteq \bigcup\limits_{i=1}^{n} \uparrow d_i$.

Let $F = \{b_1,\ b_2, \ldots, b_k\}$. By 1, for each $1 \leq i \leq n$, there exists some $b_i \in F$ such that $D_i \to b_i$. If $F \setminus \{b_1, \ldots, b_n\} \neq \emptyset$, we let $G = \{a_1,\ a_2 \ldots,\ a_s\} = F \setminus \{b_1, \ldots, b_n\}$. By 2, For each $a_j \in G$, there exists $1 \leq i_j \leq n$ such that $D_{i_j} \to a_j$. By the continuity of f, $f(D_i) \to f(b_i)$ for each $i = 1, \ldots, n$ and $f(D_{i_j}) \to f(a_j)$ for each $j = 1, 2, \ldots, s$. Since the meet operation $\wedge$ on Y is continuous, the following convergence holds:

$$f(D_1) \wedge \cdots \wedge f(D_n) \wedge f(D_{i_1}) \wedge \cdots \wedge f(D_{i_s}) \to f(b_1) \wedge \cdots \wedge f(b_n) \wedge f(a_{i_1}) \wedge \cdots \wedge f(a_{i_s}). \qquad (*)$$

Here, $f(D_1) \wedge \cdots \wedge f(D_n) \wedge f(D_{i_1}) \wedge \cdots \wedge f(D_{i_s}) = \{f(d_1) \wedge \cdots \wedge f(d_n) \wedge f(d_{i_1}) \wedge \cdots \wedge f(d_{i_s}) : (d_1,\ \ldots,\ d_k,\ d_{i_1},\ \ldots,\ d_{i_s}) \in (\prod\limits_{i=1}^{n} D_i) \times (\prod\limits_{j=1}^{s} D_{i_j})\}$. Let U be an arbitrary open neighborhood of $\wedge f(F)$. By $(*)$, there exists some $(d_1,\ \ldots,\ d_n,\ d_{i_1},\ \ldots,\ d_{i_s}) \in (\prod\limits_{i=1}^{n} D_i) \times (\prod\limits_{j=1}^{s} D_{i_j})$ such that $f(d_1) \wedge \cdots \wedge f(d_n) \wedge f(d_{i_1}) \wedge \cdots \wedge f(d_{i_s}) \in U$.

Since D_{i_j} repeats D_i and each D_i is directed, there exists some $(d_1',\ d_2',\ \ldots,\ d_n') \in \prod\limits_{i=1}^{n} D_i$ such that $f(d_1') \wedge \cdots \wedge f(d_n') \sqsupseteq f(d_1) \wedge \cdots \wedge f(d_n) \wedge f(d_{i_1}) \wedge \cdots \wedge f(d_{i_s})$. By 3, there exists some $\uparrow F' \in \mathcal{F}$ such that $\uparrow F' \subseteq \bigcup\limits_{i=1}^{n} \uparrow d_i'$. Thus $\bar{f}(\uparrow F') = \wedge f(F') \sqsupseteq f(d_1') \wedge \cdots \wedge f(d_n')$. Since U is an upper set, it follows that $\wedge f(F') = \bar{f}(\uparrow F') \in U$, then $\bar{f}(\mathcal{F}) = \{\wedge f(F') : \uparrow F' \in \mathcal{F}\} \to \wedge f(F)$. By Proposition 3.7, $\bar{f}$ is continuous.

(3) Homomorphism $\bar{f}$ is unique.

Suppose we have a deflationary homomorphism $g : (UX, \cup) \to (Y, \wedge)$ such that $f = g \circ i$, then $g(\uparrow x) = f(x) = \bar{f}(\uparrow x)$. For each $\uparrow F \in UX$ with $F = (a_1, \ldots, a_n)$,

$$\begin{aligned} g(\uparrow F) &= g(\uparrow a_1 \cup \uparrow a_2 \cup \cdots \cup \uparrow a_n) \\ &= g(\uparrow a_1) \wedge g(\uparrow a_2) \wedge \cdots \wedge g(\uparrow a_n) \\ &= \bar{f}(\uparrow a_1) \wedge \bar{f}(\uparrow a_2) \wedge \cdots \wedge \bar{f}(\uparrow a_n) \\ &= \bar{f}(\uparrow a_1 \cup \uparrow a_2 \cup \cdots \cup \uparrow a_n) \\ &= \bar{f}(\uparrow F). \end{aligned}$$

Thus $\bar{f}$ is unique.

In conclusion, according to definition 3.1, endowed with topology $O_{\Rightarrow_U}(UX)$, the directed deflationary semilattice $(UX, \cup)$ is the directed upper powerspace of X, that is, $P_U(X) \cong (UX, \cup)$. □

Suppose that X, Y are two directed spaces and $f : X \to Y$ is a continuous map. Define map $P_U(f)$:

$P_U(X) \to P_U(Y)$ as follows:

$$\forall{\uparrow}F \in UX,\ P_U(f)({\uparrow}F) = {\uparrow}f(F).$$

Then $P_U(f)$ is well-defined and order preserving. It is easy to check that $P_U(f)$ is a deflationary homomorphism. If id_X is the identity map and $g : Y \to Z$ is an arbitrary continuous map from Y to a directed space Z, then $P_U(id_X) = id_{P_U(X)},\ P_U(g \circ f) = P_U(g) \circ P_U(f)$. Thus, $P_U : \mathbf{Dtop} \to \mathbf{Ddsl}$ is a functor from **Dtop** to **Ddsl**. Let $U : \mathbf{Ddsl} \to \mathbf{Dtop}$ be the forgetful functor. By Theorem 3.9, we have the following result.

