Practical h-stability of non-autonomous evolution equations with Lipschitz nonlinearities

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Abstract

This paper deals with the problem of practical h-stability for certain classes of non-autonomous semilinear systems in Banach spaces with Lipschitz nonlinearities in Banach spaces. Our original results generalize well-known fundamental results: practical stability, practical asymptotic stability, and practical exponential stability for non-autonomous infinite-dimensional systems. The main aim is to give necessary and sufficient conditions on the linear and the perturbed term for the global existence of solutions and the practical h-stability based on some nonlinear integral Gronwall type inequalities. Two examples are given to illustrate the effectiveness and advantages of the obtained results.

Keywords: Evolution operator, Non-autonomous infinite-dimensional systems, Gronwall's inequalities, h-stability, Practical h-stability.

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1 Introduction

Many problems in partial differential equations which arise from physical models can be considered as ordinary differential equations in appropriate infinite dimensional spaces, for which elegant theories and powerful techniques have recently been established, see [6, 7, 13, 19, 20] and the references cited therein. The evolution operators and its neighboring areas have developed into an abstract theory that has become a necessary discipline in functional analysis and differential equations. In the last century, a general concept of stability was introduced by M. Pinto in [22, 23] called h-stability. The intention of this notion is to obtain stability result for evolution operators in Banach spaces, see [14, 16, 17, 18]. Moreover, The stability theory of semilinear evolution equations is well developed and attract the attention of many specialists despite its long history in [10, 15, 24] and other papers. However, there are some systems that may be unstable and yet these systems may oscillate sufficiently near this state that its performance is acceptable. To deal with this situation, we need a notion of stability that is more suitable than Lyapunov stability such a concept is called practical stability, see [5, 12]. This approach leads to a new result called:

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practical uniform h-stability, see [8, 9]. This notion is quite flexible because it includes the classical notion of practical exponential stability within one common framework. Damak in [9] studied practical h-stability of non-autonomous evolution equations in Banach spaces. Many works have shown the effectiveness of the theory of inequalities, which is one of the analytic methods of the perturbation theory, see [3, 8, 11].

Motivated by the above consideration and particularly inspired by the work of [9, 11], a natural and interesting idea is to study the practical h-stability of non-autonomous evolutions equations with Lipschitz nonlinarities. The main tool used to prove our results is some Gronwall type inequalities. They can be used in the study of existence, uniqueness, continuation, boundedness, oscillation and stability properties of the solutions of differential equations. Then, we obtain necessary and sufficient conditions to ensure the practical h-stability of non-autonomous semilinear systems. A practical stability approach is obtained.

The rest of this paper is organized as follows. In section 2, we recall some definitions and some tools used in the proofs. The main results are provided in Section 3. In Section 4, two examples are given to show the effectiveness of our result. Our conclusion is given in Section 5.

2 Mathematical Preliminaries

Throughout this paper we adopt the following notations:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : \ x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}_+^* = \{x \in \mathbb{R} : \ x > 0\}$.
- X denotes a real or complex Banach space with the norm $\|.\|_X$.
- $L(X)$ denotes the Banach space of all linear bounded operators P mapping X into X endowed with the norm

$$\|P\| = \sup\{\|Px\| : x \in X, \ \|x\| \leq 1\}.$$

- $D(A)$ denote the domain of the operator A.
- clM denotes the closure of a set M and I the identity operator.
- $\mathcal{BC}(J, \Omega)$ is the space of Ω-valued bounded functions endowed with the norm

$$\|h\|_\infty = \sup_{t \in I} |h(t)|.$$

We consider non-autonomous evolutions equations, defined in Banach space X, of the form:

$$\begin{cases} \dot{u} = A(t)u + f(t,u), \qquad t \geq t_0 \geq 0, \\ \\ u(t_0) = u_0 \end{cases} \tag{1}$$

where $u \in X$ is the system state, $\{A(t)\}_{t\geq 0}$ is a linear operator acting on X and $f(t,u) : \mathbb{R}_+ \times X \longrightarrow X$ is a nonlinear operator continuous in t and locally Lipschitz continuous in

x, uniformly in t on bounded intervals, that is, for every $\tilde{t} \geq 0$ and constant $c \geq 0$, there is a constant $K(c, \tilde{t})$, such that

$$\|f(t,u) - f(t,v)\|_X \leq K(c,\tilde{t})\|u - v\|_X$$

holds for all $u, v \in X$, with $\|u\| \leq c$, $\|v\| \leq c$, and $t \in [0, \tilde{t}]$. We assume that $A(t) : D(A(t)) \subset X \to X$ is a family of (possibly unbounded) linear operators depending on time with $clD(A(t)) = X$ for all t and generates a strongly continuous evolution operator $(R(t,s))_{t \geq s}$. That is, there exists a bounded linear operator $R(t,s) : X \to X$ satisfying the following properties.

