

REFINED INEQUALITIES FOR THE NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

PINTU BHUNIA, SUVENDU JANA AND KALLOL PAUL

ABSTRACT. We present some new upper and lower bounds for the numerical radius of bounded linear operators on a complex Hilbert space and show that the bounds are stronger than the existing ones. In particular, we prove that if A is a bounded linear operator on a complex Hilbert space \mathcal{H} and if $\Re(A)$, $\Im(A)$ are the real part, the imaginary part of A , respectively, then

$$w(A) \geq \frac{\|A\|}{2} + \frac{1}{2\sqrt{2}} \left| \|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\| \right|$$

and

$$w^2(A) \geq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{4} \left| \|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2 \right|,$$

where $w(\cdot)$ and $\|\cdot\|$ denote the numerical radius and the operator norm, respectively. Further, we obtain refinements of the inequalities for the numerical radius of the product of two operators. Finally, as an application of the second inequality mentioned above, we obtain an improvement of upper bound for the numerical radius of the commutators of operators.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the \mathcal{C}^* -algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let A^* be the adjoint of A and $|A| = (A^*A)^{\frac{1}{2}}$. Let $\Re(A)$ and $\Im(A)$ denote the real part and the imaginary part of A , respectively, i.e., $\Re(A) = \frac{1}{2}(A + A^*)$ and $\Im(A) = \frac{1}{2i}(A - A^*)$. Let $\|A\|$ and $w(A)$ denote the operator norm and the numerical radius of A , respectively. Recall that

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well known that the numerical radius, i.e., $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$, we have

$$(1.1) \quad \frac{1}{2} \|A\| \leq w(A) \leq \|A\|.$$

The inequalities in (1.1) are sharp, $w(A) = \frac{\|A\|}{2}$ if $A^2 = 0$ and $w(A) = \|A\|$ if $A^*A = AA^*$. In [16], Kittaneh improved the inequalities in (1.1) and proved that

$$(1.2) \quad \frac{1}{4} \|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|.$$

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1 Also in [17], Kittaneh refined the second inequality in (1.1) to prove that

$$2 \quad (1.3) \quad w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}).$$

3 The Aluthge transform of A , denoted as \tilde{A} , is defined as $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, where U is the partial isometry
4 appearing in the polar decomposition $A = U|A|$ of A with $\ker A = \ker U$. It follows from the definition
5 of \tilde{A} that $\|\tilde{A}\| \leq \|A\|$. Also, $w(\tilde{A}) \leq w(A)$. In [20], Yamazaki proved that if $A \in \mathcal{B}(\mathcal{H})$, then
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$$7 \quad (1.4) \quad w(A) \leq \frac{1}{2}(\|A\| + w(\tilde{A})).$$

8 He also proved that (1.4) refine (1.3). For more recent results on the refinement of the numerical radius
9 inequalities of bounded linear operators we refer the readers to see [2, 3, 4, 5, 9, 10]. Next we note
10 some well known inequalities for the numerical radius of the product of two operators. Dragomir [12]
11 proved that if $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, then
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$$13 \quad (1.5) \quad w^r(B^*A) \leq \frac{1}{2}(\| |A|^{2r} + |B|^{2r} \|).$$

14 In [15], Hedarbeygi et al. established a refinement of (1.5) and proved that if $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$,
15 then
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$$17 \quad (1.6) \quad w^{2r}(B^*A) \leq \frac{1}{2}w^r(|B|^2|A|^2) + \frac{1}{4}\| |B|^{4r} + |A|^{4r} \|.$$

18 In this paper, we obtain several upper and lower bounds for the numerical radius of bounded linear
19 operators on \mathcal{H} , which improve on the existing ones in (1.1), (1.2), (1.3) and (1.4). Further, we obtain
20 upper bounds for the numerical radius of the product of two operators, which refine (1.5) and (1.6). As
21 a direct application of the results obtained here, we obtain an improvement of an existing upper bound
22 for the numerical radius of the commutators of operators.
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24 2. Numerical radius inequalities of operators