Corollary 3.10. *P_U is a left adjoint of the forgetful functor U, that is,* **Ddsl** *is a reflective subcategory of* **Dtop**.

4. Relations between upper powerspaces

In this section, we will discuss the relations between the upper powerdomains, the observationally-induced upper powerspaces and the directed upper powerspaces.

Let $(X, \mathcal{O}(X))$ be a topological space. We say that a nonempty set $A \subseteq X$ is a saturated set, if $A = \bigcap\{U \in \mathcal{O}(X) : A \subseteq U\}$. Denote $Q(X)$ the set of all nonempty compact saturated sets of X. For each $U \in \mathcal{O}(X)$, let

$$[U] = \{K \in Q(X) : K \subseteq U\}.$$

Denote $\mathcal{B}_X = \{[U] : U \in \mathcal{O}(X)\}$. The upper Vietoris topology on $Q(X)$ is generated by the subbase $\mathcal{B}_X$, denote by $V_U(Q(X))$. Particularly, for a dcpo endowed with the Scott topology, the compact saturated sets are just the compact upper sets.

Theorem 4.1. [2] *Suppose X is a sober and locally compact space, $(Q(X), V_U(Q(X)))$ is order isomorphic and topological homomorphic to the observationally-induced upper space of X with respect to the union operation of sets. Under this condition, we have $V_U(Q(X)) = \sigma(Q(X))$. Here, $Q(X)$ is endowed with the order reverse to containment, and $\sigma(Q(X))$ denotes the Scott topology.*

Theorem 4.2. [9] *Let P be a continuous domain. Then $(Q(P), \supseteq)$, endowed with the Scott topology, is isomorphic to the upper powerdomain $S(P)$ over P (which is also called Smyth powerdomain). Besides, the following statements hold:*

(1) $(Q(P), \supseteq)$ is a continuous meet semilattice;
(2) $\forall K \in Q(P)$, $K = \bigcap\{{\uparrow}F : 1 \le |F| < \omega \ \&\ K \subseteq ({\uparrow}F)^{\circ}\}$.

According to the two theorems above, for each continuous domain endowed with the Scott topology, its observationally-induced upper powerspace is isomorphic to the upper powerdomain. Next, we discuss the directed upper powerspaces of continuous domains endowed wit the Scott topology.

Let X be a continuous domain. Then $(X, \sigma(X))$ is a directed space, and each upper set generated by a finite set is a compact saturated set. Thus, $UX \subseteq Q(X)$. Denote $\sigma(Q(X))|_{UX}$ the induced topology from the Scott topology on $Q(X)$.

Proposition 4.3. *Let X be a continuous domain endowed with the Scott topology $\sigma(X)$.*

(1) For each given directed set $\mathcal{F} \subseteq UX$ and ${\uparrow}F \in UX$, we have $\mathcal{F} \Rightarrow_U {\uparrow}F \ \Leftrightarrow\ \bigcap\{{\uparrow}G : {\uparrow}G \in \mathcal{F}\} \subseteq {\uparrow}F$.
(2) $O_{\Rightarrow_U}(UX) = \sigma(Q(X))|_{UX}$.

Proof. (1) Suppose that $\mathcal{F} \subseteq UX$ is a directed set of UX, ${\uparrow}F \in UX$, and $\mathcal{F} \Rightarrow_U {\uparrow}F$. By definition, there exist finite directed sets $D_1 \ldots, D_n \subseteq X$ such that

1. $F \cap \lim D_i \neq \emptyset,\ i = 1, 2, \ldots,\ n;$
2. $F \subseteq \bigcup_{i=1}^{n} \lim D_i;$

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3. $\forall(d_1,\ldots,d_n)\in\prod_{i=1}^{n}D_i$, there exists some $\uparrow F^{'}\in\mathcal{F}$, such that $\uparrow F^{'}\subseteq\bigcup_{i=1}^{n}\uparrow d_i$.

By contradiction, suppose $\bigcap\{\uparrow G:\uparrow G\in\mathcal{F}\}\not\subseteq\uparrow F$. Then there exists some $a\in\bigcap\{\uparrow G:\uparrow G\in\mathcal{F}\}$ such that $a\notin\uparrow F$. Thus, $F\subseteq X\backslash\downarrow a$. By 1 and 2, for each i, there exists some $d_i\in D_i$ such that $d_i\in X\backslash\downarrow a$. By 3, there exists some $\uparrow F^{'}\in\mathcal{F}$ such that $\uparrow F^{'}\subseteq\bigcup_{i=1}^{n}\uparrow d_i$, which contradicts with $a\notin\bigcup_{i=1}^{n}\uparrow d_i$. Thus $\bigcap\{\uparrow G:\uparrow G\in\mathcal{F}\}\subseteq\uparrow F$.