(i) $R(s,s) = I$, $R(t,s) = R(t,r)R(r,s)$ for all $t \geq r \geq s \geq t_0$.

(ii) $(t,s) \to R(t,s)$ is strongly continuous for $t \geq s \geq t_0$.

(iii) For every $\nu \in D(A(s))$, $R(t,s)\nu$ is differentiable both in t and s satisfying

$$\frac{\partial}{\partial t} R(t,s)\nu = A(t)R(t,s)\nu,$$

$$\frac{\partial}{\partial s} R(t,s)\nu = -R(t,s)A(s)\nu.$$

Let $u(t) = u(t, u_0)$ be denoted by the solution of (1) at moment $t \geq t_0 \geq 0$ associated with an initial condition $u_0 \in X$ at $t = t_0$.
Let Δ the set of all the pair $(t,s) \in \mathbb{R}_+^2$ with $t \geq s$.
We also need to recall some notations about evolution family (see [17]).

Definition 1 *Let $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$. An evolution operator $R : \Delta \to L(X)$ is said to be uniformly h-stable if there exists $S \geq 1$, such that*

$$\|R(t,s)\| \leq S h(t) h(s)^{-1}, \ \textit{for all}\, t \geq s \geq 0.$$

Here $h(t)^{-1} = \dfrac{1}{h(t)}$.

Remark 1 *If $h(t) = e^{-\beta t}$, with $\beta > 0$, $t \geq 0$ then h-stability coincide with exponential stability, if $h(t) = \dfrac{1}{(1+t)^\gamma}$, with $\gamma > 0$, $t \geq 0$, then we coincide with uniform polynomial stability.*

Remark 2 *There is no connection between the concept of polynomial stability and exponential stability as shown in [9, 16]. If $A(t) = A$ is independent of t and generates the C_0- strong continuous semi-group $T(t)$, then the evolution operator $T(t) = R(t,0) = e^{tA}$, for all $t \geq 0$. For instance, if the operator $A(t)$, $t \geq 0$ is bounded on $\mathbb{R}_+$, then the semi-group evolution operator $R(t,s)$ satisfying the above conditions is always exists. Indeed, if $A(t)$, $t \geq 0$ is unbounded, then the evolution operator $R(t,s)$ exists provided additional conditions see [1, 2, 4, 20] for the details.*

The nominal system of (1) is described by

$$\dot{u} = A(t)u(t), \qquad u(t_0) = u_0, \qquad t \geq t_0 \geq 0, \tag{2}$$

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where $u \in X$ is the system state, $A(t)$ is a linear operator acting on a Banach space X. Note that, the solution of (2) through (t_0, u_0) can be represented as

$$u(t) = u(t, t_0, u_0) = R(t, t_0)u_0, \qquad t \geq t_0 \geq 0.$$

We present now the notion of uniform h-stability for system (2) which is recently introduced in [21].

Definition 2 *Let $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$. The system (2) is said to be globally uniformly h-stable if there exists $S \geq 1$, such that for all $t_0 \in \mathbb{R}_+$ and all $u_0 \in X$,*

$$\|u(t)\| \leq S\|u_0\|h(t)h(t_0)^{-1}, \quad \textit{for all}\, t \geq t_0 \geq 0.$$

Here $h(t)^{-1} = \dfrac{1}{h(t)}$.

Remark 3 *If $h(t) = e^{-\beta t}$, with $\beta > 0,\ t \geq 0$ then we recover the concept of uniform exponential stability and if $h(t) = \dfrac{1}{(1+t)^\gamma}$, with $\gamma > 0,\ t \geq 0$ then we recover the concept of uniform polynomial stability.*

Now, we recall some definitions about practical stability due to ([12]) when the origin will not be an equilibrium point of the semilinear non-autonomous system (1), that is, the trajectory tends towards a small neighborhood of the origin.