25 We start our work with the following improvement of the first inequality in (1.1).
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27 **Theorem 2.1.** *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$28 \quad w(A) \geq \frac{1}{2}\|A\| + \frac{1}{2\sqrt{2}} \left| \|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\| \right|.$$

29 *Proof.* Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by the Cartesian decomposition of A , we have
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$$31 \quad \begin{aligned} 32 \quad |\langle Ax, x \rangle| &= \sqrt{\langle \Re(A)x, x \rangle^2 + \langle \Im(A)x, x \rangle^2} \\ 33 &\geq \frac{1}{\sqrt{2}}(|\langle \Re(A)x, x \rangle| + |\langle \Im(A)x, x \rangle|) \\ 34 &\geq \frac{1}{\sqrt{2}} |\langle (\Re(A) \pm \Im(A))x, x \rangle|. \end{aligned}$$

35 Taking supremum over all $x \in \mathcal{H}$, $\|x\| = 1$, we get
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$$37 \quad w(A) \geq \frac{1}{\sqrt{2}} \|\Re(A) \pm \Im(A)\|.$$

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1 Thus,

$$2 \quad w(A) \geq \frac{1}{\sqrt{2}} \max\{\|\Re(A) + \Im(A)\|, \|\Re(A) - \Im(A)\|\}.$$

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4 Now,

$$5 \quad \frac{1}{\sqrt{2}} \max\{\|\Re(A) + \Im(A)\|, \|\Re(A) - \Im(A)\|\}$$

$$6 \quad = \frac{1}{\sqrt{2}} \left\{ \frac{\|\Re(A) + \Im(A)\| + \|\Re(A) - \Im(A)\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2} \right\}$$

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$$8 \quad \geq \frac{1}{\sqrt{2}} \left\{ \frac{\|(1+i)A^*\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2} \right\}$$

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$$10 \quad = \frac{1}{\sqrt{2}} \left\{ \frac{\|(1+i)A^*\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2} \right\}$$

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$$12 \quad = \frac{\|A\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2\sqrt{2}}.$$

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17 This completes the proof. □

18 **Remark 2.2.** (i) Clearly, the inequality in Theorem 2.1 refines the first inequality in (1.1).

19 (ii) If $w(A) = \frac{\|A\|}{2}$, then $\|\Re(A) + \Im(A)\| = \|\Re(A) - \Im(A)\|$, but the converse is not necessarily true.

20 For example, if A is (non-zero) self-adjoint, then $\|\Re(A) + \Im(A)\| = \|\Re(A) - \Im(A)\|$ and $w(A) \neq \frac{\|A\|}{2}$.

21 (iii) In [8, Th. 2.1], Bhunia and Paul proved that

$$22 \quad (2.1) \quad w(A) \geq \frac{1}{2} \|A\| + \frac{1}{2} |\|\Re(A)\| - \|\Im(A)\||.$$

23 Clearly, if $A \in \mathcal{B}(\mathcal{H})$ is (non-zero) self-adjoint, then

$$24 \quad \frac{1}{2} \|A\| + \frac{1}{2} |\|\Re(A)\| - \|\Im(A)\|| > \frac{\|A\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2\sqrt{2}}$$

25 and if $A = \begin{pmatrix} 1+i & 0 \\ 0 & 0 \end{pmatrix}$, then

$$26 \quad \frac{1}{2} \|A\| + \frac{1}{2} |\|\Re(A)\| - \|\Im(A)\|| < \frac{\|A\|}{2} + \frac{|\|\Re(A) + \Im(A)\| - \|\Re(A) - \Im(A)\||}{2\sqrt{2}}.$$

27 Thus, we conclude that the inequality in Theorem 2.1 and the inequality (2.1) are not comparable, in
28 general.

29 Next we prove the following inequality which improve on the first inequality in (1.2).

30 **Theorem 2.3.** If $A \in \mathcal{B}(\mathcal{H})$, then

$$31 \quad w^2(A) \geq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{4} |\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|.$$