On the other hand, suppose that $\bigcap\{\uparrow G:\uparrow G\in\mathcal{F}\}\subseteq\uparrow F$. Let $F=\{a_1,\ a_2,\ldots,a_n\}$. Since X is a continuous domain, then each $\twoheaddownarrow a_i$ is directed and $a_i=\bigvee\twoheaddownarrow a_i$ for $i=1,2,\ldots,n$. Let $D_i=\twoheaddownarrow a_i$ amd $D_i\to a_i$. Then, each D_i satisfy 1 and 2 in the definition of $\Rightarrow_U$. For arbitrary $(d_1,d_2,\ldots,\ d_n)\in\prod_{i=1}^{n}D_i$, we have $\uparrow F\subseteq\bigcup_{i=1}^{n}\twoheaduparrow d_i=(\bigcup_{i=1}^{n}\uparrow d_i)^{\circ}$. Since each continuous domain is well-filtered, it follows that there exists some $\uparrow G\in\mathcal{F}$ such that $\uparrow G\subseteq(\bigcup_{i=1}^{n}\uparrow d_i)^{\circ}$, and then, 3 in the definition of $\Rightarrow_U$ holds. Therefore $\mathcal{F}\Rightarrow_U\uparrow F$.

(2) Suppose $\mathcal{U}\in O_{\Rightarrow_U}(UX)$. Let $\mathcal{U}_Q=\{K\in Q(X):\exists\uparrow F\in\mathcal{U},\ K\subseteq\uparrow F\}$. Obviously, $\mathcal{U}=\mathcal{U}_Q\cap Q(X)$. Let $\mathcal{K}\subseteq Q(X)$ be a directed set with respect to the reverse inclusion order and $\bigcap\{K:K\in\mathcal{K}\}\in\mathcal{U}_Q$. There exists some $\uparrow F\in\mathcal{U}$ such that $\bigcap\{K:K\in\mathcal{K}\}\subseteq\uparrow F$. Let $F=\{a_1,a_2,\ldots,\ a_n\}$. Since X is a continuous domain, then each $\twoheaddownarrow a_i$ is directed and $a_i=\bigvee\twoheaddownarrow a_i$ for $i=1,2,\ldots,n$. Let $\mathcal{F}=\{\bigcup_{i=1}^{n}\uparrow d_i:d_i\ll a_i,\ i=1,2,\ldots,n\}$. Since X is a continuous domain, then $Q(X)$ is a continuous domain. We have $\uparrow F\subseteq(\bigcup_{i=1}^{n}\uparrow d_i)^{\circ}\subseteq\bigcup_{i=1}^{n}\uparrow d_i$ for $i=1,2,\ldots,n$. Besides, $\cap\mathcal{F}=\uparrow F$. By (1), $\mathcal{F}\Rightarrow_U\uparrow F$. Thus, there exists some $(d_1,d_2,\ldots,d_n)\in\prod_{1\leq i\leq n}\twoheaddownarrow a_i$ such that $\bigcup_{i=1}^{n}\uparrow d_i\in\mathcal{U}$. Noticing that $\bigcap\{K:K\in\mathcal{K}\}\subseteq\uparrow F\subseteq(\bigcup_{i=1}^{n}\uparrow d_i)^{\circ}$, there exists some $K\in\mathcal{K}$ such that $K\subseteq(\bigcup_{i=1}^{n}\uparrow d_i)^{\circ}$. By the definition of $\mathcal{U}_Q$, we have $K\in\mathcal{U}$, that is, $\mathcal{U}_Q$ is a Scott open set in $Q(X)$. Therefore, $O_{\Rightarrow_U}(UX)\subseteq\sigma(Q(X))|_{UX}$.

On the other hand, let $\mathcal{V}\in\sigma(Q(X))$, and $\mathcal{F}\subseteq UX$ be a directed set with $\mathcal{F}\Rightarrow_U\uparrow F\in\mathcal{V}\cap UX$. By (1), $\bigcap\{\uparrow G:\uparrow G\in\mathcal{F}\}\in\mathcal{V}$, there exists some $\uparrow G\in\mathcal{F}\cap\mathcal{V}$, that is, $\mathcal{F}\cap\mathcal{V}\cap UX\neq\emptyset$. Thus, $\sigma(Q(X))|_{UX}\subseteq O_{\Rightarrow_U}(UX)$. □

Example 4.4. Let $X=\mathbb{R}^n$ be the n-dimensional Euclidean space. Then X is a locally compact T_2 space, thus a locally compact sober space. Denote the observationally-induced upper space over X by $P_O(X)$. By Theorem 4.1, $P_O(X)=\{K\subseteq X:K$ is a nonempty compact set of X $\}$ for which the topology is the Scott topology. It is easy to check that, $P_O(X)$ is a continuous domain and for each nonempty compact set $K\subseteq X$, $\{\{a\}:a\in K\}$ is a compact saturated set of $P_O(X)$. It follows that, for the directed space $P_O(X)$, its directed upper powerspace $P_U(P_O(X))\neq Q(P_O(X))$.

5. The directed lower powerspaces of directed spaces

In this section, we introduce the directed lower powerspace of a directed space, which is a free algebra generated by the inflationary operation of the directed space. These results can be found in [15]. And the proofs are similar to the directed upper powerspaces. We just represent some results as basic knowledge for the following work.

Definition 5.1. [15] Let X be a directed space.