Definition 3 *The system (1) is said to be*
(i) practically stable if given (λ, η) with $0 < \lambda < \eta$, we have

$$\|u_0\| < \lambda \Longrightarrow \|u(t)\| < \eta, \qquad \forall t \geq t_0.$$

(ii) quasi-asymptotically stable (in the large) if for each $\varepsilon > 0$, $\alpha > 0$, there exists a positive number $T = T(\varepsilon, \alpha)$, such that

$$\|u_0\| \leq \alpha \Longrightarrow \|u(t)\| < \varepsilon, \quad \forall t \geq T.$$

(iii) practically asymptotically stable if (i) and (ii) hold at the same time with $\alpha = \lambda$.

A precise definition of the practical uniform h-stability is given as follows, which will be used in subsequent main result.

Definition 4 *([9]) Let $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$. The system (1) is called globally uniformly practically h-stable if there exist $S \geq 1$ and $r > 0$, such that for all $t_0 \in \mathbb{R}_+$ and all $u_0 \in X$,*

$$\|u(t)\| \leq S\|u_0\|h(t)h(t_0)^{-1} + r, \ \textit{for all}\, t \geq t_0 \geq 0. \tag{3}$$

In this case, the positive number r is called the convergence radius.

Remark 4 *The inequality (3) implies that $u(t)$ will be globally uniformly bounded by a small bound $r > 0$, that is, $\|u(t)\|$ will be small for sufficiently large t. Moreover, for some special cases of h, the h-stability coincides with known types of stability:*

- *If $h(t) = e^{-\lambda t}$ for $\lambda > 0$, then the system (1) is practically exponentially stable.*
- *If $h(t) = \dfrac{1}{(1+t)^\gamma}$ for $\gamma > 0$, the system (1) is practically polynomially stable.*

– *If $h(t)$ is a strictly decreasing function, such that $h(t)$ tends to 0 when $t \to +\infty$, then the system (1) is practically asymptotically stable.*

Now, we recall the Gronwall-Bellman Type Integral Inequality [11] which will be use in our work.

Lemma 1 *Let φ, λ and ϱ be non-negative continuous functions on $\mathbb{R}_+$, for which the following inequality holds*

$$\varphi(t) \leq c + \int_{t_0}^{t} \lambda(s)\varphi(s) + \varrho(s)ds, \qquad c \geq 0, \qquad \forall\, t \geq t_0. \tag{4}$$

Then,

$$\varphi(t) \leq c \exp\left(\int_{t_0}^{t} \lambda(s)ds\right) + \xi \exp\left(\int_{t_0}^{t} \lambda(s) + \frac{1}{\xi}\varrho(s)ds\right), \qquad \forall\, t \geq t_0, \qquad \forall\, \xi > 0.$$

We point out a new nonlinear integral inequality which can be used in applications as handy tools, and in the analysis of various problems in the theory of the partial differential equations (PDEs) and ordinary differential equations (ODEs).

Lemma 2 *Let $\varphi, a, b, \theta,$ and ϱ be non-negative continuous functions on $\mathbb{R}_+$, for which the following inequality holds*

$$\varphi(t) \leq a(t) + b(t)\int_{t_0}^{t} \theta(s)\varphi^{\alpha}(s) + \varrho(s)ds, \qquad \forall t \geq t_0, \quad \alpha \in]0,1[.$$

Then, for all $t \geq t_0$ and all $\zeta > 0$

$$\varphi(t) \leq a(t) + b(t)\int_{t_0}^{t} [\theta(s)\zeta^{\alpha-1}(\alpha a(s) + \zeta(1-\alpha)) + \varrho(s)] \exp\left(\int_{s}^{t} \alpha\zeta^{\alpha-1}\theta(\tau)b(\tau)d\tau\right) ds.$$

Proof. Let

$$w(t) = \int_{t_0}^{t} \theta(s)\varphi^{\alpha}(s) + \varrho(s)ds, \qquad t \geq t_0 \geq 0, \qquad 0 \leq \alpha < 1.$$