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1 *Proof.* As the proof of Theorem 2.1, we have

$$2 \quad w^2(A) \geq \frac{1}{2} \max\{\|\Re(A) + \Im(A)\|^2, \|\Re(A) - \Im(A)\|^2\}.$$

4 Now,

$$\begin{aligned} 5 & \frac{1}{2} \max\{\|\Re(A) + \Im(A)\|^2, \|\Re(A) - \Im(A)\|^2\} \\ 6 & = \frac{1}{2} \left\{ \frac{\|\Re(A) + \Im(A)\|^2 + \|\Re(A) - \Im(A)\|^2}{2} + \frac{|\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|}{2} \right\} \\ 7 & \geq \frac{1}{2} \left\{ \frac{\|(\Re(A) + \Im(A))^2 + (\Re(A) - \Im(A))^2\|}{2} + \frac{|\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|}{2} \right\} \\ 8 & = \frac{\|A^*A + AA^*\|}{4} + \frac{|\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|}{4}. \end{aligned}$$

14 This completes the proof. □

16 **Remark 2.4.** (i) Clearly, the inequality in Theorem 2.3 refines the first inequality in (1.2).

17 (ii) If $w^2(A) = \frac{\|A^*A + AA^*\|}{4}$, then $\|\Re(A) + \Im(A)\| = \|\Re(A) - \Im(A)\|$, but the converse is not necessarily

19 true. (iii) In [8, Th. 2.9], Bhunia and Paul proved that if $A \in \mathcal{B}(\mathcal{H})$, then

$$21 \quad (2.2) \quad w^2(A) \geq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{2} |\|\Re(A)\|^2 - \|\Im(A)\|^2|.$$

23 Considering the same examples as in Remark 2.2 (iii), we conclude that the inequality in Theorem 2.3 and the inequality (2.2) are not comparable, in general.

25 Next we obtain an upper bound for the numerical radius which improve on (1.4). For this purpose, first we need the following lemma, based on polarization principle.

28 **Lemma 2.5.** Let $A \in \mathcal{B}(\mathcal{H})$, and let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} 30 \quad \langle Ax, x \rangle & = \frac{\langle A(x+y), x+y \rangle - \langle A(x-y), x-y \rangle}{4} \\ 31 & \quad + i \frac{\langle A(x+iy), x+iy \rangle - \langle A(x-iy), x-iy \rangle}{4}. \end{aligned}$$

35 **Theorem 2.6.** If $A \in \mathcal{B}(\mathcal{H})$, then

$$36 \quad w(A) \leq \frac{1}{2} \left(\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A} + \tilde{A}|A|) \right)^{\frac{1}{2}}.$$

40 *Proof.* First we note that

$$41 \quad w(A) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}A)\| = \sup_{\theta \in \mathbb{R}} w(\Re(e^{i\theta}A)).$$

1 Let $A = U|A|$ be the polar decomposition of A . Then, by lemma 2.5, we have

$$\begin{aligned} 2 & \langle e^{i\theta}Ax, x \rangle = \langle e^{i\theta}|A|x, U^*x \rangle \\ 3 & = \frac{1}{4} \left(\langle |A|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x \rangle - \langle |A|(e^{i\theta}x - U^*x), e^{i\theta}x - U^*x \rangle \right) \\ 4 & + \frac{i}{4} \left(\langle |A|(e^{i\theta}x + iU^*x), e^{i\theta}x + iU^*x \rangle - \langle |A|(e^{i\theta}x - iU^*x), e^{i\theta}x - iU^*x \rangle \right). \end{aligned}$$