(1) A binary operation $\oplus:X\otimes X\to X$ on X is called an inflationary operation if it is continuous and satisfies the following four conditions: $\forall x,\ y,\ z\in X$,
   (a) $x\oplus x=x$,
   (b) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$,
   (c) $x\oplus y=y\oplus x$,
   (d) $x\oplus y\geq x$.

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(2) If $\oplus$ is a inflationary operation on X, then $(X, \oplus)$ is called a directed inflationary semilattice, that is, directed inflationary semilattices are those directed spaces with inflationary operations.

By Theorem 2.6(2), the operation $\oplus$ on a directed space X is continuous if and only if it is monotone and for each given $x,\ y \in X$ and directed set $D \subseteq X,\ \ x \equiv \lim D$ implies $x \oplus y \equiv \lim(D \oplus y)$. Here, $D \oplus y = \{d \oplus y : d \in D\}$.

Definition 5.2. [15] Let $(X, \oplus),\ (Y, \uplus)$ be two directed inflationary semilattices, $f : (X, \oplus) \to (Y, \uplus)$ is called an inflationary homomorphism between X and Y, if f is continuous and $\forall x, y \in X,\ f(x \oplus y) = f(x) \uplus f(y)$.

Denote the category of all directed inflationary semilattices and inflationary homomorphisms by **Disl**. Then **Disl** is a subcategory of **Dtop**.

Lemma 5.3. [15] *Suppose that $(X, \oplus)$ is a directed inflationary semilattice. Then $\oplus = \vee_{\sqsubseteq}$. Here, $x \vee_{\sqsubseteq} y$ means the supremum of x and y with respect to the specialization order $\sqsubseteq$ of X (called the sup operation). Conversely, suppose that X is a directed space such that for each $x, y \in X$, $x \vee_{\sqsubseteq} y$ exists. Continuity of $\vee_{\sqsubseteq}$ will naturally imply that $(X, \vee_{\sqsubseteq})$ is a directed inflationary semilattice.*

We see that a directed inflationary semilattice $(X, \oplus)$ is just a directed space with a continuous sup operation satisfy $\oplus = \vee_{\sqsubseteq}$. We use the symbol $\vee$ instead of $\vee_{\sqsubseteq}$ if there is no confusion. Therefore, a directed inflationary semilattice can be represented by a tuple of the form $(X, \vee)$, where X is a directed space and $\vee$ represents the continuous sup operation on X.

Next, we give the definition of a directed lower powerspace.

Definition 5.4. [15] Suppose that X is a directed space. A directed space Z is called the directed lower powerspace over X if and only if the following two conditions are satisfied:

(1) Z is a directed inflationary semilattice, i.e., the sup operation $\vee$ on Z exists and is continuous,
(2) There is a continuous map $i : X \longrightarrow Z$ satisfying: for any directed inflationary semilattice $(Y, \vee)$ and any continuous map $f : X \longrightarrow Y$, there exists a unique inflationary homomorphism $\bar{f} : (Z, \vee) \to (Y, \vee)$ such that $f = \bar{f} \circ i$.

Let X be a directed space. Set $LX = \{\downarrow F : F \subseteq_{fin} X\}$, where $F \subseteq_{fin} X$ is an arbitrary nonempty finite subset of X. Define an order $\leq_L$ on LX as follows:

$$\downarrow F_1 \leq_L \downarrow F_2 \iff \downarrow F_1 \subseteq \downarrow F_2.$$

Let $\mathcal{D} \subseteq LX$ be a directed set (respect to order $\leq_L$) and $\downarrow F \in LX$. Define a convergence relation $\Rightarrow_L$ as follows:

$$\mathcal{D} \Rightarrow_L \downarrow F \iff \forall a \in F, \text{ there exists a directed set } D_a \text{ of } X \text{ such that } D_a \subseteq \bigcup \mathcal{D} \text{ and } D_a \to a.$$

A subset $\mathcal{U} \subseteq LX$ is called a $\Rightarrow_L$ convergence open set of LX if and only if for each directed subset $\mathcal{D}$ of LX and $\downarrow F \in LX,\ \ \mathcal{D} \Rightarrow_L \downarrow F \in \mathcal{U}$ implies $\mathcal{D} \cap \mathcal{U} \neq \emptyset$. Denote all $\Rightarrow_L$ convergence open sets of LX by $O_{\Rightarrow_L}(LX)$.

Theorem 5.5. *[15] Let X be a directed space. Then $(LX, O_{\Rightarrow_L}(LX))$ is the lower powerspace of X.*

The lower powerspace is unique up to isomorphism, we denote the lower powerspace of each directed space X by $P_L(X) = (LX,\ \cup)$.

Suppose that X, Y are two directed spaces and $f : X \to Y$ is a continuous map. Define the map $P_L(f) : P_L(X) \to P_L(Y)$ as follows: $\forall \downarrow F \in LX,\ P_L(f)(\downarrow F) = \downarrow f(F)$. It is evident that $P_L(f)$ is well-defined and order preserving. According to 5.5, it is easy to check that $P_L(f)$ is an inflationary homomorphism between the two directed lower powerspaces.

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Proposition 5.6. *[15] P_L is a left adjoint of the forgetful functor U, that is,* **Disl** *is a reflective subcategory of* **Dtop**.