Then,

$$\varphi(t) \leq a(t) + b(t)w(t), \qquad t \geq t_0 \geq 0. \tag{5}$$

We have,

$$w'(t) = \theta(t)\varphi^{\alpha}(t) + \varrho(t) \leq \theta(t)(a(t) + b(t)w(t))^{\alpha} + \varrho(t).$$

Using that $\varepsilon \to \varepsilon^{\alpha}$ is concave if $\alpha \in (0,1)$, we deduce that $\varepsilon^{\alpha} \leq \zeta^{\alpha-1}(\alpha\varepsilon + \zeta(1-\alpha))$ for any $\zeta > 0$. Thus, for all $t \geq t_0$ and all $\zeta > 0$

$$w'(t) \leq \theta(t)\zeta^{\alpha-1}(\alpha(a(t) + b(t)w(t)) + \zeta(1-\alpha)) + \varrho(t).$$

It follows that, for all $t \geq t_0$ and all $\zeta > 0$

$$w'(t) \leq \alpha\zeta^{\alpha-1}\theta(t)b(t)w(t) + \theta(t)\zeta^{\alpha-1}(\alpha a(t) + \zeta(1-\alpha)) + \varrho(t).$$

By the Generalized Gronwall-Bellman Inequality ([8]), we deduce that for all $t \geq t_0$ and all $\zeta > 0$

$$w(t) \leq \int_{t_0}^{t} [\theta(s)\zeta^{\alpha-1}(\alpha a(s) + \zeta(1-\alpha)) + \varrho(s)] \exp\left(\int_{s}^{t} \alpha\zeta^{\alpha-1}\theta(\tau)b(\tau)d\tau\right) ds. \tag{6}$$

Substituting (6) into (5), yields the desired inequality. This completes the proof. □

3 Main Results

Pinto [22] obtained the characterization of uniform h-stability of linear systems in finite-dimensional spaces. Also, we have the same characterization for the linear non-autonomous systems (2) in what follows.

Lemma 3 *([9]) The system* (2) *is uniformly h-stable if and only if the evolution operator of the linear system (2) is uniformly h-stable.*

Remark 5 *Note that, if $A(t) = A$ and generates the C_0-strong continuous semi-group, then we have the following result for linear time-invariant systems.*

Example 1 *We consider the linear system*

$$\dot{u} = \frac{3 - 5v(t)}{(t+1)v(t)}u(t), \qquad u \in X, \qquad u(t_0) = u_0, \qquad t \geq t_0 \geq 0 \tag{7}$$

where $v : \mathbb{R}_+ \to \mathbb{R}_+^*$, such that $v(t) = (t+1)^3 + 1$. The evolution operator $R : \Delta \to L(X)$ is given by

$$R(t,s) = \frac{(s+1)^2 v(s)}{(t+1)^2 v(t)} I, \quad \forall (t,s) \in \Delta.$$

Then,

$$\|R(t,s)\| \leq \frac{(s+1)^2}{(t+1)^2}, \quad \forall (t,s) \in \Delta.$$

Therefore, by applying Lemma 3 the linear system (7) *is globally uniformly h-stable with $S = 1$ and $h(t) = \dfrac{1}{(1+t)^2}$, is a positive and continuous function.*

Let us consider the differential equation (1) when the origin may not be necessary an equilibrium point. In what follows, we will investigate the global existence of solutions and the practical uniform h-stability under different conditions on the perturbed term.
The existence of the trajectories of this class of models (1) has been studied by several authors. For instance, Pazy [20] contains an investigation of the general abstract model (1) where $A(t),\ t \geq 0$ is the generator of linear evolution operator on a abstract Banach space X, and the nonlinear operator f is continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals. It is shown in Pazy [20] that equation (1) has a unique local mild solution on some interval $[t_0, t_0 + \tau],\ \tau > 0,\ t_0 \geq 0$ given by

$$u(t) = R(t,t_0)u_0 + \int_{t_0}^{t} R(t,s) f(s,u(s)) ds, \quad t_0 \leq t \leq t_0 + \tau, \tag{8}$$

with $\tau > 0$ and $R(.,.)$ is the evolution operator of the linear system (2).
Moreover, if $T = t_0 + \tau < \infty$, then $\lim_{t \to T} \|u(t)\| = \infty$.