6. Properties preserved by directed power functors

We first introduce the notion of a c-space and an FS-space.

Definition 6.1. A T_0 topological space X is called a c-space if for any open subset V of X and $x \in V$, there exists a $y \in V$ such that $x \in (\uparrow y)^\circ \subseteq \uparrow y$.

The c-spaces have another characterization by a generalized way-below relation on topological spaces called d-approximation.

Definition 6.2. We say that x *d-approximates* y, denoted by $x \ll_d y$, if for any directed subset $D \subseteq X$, $D \to y$ implies $x \sqsubseteq d$ for some $d \in D$. A topological space X is called a continuous space if it is a directed space such that $\Downarrow_d x$ is directed and $\Downarrow_d x \to x$ for all $x \in X$, where $\Downarrow_d x = \{y \in X : y \ll_d x\}$.

Proposition 6.3. [16] A T_0 topological space is a c-space iff it is a continuous space.

Definition 6.4. [16] An approximate identity for a directed space X is a directed set $\mathcal{D} \subseteq [X \to X]$ satisfying $\mathcal{D} \to 1_X$, where 1_X is the identity map of X, and $\mathcal{D} \to 1_X$ means the pointwise convergence, i.e., $\forall x \in X, (f(x))_{f \in \mathcal{D}} \to x \in X$.

Definition 6.5. [16] A continuous map $\delta : X \to X$ on a directed space X is called finitely separating if there exists a finite set F_δ such that for each $x \in X$, there exists a $y \in F_\delta$ such that $\delta(x) \leq y \leq x$. A directed space is finitely seperated if there is an approximate identity for X consisting of finitely seperating maps. A finitely separated directed space is called an FS-space.

All FS-spaces with continuous maps form a category denoted by **FS**. Denote the category of all c-spaces by **CS**. They are very natural extensions for FS-domains and continuous domains. Moreover, **FS** is cartesian closed [16]. Thus, as in domain theory, it is meaningful to investigate the directed powerspaces over FS-spaces and c-spaces. In [4], it was shown that the free algebras (in the category **Dtop**) over c-spaces are still c-spaces by a categorical method. As concrete examples of free algebras, the directed upper powerspaces and directed lower powerspaces of c-spaces will be c-spaces. Here, we prove this by a topological method, which is more concrete and easier to understand.

Let X be a directed space. Given any element $\downarrow F$ of LX, we denote by $\Uparrow(\downarrow F)$ the upper subset of LX generated by the element $\downarrow F$. That is, $\Uparrow(\downarrow F) = \{\downarrow G \in LX : \downarrow F \leq_L \downarrow G\}$. For a subset $\{\downarrow F_i : i \in I\}$ of LX, denote by $\Uparrow\{\downarrow F_i : i \in I\}$ the upper subset of LX generated by $\{\downarrow F_i : i \in I\}$. The same, for any $\uparrow F \in UX$, $\Uparrow(\uparrow F) = \{\uparrow G \in UX : \uparrow F \leq_U \uparrow G\}\}$.

Proposition 6.6. *Let X be a c-space and $D_0 = \{d_1, \ldots, d_n\}$ be any finite subset of X.*

(1) Define $\mathcal{V}^1_{D_0} = \Uparrow \{\downarrow\{x_1, \ldots, x_n\} : x_i \in \mathrm{int}(\uparrow d_i), i = 1, \ldots, n.\}$. Then $\mathcal{V}^1_{D_0} \subseteq \Uparrow\{\downarrow\{d_1, \ldots, d_n\}\}$ and $\mathcal{V}^1_{D_0}$ is an open subset of $P_L(X)$.

(2) Define $\mathcal{V}^2_{D_0} = \Uparrow \{\uparrow F : F \subseteq_{fin} X, F \subseteq \bigcup_{d \in D_0} \mathrm{int}(\uparrow d)\}$. Then $\mathcal{V}^2_{D_0} \subseteq \Uparrow\{\uparrow\{d_1, \ldots, d_n\}\}$ and $\mathcal{V}^2_{D_0}$ is an open subset of $P_U(X)$.

Proof. (1) The inclusion relation is obvious. We need only to prove that $\mathcal{V}^1_{D_0}$ is a directed open subset of $P_L(X)$. Supposet that $\mathcal{D} \Rightarrow_L \downarrow F \in \mathcal{V}^1_{D_0}$. Let $F = \{a_1, \ldots, a_m\}$. By the definition of $\Rightarrow_L$ convergence, we know that there exist finite directed subsets $D_1, \ldots, D_m$ of X satisfying

$$D_j \to a_j, 1 \leq j \leq m. \tag{$*$}$$

Since $\downarrow F \in \mathcal{V}^1_{D_0}$, there exist some $x_i \in \mathrm{int}(\uparrow d_i)$ for each $1 \leq i \leq n$ such that $\downarrow\{x_1 \ldots x_n\} \subseteq \downarrow F$. Thus for any $1 \leq i \leq n$, there exists some $a'_i \in F$ such that $x_i \leq a'_i$. Denote the corresponding directed subsets in

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$(*)$ as D_i'. Then $D_i' \to a_i', 1 \le i \le n$. By $a_i' \in \mathrm{int}(\uparrow d_i)$, there exists $d_i' \in \mathrm{int}(\uparrow d_i)$ for each $1 \le i \le n$. By the definition of $\Rightarrow_L$ convergence, we know that $D_i \subseteq \cup\mathcal{D}$. Since $\mathcal{D}$ is directed, there exists a $\downarrow F_1 \in \mathcal{D}$ such that $\downarrow\{d_1^{'}, \dots, d_n^{'}\} \subseteq \downarrow F_1$, i.e., $\downarrow F \in \mathcal{V}^1_{D_0} \cap \mathcal{D}$. Thus, $\mathcal{V}^1_{D_0}$ is an open subset of $P_L(X)$.

(2) The inclusion relation is obvious. We need to show that $\mathcal{V}^2_{D_0}$ is a directed open subset of $P_U(X)$. Assume that $\mathcal{F} \Rightarrow_U \uparrow F \in \mathcal{V}^2_{D_0}$. Let $F = \{a_1, \dots, a_m\}$. By the definition of $\Rightarrow_U$ convergence, there exist finite directed subsets $D_1, \dots, D_m$ of X satisfying

$$D_j \to a_j, 1 \le j \le m. \tag{$**$}$$

Since $\uparrow F \in \mathcal{V}^2_{D_0}$, there exists a finite subset $G = \{x_1, \dots, x_k\} \subseteq \bigcup_{d \in D_0} \mathrm{int}(\uparrow d)$ in X such that $\uparrow F \subseteq \uparrow\{x_1 \dots . x_k\}$. Thus, for any $1 \le j \le m$, there exits an $x_j' \in \{x_1, \dots, x_k\}$ such that $x_j' \le a_j$.

- if $\{x_1, \dots, x_k\} \setminus \{x_1', \dots, x_m'\} \ne \emptyset$, By $(**)$, for any $1 \le j \le m$, there exits some $d_j' \in D_j \cap \cup_{d \in D_0}\mathrm{int}(\uparrow d), 1 \le j \le m$. By condition 2 of definition of $\Rightarrow_U$ convergence, we have that for $\uparrow\{d_1', \dots, d_m'\}$, there exists some $\uparrow F_1 \in \mathcal{D}$ such that $\uparrow F_1 \subseteq \uparrow\{d_1', \dots, d_m'\}$. By the construction of $\mathcal{V}^2_{D_0}$, $\uparrow F_1 \in \mathcal{D} \cap \mathcal{V}^2_{D_0}$. Thus $\mathcal{V}^2_{D_0}$ is open in $P_U(X)$.
- If $\{x_1', \dots, x_m'\}$ has already covered by $\{x_1, \dots, x_k\}$, then for any $1 \le i \le k$, there exists $a_i' \in F$ such that $x_i \le a_i'$. At this time, we have $k \le m$. Let $F \setminus \{a_1', \dots, a_k'\} = \{a_1'', \dots, a_s''\}$, $0 \le s$. Denote $D_1', \dots, D_k', D_1'', \dots, D_s''$ the corresponding directed subset in $(**)$, then $D_i' \to a_i', 1 \le i \le k; D_t'' \to a_t''$, $0 \le t \le s$.

  Since $a_i', a_t'' \in \cup_{d \in D_0}\mathrm{int}(\uparrow d), 1 \le i \le k, 0 \le t \le s$, there exist

$$d_i' \in D_i' \cap \bigcup_{d \in D_0} \mathrm{int}(\uparrow d), 1 \le i \le k,$$
$$d_t'' \in D_t'' \cap \bigcup_{d \in D_0} \mathrm{int}(\uparrow d), 0 \le t \le s.$$

  By condition 2 in defintion of $\Rightarrow_U$ convergence, for any $\uparrow\{d_1', \dots, d_k', d_1'', \dots, d_s''\}$, there exists some $\uparrow F_1 \in \mathcal{D}$ such that $\uparrow F_1 \subseteq \uparrow\{d_1', \dots, d_k', d_1'', \dots, d_s''\}$. By the construction of $\mathcal{V}^2_{D_0}$, we know that $\uparrow F_1 \in \mathcal{D} \cap \mathcal{V}^2_{D_0}$, and $\mathcal{V}^2_{D_0}$ is open in $P_U(X)$. $\square$

Theorem 6.7. *If X is a c-space, then $P_L(X)$ and $P_U(X)$ are both c-spaces.*

Proof. (1) Let X be a c-space. To prove that $P_L(X)$ is a c-space, for any directed open subset $\mathcal{U}$ of $P_L(X)$ and $\downarrow F = \downarrow\{a_1, \dots, a_n\} \in \mathcal{U}$, we need to find out a $\downarrow F' \in \mathcal{U}$ such that $\downarrow F \in \mathrm{int}\left(\Uparrow(\downarrow F')\right) \subseteq \mathcal{U}$.

Since X is a c-space, for any $a_i \in F$, there exists a directed subset D_{a_i} of X such that $D_{a_i} \to a_i$, where $\forall d \in D_{a_i},\ a_i \in \mathrm{int}(\uparrow d), i = 1, \dots, n$. Let

$$\mathcal{D} = \left\{ \downarrow\{d_1, \dots, d_n\} : (d_1, \dots, d_n) \in \prod_{i=1}^{n} D_{a_i} \right\}.$$

Then it is a directed subset in $P_L(X)$, by the definition of $\Rightarrow_L$ convergence, we can see that $\mathcal{D} \Rightarrow_L \downarrow F$. Thus there exists $\downarrow\{d_1, \dots, d_n\} \in \mathcal{D} \cap \mathcal{U}$ and $\downarrow F \in \Uparrow\{\downarrow\{d_1, \dots, d_n\}\} \subseteq \mathcal{U}$. We need only to prove that $\downarrow F \in \mathrm{int}\left(\Uparrow\{\downarrow\{d_1, \dots, d_n\}\}\right)$.