Next, we will study the global practical uniform h-stability of non-autonomous evolution equation (1) under conditions on the perturbed terms via Gronwall-Bellman Type Integral Inequality. This result can be considered as further extensions of Hammi and Hammami [11] in finite-dimensional spaces when $h(t) = e^{-\beta t},\ \beta > 0,\ t \geq 0$: practical exponential stability.

Theorem 1 *Assume that the nominal system (2) is globally uniformly h-stable and suppose that*

$$\|f(t,u)\| \leq \upsilon(t)\|u\| + \rho(t), \qquad \forall\, u \in X, \qquad \forall\, t \geq 0, \tag{9}$$

where υ and ρ are non-negative continuous functions on $\mathbb{R}_+$, with υ is integrable on $\mathbb{R}_+$ and

$$\int_0^t h(s)^{-1}\rho(s)ds \leq M_1, \qquad M_1 > 0, \qquad \forall\, t \geq 0. \tag{10}$$

Then,

(*i*) *The mild solution u of (1), if there exists, is defined on $[t_0, \infty)$.*

(*ii*) *The perturbed system (1) is globally uniformly practically h-stable.*

Proof. Let $T = t_0 + \tau$, $\tau > 0$ be such that $I = [t_0, T)$ is the maximal interval of existence of the mild solution of (1). By h-stability of system (2), there exist $S \geq 1$ and a positive bounded function h, such that

$$\|R(t,t_0)\| \leq Sh(t)h(t_0)^{-1}, \qquad \forall t \geq t_0. \tag{11}$$

From the condition (9) and (8), we get

$$\|u(t)\| \leq S\|u_0\|h(t)h(t_0)^{-1} + S\int_{t_0}^t h(t)h(s)^{-1}\Big(\upsilon(s)\|u(s)\| + \rho(s)\Big)ds.$$

Dividing both sides by $h(t)$ and denote $\varrho(t) = h(t)^{-1}\|u(t)\|$, yields

$$\varrho(t) \leq S\varrho(t_0) + \int_{t_0}^t S\upsilon(s)\varrho(s) + Sh(s)^{-1}\rho(s)ds.$$

Applying Lemma 1, we obtain

$$\begin{aligned}\varrho(t) &\leq S\varrho(t_0)\exp\left(\int_{t_0}^t S\upsilon(s)ds\right) + \xi\exp\left(S\int_{t_0}^t \upsilon(s)ds + \frac{S}{\xi}\int_{t_0}^t h(s)^{-1}\rho(s)ds\right)\\ &\leq SM_2\varrho(t_0) + \xi M_2\exp\left(\frac{S}{\xi}M_1\right), \quad \forall\xi > 0, \quad \forall t \geq t_0,\end{aligned}$$

where $M_2 = \exp\left(S\int_0^\infty \upsilon(s)ds\right)$. Then,

$$\|u(t)\| \leq SM_2\|u_0\|h(t)h(t_0)^{-1} + \xi M_2\|h\|_\infty\exp\left(\frac{S}{\xi}M_1\right), \quad \forall\xi > 0.$$

Therefore, the solution $u(t, t_0, u_0)$ verifies the estimation

$$\|u(t)\| \leq S_1\|u_0\|h(t)h(t_0)^{-1} + r, \tag{12}$$

with $S_1 = SM_2$ and $r = \xi M_2\|h\|_\infty\exp\left(\frac{S}{\xi}M_1\right)$. Thus, using Theorem 1.4 in [20] the solution $u(t)$ of (1) is defined on $[t_0, \infty)$. From (12), we can see that the system (1) is globally uniformly h-stable. The proof is complete. □

Remark 6 *If we replace the condition* (10) *by* $\rho(s) \leq h(s)e^{-\beta s}$, *for* $s \in \mathbb{R}_+$, *and* $\beta > 0$, *we obtain the global practical* h*-stability of system* (1). *Moreover, one can see that the theorem 1 generalizes the one's given in [8] in finite-dimensional spaces. Also, Theorem 1 in [9] can be regarded as a special case of Theorem 1 when* $\rho(t) = 0$.

Now, we obtain some result about the practical h-stability of non-autonomous evolution equation (1) in the following corollaries.

Corollary 1 *Suppose that the system* (2) *is globally uniformly* h*-stable and the map* f *verifies the following condition:*

$$\|f(t,u)\| \leq \rho(t), \qquad \forall\, t \geq 0,$$

where ρ *is non-negative continuous on* $\mathbb{R}_+$, *and satisfies the condition* (10). *Then, the system* (1) *is globally uniformly practically* h*-stable.*

Corollary 2 *We consider the perturbed system in Banach space* X :

$$\dot{u} = Au(t) + B(t)u + \varsigma(t)I, \qquad u(t_0) = u_0, \qquad t \geq t_0 \geq 0. \tag{13}$$

where A *is a linear operator in* X *with domain* $D(A)$, *generating a strongly continuous* $h-$ *stable semigroup* $T(t)$, *that is,*

$$A = \lim_{h \longrightarrow 0} \frac{T(h) - I}{h}$$

in the strong topology and $B(t)$, $t \geq 0$ *is a linear bounded operator on* X. ς *is a non-negative continuous function on* $\mathbb{R}_+$, *such that* $h^{-1}\varsigma$ *is integrable on* $\mathbb{R}_+$, *and* $B(t)$ *satisfies the following condition:*

$$\int_0^{+\infty} \|B(s)\| ds < \infty, \qquad \forall\, t \geq 0.$$

Then, the system (13) *is globally uniformly practically* h*-stable.*

Next, we will show the global existence, uniqueness of solutions and the practical uniform h-stability of the system (1) under a general condition on the right-hand side of the system via nonlinear integral Gronwall type inequality.

Theorem 2 *Let* $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ *be a decreasing function. Assume that the system* (2) *is globally uniformly* h*-stable and suppose that there exists* $\alpha \in]0,1[$, *such that*

$$\|f(t,u)\| \leq \sigma(\upsilon(t)\|u\|^\alpha + \rho(t)), \quad \forall\, u \in X, \quad \forall\, t \geq 0, \tag{14}$$

where

- $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0, +\infty[$ *with continuous nonincreasing first derivative* σ' *on* $]0, +\infty[$.
- υ *and* ρ *are continuous non-negative on* $\mathbb{R}_+$, *and there exist positive constants* M *and* N, *such that*

$$\int_0^{+\infty} \sigma'(\rho(s))\upsilon(s) ds \leq M < +\infty, \qquad \int_0^{+\infty} \sigma(\rho(s)) ds \leq N < +\infty. \tag{15}$$

Then,

(i) *The mild solution u of* (1), *if there exists, is defined on* $[t_0, \infty)$.

(ii) *The perturbed system* (1) *is globally uniformly practically h-stable.*

Proof. Let $T = t_0 + \tau,\ \tau > 0$ be such that $I = [t_0, T)$ is the maximal interval of existence of the mild solution of (1). Using (8) and (11), we get that for every $t \in I$,

$$\|u(t)\| \leq S\|u_0\|h(t)h(t_0)^{-1} + Sh(t)\int_{t_0}^{t} h(s)^{-1}\sigma((\upsilon(s)\|u(s)\|^{\alpha} + \rho(s)))ds. \qquad (16)$$

Applying the mean value Theorem for the function σ, then for any $y \geq x > 0$, there exists $k \in]x, y[$, such that

$$\sigma(y) - \sigma(x) = \sigma'(k)(y - x) \leq \sigma'(x)(y - x).$$

We obtain,

$$\sigma((\upsilon(s)\|u(s)\|^{\alpha} + \rho(s))) \leq \sigma'(\rho(s))\upsilon(s)\|u(s)\|^{\alpha} + \sigma(\rho(s)). \qquad (17)$$

Hence, from (16) and (17), one gets that

$$\|u(t)\| \leq S\|u_0\|h(t)h(t_0)^{-1} + Sh(t)\int_{t_0}^{t} [\sigma'(\rho(s))\upsilon(s)h(s)^{-1}\|u(s)\|^{\alpha} + \sigma(\rho(s))h(s)^{-1}]ds. \qquad (18)$$