Let $D_0 = \{d_1, \dots, d_n\}$, By Proposition 6.6, $\mathcal{V}^1_{D_0}$ is open in $P_L(X)$ and $\downarrow F \in \mathcal{V}^1_{D_0} \subseteq \Uparrow\{\downarrow\{d_1, \dots, d_n\}\}$, i.e., $\downarrow F \in \mathrm{int}\left(\Uparrow\{\downarrow\{d_1, \dots, d_n\}\}\right)$.

(2) To prove that $P_U(X)$ is a c-space, for any directed open subset $\mathcal{U}$ of $P_U(X)$ and $\uparrow F = \uparrow\{a_1, \dots, a_n\} \in \mathcal{U}$, we need to find out a $\uparrow F' \in \mathcal{U}$ such that $\uparrow F \in \mathrm{int}\left(\Uparrow(\uparrow F')\right) \subseteq \mathcal{U}$.

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Since X is a c-space, for any $a_i \in F$, there exists a directed subset D_{a_i} of X such that $D_{a_i} \to a_i$, where $\forall d \in D_{a_i},\ a_i \in \mathrm{int}(\uparrow d), i = 1, \ldots, n$. Let

$$\mathcal{D} = \left\{ \uparrow\{d_1, \ldots, d_n\} : (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_{a_i} \right\}.$$

Then it is a directed subset in $P_U(X)$, by the definition of $\Rightarrow_U$ convergence, we can see that $\mathcal{D} \Rightarrow_U \uparrow F$ by letting those D_i in the convergence conditions to be those D_{a_i}. Thus there exists $\uparrow\{d_1, \ldots, d_n\} \in \mathcal{D} \cap \mathcal{U}$ and $\uparrow F \in \Uparrow\{\uparrow\{d_1, \ldots, d_n\}\} \subseteq \mathcal{U}$. We need only to prove that $\uparrow F \in \mathrm{int}\left(\Uparrow\{\uparrow\{d_1, \ldots, d_n\}\}\right)$.

Let $D_0 = \{d_1, \ldots, d_n\}$, By Proposition 6.6, $\mathcal{V}^2_{D_0}$ is open in $P_U(X)$ and $\uparrow F \in \mathcal{V}^2_{D_0} \subseteq \Uparrow\{\uparrow\{d_1, \ldots, d_n\}\}$, i.e., $\uparrow F \in \mathrm{int}\left(\Uparrow\{\uparrow\{d_1, \ldots, d_n\}\}\right)$. □

Now, we show that the directed upper powerspace and the directed lower powerspace of an FS-space are FS-spaces.

Theorem 6.8. *Let X be a directed space, $\mathcal{D} = \{f_i\}_{i \in I} \subseteq [X \to X]$ be a directed subset and $\mathcal{D} \to f \in [X \to X]$. Then $P_L(\mathcal{D}) \to P_L(f)$, $P_U(\mathcal{D}) \to P_U(f)$, i.e., functors P_L and P_U are continuous.*

Proof. Let X be a directed space and the directed subset $\mathcal{D}$ and f satisfying the supposed condition.

(1) For P_L: $P_L(\mathcal{D}) = \{P_L(f_i) : f_i \in \mathcal{D}\}$ is a directed subset in $[P_L(X) \to P_L(X)]$. $P_L(f)(\downarrow F) = \downarrow f(F)$. We need only to prove that $P_L(\mathcal{D}) \Rightarrow_L P_L(f)$. It is equivalent to showing that

$$\{\downarrow f_i(F)\}_{i \in I} \Rightarrow_L \downarrow f(F),\ \forall \downarrow F \in P_L(X).$$

Given any $\downarrow F \in P_L(X)$ with $F = \{a_1, \ldots, a_n\}$, let $D_j = \{f_i(a_j)\}_{i \in I}$. Then it is a directed subset of X and

$$D_j \to f(a_j),\ j = 1, \ldots, n.$$

This means that $\{\downarrow f_i(F)\}_{i \in I} \Rightarrow_L \downarrow f(F)$.

(2) For P_U: similarly with (1), we need only to show that

$$\{\uparrow f_i(F)\}_{i \in I} \Rightarrow_U \uparrow f(F),\ \forall \uparrow F \in P_U(X).$$

Given any $\uparrow F \in P_U(X)$ with $F = \{a_1, \ldots, a_n\}$, let $D_j = \{f_i(a_j)\}_{i \in I}$. Then it is a directed subset of X and

$$D_j \to f(a_j),\ j = 1, \ldots, n.$$

For any $(f_{i_1}(a_1), \ldots, f_{i_n}(a_n)) \in \prod_{j=1}^{n} D_j$, since $\mathcal{D}$ is directed, we can pick $i_0 \in I$ such that $f_i \leq f_{i_0},\ i = i_1, \ldots, i_n$. Thus,

$$\uparrow(f_{i_0}(a_1), \ldots, f_{i_0}(a_n)) \subseteq \uparrow(f_{i_1}(a_1), \ldots, f_{i_n}(a_n)).$$

This means that $\{\uparrow f_i(F)\}_{i \in I} \Rightarrow_U \uparrow f(F)$.