Using Lemma 2, we obtain from (18)

$$\begin{aligned} \|u(t)\| &\leq S\|u_0\|h(t)h(t_0)^{-1} + Sh(t)\int_{t_0}^{t} \sigma'(\rho(s))h(s)^{-1}\upsilon(s)\zeta^{\alpha-1}[\alpha Sh(s)h(t_0)^{-1}\|u_0\| \\ &+ \zeta(1-\alpha)] + \sigma(\rho(s))h(s)^{-1})\exp\left(\int_{\tau}^{t} \alpha\zeta^{\alpha-1}S\sigma'(\rho(\tau))\upsilon(\tau)d\tau\right)ds, \quad \forall \zeta > 0. \end{aligned}$$

Then,

$$\|u(t)\| \leq (S + S^2\zeta^{\alpha-1}\alpha M e^{\alpha S\zeta^{\alpha-1}M})\|u_0\|h(t)h(t_0)^{-1} + (S\zeta^{\alpha}(1-\alpha)M + SN)e^{\alpha S\zeta^{\alpha-1}M}, \quad \forall \zeta > 0. \qquad (19)$$

Hence, using Theorem 1.4 in [20] we have the solution $u(t)$ exists in $[t_0, \infty)$. From (19), we can see that the system (1) is globally practically uniformly practically h-stable. This ends the proof. □

Remark 7 *Theorems 1 and 2 are hold for practical exponential stability:* $h(t) = e^{-\beta t}$, *practical polynomial stability:* $h(t) = \dfrac{1}{(1+t)^{\gamma}}$, *and practical logarithmic stability* $h(t) = \dfrac{1}{1 + \ln(1+t)}$. *for all* $t \in \mathbb{R}_+$ *and* $\gamma,\ \beta > 0$.

4 Examples

In this section, two examples are given to illustrate the effectiveness and advantages of the results obtained

Example 2 *Consider the following non-autonomous evolution equation:*

$$\dot{u} = tAu + \frac{2u}{1+\|u\|^2}\ln(e^{-t}\sqrt{\|u\|} + \frac{1}{1+t^2} + 1), \qquad t \geq t_0 \geq 0, \tag{20}$$

where A is a bounded linear operator on a Banach space X. We suppose that A is exponentially stable, that is, there exist $M, c > 0$, such that

$$\|e^{At}\| \leq Me^{-ct}, \quad t \geq 0.$$

System (20) can be written as system (1), with

$$A(t) = tA$$

and

$$f(t,u) = \frac{2u}{1+\|u\|^2}\ln(e^{-t}\sqrt{\|u\|} + \frac{1}{1+t^2} + 1).$$

The evolution operator generated by $A(.) \in L(X)$,

$$R(t,s) = e^{A(t^2-s^2)/2}, \quad \forall t \geq s \geq 0,$$

satisfies

$$\|R(t,s)\| \leq Me^{-\frac{c}{2}(t^2-s^2)}, \quad \forall t \geq s \geq 0.$$

Then, the evolution operator of the unperturbed system is h-stable with $h(t) = e^{-\frac{c}{2}t^2} \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function. We have,

$$\|f(t,u)\| \leq \ln(e^{-t}\sqrt{\|u\|} + \frac{1}{1+t^2} + 1).$$

Hence, the perturbation term satisfies condition (14) of Theorem 2 with $\sigma(x) = \ln(x+1)$ is a differentiable increasing function on $]0, +\infty[$ with continuous nonincreasing first derivative, $\upsilon(t) = e^{-t}$, $\alpha = \frac{1}{2}$ and $\rho(t) = \frac{1}{1+t^2}$. All the assumptions of Theorem 2 are satisfied, then the system (20) is globally uniformly practically h-stable.

Example 3 *We Consider the perturbed heat equation with Dirichlet boundary conditions:*

$$\begin{cases} \dfrac{\partial u(\zeta,t)}{\partial t} = \dfrac{\partial^2 u(\zeta,t)}{\partial^2 \zeta} - \dfrac{2}{(1+t)(2+t)}u(\zeta,t) + \dfrac{1}{1+t^2}u(\zeta,t) + e^{-t}, \\ \\ u(0,t) = 0 = u(\pi,t), \quad t \geq 0, \quad u(\zeta,0) = u_0(\zeta) \end{cases} \tag{21}$$

$u(\zeta,t)$ represents the temperature at position $\zeta \in [0,\pi]$ at time $t \geq 0$ and u_0 represents the initial temperature profile. The two boundary conditions state that there is no heat flow at the boundary. The partial differential equation can be formulated to an abstract differential equation on $X = L^2(0,\pi)$ of the form

$$\dot{u} = A(t)u + f(t,u),$$

where the family of operator $\{A(t)\}_{t\geq 0}$ is defined as $A(t)v = Av - \dfrac{2}{(1+t)(2+t)}v$ where the operator $A = \dfrac{\partial^2}{\partial^2\zeta}$ with

$$D(A) = \{v \in L^2(0,\pi), \frac{\partial v}{\partial \zeta} \text{ are absolutely continuous}, \frac{\partial^2 v}{\partial^2 \zeta} \in L^2(0,\pi) \text{ and } v(0) = 0 = v(\pi)\},$$

and

$$f(t,u(\zeta,t)) = \frac{1}{1+t^2}u(\zeta,t) + e^{-t}.$$

Noting that A *is a diagonal operator with the set of eigenvalues* $\{-n^2\}_{n\in\mathbb{N}}$ *and the normal eigenvector* $\phi_n(\zeta) = \sqrt{\frac{2}{\pi}}\sin(n\zeta), \quad n \geq 0.$

From [20], we can verify that $A(t)$ *generates an evolution operator* $(R(t,s))_{t\geq s}$ *generated by* $A(t)$ *of the form:*

$$R(t,s) = e^{(t-s)A}\exp\left(\int_s^t -\frac{2}{(1+r)(2+r)}dr\right), \quad \forall t \geq s \geq 0.$$

Then, for any $v \in X$*, we obtain*

$$R(t,s)v = \sum_{n=1}^{\infty}(\frac{2+s}{1+t})^2 e^{-n^2(t-s)} < v, \phi_n > \phi_n, \quad \forall t \geq s \geq 0.$$

Therefore, by applying Lemma 3, we have the linear system (2) *is globally uniformly h-stable with* $S = 1$ *and* $h(t) = \left(\frac{2+t}{1+t}\right)^2$*, is a positive and continuous function.*

Moreover, the perturbation $f(t,u)$ *satisfies the condition* (9) *with* $\upsilon(t) = \frac{1}{1+t^2}$ *and* $\rho(t) = e^{-t}$*, that are non-negative and continuous functions on* $\mathbb{R}_+$*, where* $\upsilon \in L^1(\mathbb{R}_+)$ *and the condition* (10) *holds. Consequently, by applying Theorem 1, we deduce that the system* (21) *is globally uniformly practically* $\left(\frac{2+t}{1+t}\right)^2$*-stable.*

Figure 1 *shows the evolution of the state* $u(\zeta,t)$ *of the system (21) with initial state* $x_0(\zeta) = \sin(\pi\zeta)$.

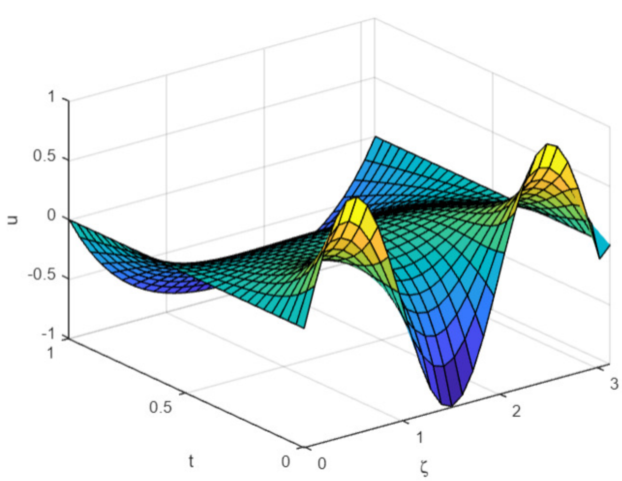

Figure 1: *Evolution of the state* $u(\zeta,t)$ *of the system (21).*

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5 Conclusion

The aim of this paper solves retention stability property of the non-autonomous evolution equations in Banach spaces, resorting to certain nonlinear integral Gronwall type inequalities based on the choice of the upper perturbation. Sufficient conditions are given to study the global existence of solutions and the practical h-stability in Banach spaces under some conditions. This result can be viewed as a generalization of practical exponential stability. Two examples are included to illustrate the effectiveness and advantages.

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