Theorem 6.9. *Let X be a directed space. If δ is a finitely separable map on X, then $P_L(f)$ and $P_U(f)$ are finitely separable maps on $P_L(X)$ and $P_U(X)$ respectively.*

Proof. We verify that $P_L(\delta)$ is a finitely separate map on $P_L(X)$, i.e., there exists a finite subset G_δ of $P_L(X)$ such that $\forall \downarrow F \in P_L(X)$, there exists a $\downarrow F' \in G_\delta$ such that

$$P_L(\delta)(\downarrow F) \leq \downarrow F' \leq \downarrow F.$$

Let F_δ be the finitely separate set of δ on X. Let $G_\delta = \{\downarrow A : A \subseteq F_\delta\}$. We claim that G_δ satisfies the needs. For any $\downarrow F \in P_L(X)$ with $F = \{a_1, \ldots, a_n\}$, we have $P_L(\delta)(\downarrow F) = \downarrow \delta(F)$. There exists some

$y_i \in F_\delta$ such that $\delta(a_i) \leq y_i \leq a_i,\ 1 \leq i \leq n$. Thus,

$$\downarrow\delta(F) \subseteq \downarrow(y_1, \ldots, y_n) \subseteq \downarrow F$$

Thus we have $P_L(\delta)(F) \leq \downarrow(y_1, \ldots, y_n) \leq \downarrow F$, proved. For $P_U(\delta)$, the proof is similar, so we omit it. □

By Theorem 6.8 and Theorem 6.9, we have the following statement.

Theorem 6.10. *Let X be a directed space. If X is an FS-space, then $P_L(X)$ and $P_U(X)$ are both FS-spaces.*

Reference

[1] Abramsky, S., Jung, A.: domain theory. In: Abramsky, S.,Gabbay, D.M., Maibaum, T.S.E.(eds.), Semantic Structures. In: Handbook of Logic in Computer Science, vol.3, pp.1-168, Clarendon Press, Oxford (1994)

[2] Battenfeld, I., Schöder, M.:Observationally-induced lower and upper powerspace constructions. Journal of Logical and Algebraic Methods in Programming. 84, 668-682 (2015)

[3] Chen, Y., Kou, H., Lyu Z.: Continuity and core compactness of topological spaces. arXiv:2203.06344 (2022)

[4] Chen, Y., Kou, H.: Free algebras over directed spaces. arXiv:2209.04125 (2022)

[5] Erné, M.: The ABC of order and topology. In: H. Herrlich, H. E. Porst (des.) Category Theory at Work, pp. 57-83, Heldermann, Berlin (1991)

[6] Erné, M.: Infinite distributive laws versus local connectedness and compactness properties. Topology and its Applications. 156, 2054-2069 (2009)

[7] Flannery, K.E, Martin,J.J.: The Hoare and Smyth power domain constructors commute under composition. Journal of Computer and System Sciences, 40(2): 125-135. (1990)

[8] Geng, J., Kou, H.: Consistent Hoare powerdomains over dcpos, Topology and its Applications. 232, 169-175 (2017)

[9] Gierz, G. et al.: Continuous Lattices and domains. Cambridge University Press, Cambridge (2003).

[10] Goubault-Larrecq, J: Non-Hausdorff topology and domain theory: Selected topics in point-set topology, Cambridge University Press, Cambridge, 2013

[11] Heckmann, R.: Power domain constructions. Sci. Comput. Program. 17, 77-117 (1991)

[12] Heckmann, R.: An upper power domain construction in terms of strongly compact sets, Lect. Notes Comput. Sci. 598, 272-293 (1992)

[13] Heckmann, R.: Lower and upper power domain constructions commute on all cpos. Information Processing Letters, 40(1): 7-11.(1991)

[14] MacLane, S.: Categories for the Working Mathematician. Springer-Verlag, New York (1971)

[15] Xie, X., Kou H.: Lower power structures of directed spaces(Chinese). Journal of Sichuan University (Natural Science Edition). 57, 211-217 (2020)

[16] Xie, X., Kou H.: The cartsian closedness of c-spaces. AIMS Mathematics. 7(9), 16315-16327 (2022)

[17] Xu, X.: Order and Topology (Chinese). Science Press, Beijing (2016)

16 Jan 2023 20:51:56 PST
220929-HuiKou Version 2 - Submitted to Rocky Mountain J. Math.

[18] Yu Y., Kou, H.: Directed spaces defined through T_0 spaces with specialization order(Chinese), Journal of Sichuan University (Natural Science Edition). 52(2), 217-222 (2015)

[19] Yuan, Y., Kou, H.: Consistent Smyth Powerdomains. Topology and its Applications. 173, 264-275 (2014)

[20] Yuan, Y., Kou, H.: Consistent Hoare Powerdomains. Topology and its Applications. 178, 40-45 (2014)

[21] Yuan, Y., Kou, H.: Consistent Plotkin powerdomains. Topology and its Applications. 178, 339-344 (2014)