7 Therefore, we have

$$\begin{aligned} 8 & \operatorname{Re} \langle e^{i\theta}Ax, x \rangle = \frac{1}{4} (\langle |A|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x \rangle - \langle |A|(e^{i\theta}x - U^*x), e^{i\theta}x - U^*x \rangle) \\ 9 & \leq \frac{1}{4} \langle |A|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x \rangle \\ 10 & = \frac{1}{4} \langle (e^{-i\theta} + U)|A|(e^{i\theta} + U^*)x, x \rangle \\ 11 & \leq \frac{1}{4} \| (e^{-i\theta} + U)|A|(e^{i\theta} + U^*) \| \\ 12 & = \frac{1}{4} \| |A|^{\frac{1}{2}}(e^{i\theta} + U^*)(e^{-i\theta} + U)|A|^{\frac{1}{2}} \| \quad (\|A^*A\| = \|AA^*\|) \\ 13 & = \frac{1}{4} \| 2|A| + e^{i\theta}\tilde{A} + e^{-i\theta}\tilde{A}^* \| \\ 14 & = \frac{1}{2} \| |A| + \Re(e^{i\theta}\tilde{A}) \| \\ 15 & = \frac{1}{2} \left\| \left(|A| + \Re(e^{i\theta}\tilde{A}) \right)^2 \right\|^{\frac{1}{2}} \\ 16 & = \frac{1}{2} \left\| |A|^2 + (\Re(e^{i\theta}\tilde{A}))^2 + |A|\Re(e^{i\theta}\tilde{A}) + \Re(e^{i\theta}\tilde{A})|A| \right\|^{\frac{1}{2}} \\ 17 & = \frac{1}{2} \left\| |A|^2 + (\Re(e^{i\theta}\tilde{A}))^2 + \Re(e^{i\theta}(|A|\tilde{A} + \tilde{A}|A|)) \right\|^{\frac{1}{2}} \\ 18 & \leq \frac{1}{2} \left(\|A\|^2 + \|\Re(e^{i\theta}\tilde{A})\|^2 + \|\Re(e^{i\theta}(|A|\tilde{A} + \tilde{A}|A|))\| \right)^{\frac{1}{2}} \\ 19 & \leq \frac{1}{2} \left(\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A} + \tilde{A}|A|) \right)^{\frac{1}{2}}. \end{aligned}$$

20 Since $\operatorname{Re} \langle e^{i\theta}Ax, x \rangle = \langle \Re(e^{i\theta}A)x, x \rangle$, so taking supremum over $\theta \in \mathbb{R}$, we get

$$21 \quad w(A) \leq \frac{1}{2} \left(\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A} + \tilde{A}|A|) \right)^{\frac{1}{2}},$$

22 as desired. □

23 **Remark 2.7.** From [14] we have if $A, X \in \mathcal{B}(\mathcal{H})$, then $w(A^*X + XA) \leq 2\|A\|w(X)$ and so

$$\begin{aligned} 24 & \|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A} + \tilde{A}|A|) \leq \|A\|^2 + w^2(\tilde{A}) + 2\|A\|w(\tilde{A}) \\ 25 & = \|A\|^2 + w^2(\tilde{A}) + 2\|A\|w(\tilde{A}) \\ 26 & = (\|A\| + w(\tilde{A}))^2. \end{aligned}$$

1 Therefore,

$$2 \quad w(A) \leq \frac{1}{2} (\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A} + \tilde{A}|A|))^{1/2} \leq \frac{1}{2} (\|A\| + w(\tilde{A})).$$

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4 Thus the inequality in Theorem 2.6 is stronger than the inequality (1.4).

5 Next we need the following two lemmas, first one is known as Heinz inequality and second one is
6 known as Buzano's inequality.

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8 **Lemma 2.8.** ([18]) Let $A \in \mathcal{B}(\mathcal{H})$. Then, for all $x, y \in \mathcal{H}$,

$$9 \quad |\langle Ax, y \rangle|^2 \leq \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} y, y \rangle,$$

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11 for all $\alpha \in [0, 1]$.

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13 **Lemma 2.9.** ([11]) Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$14 \quad |\langle a, e \rangle \langle e, b \rangle| \leq \frac{1}{2} (|\langle a, b \rangle| + \|a\| \|b\|).$$

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16 Now we are in a position to prove an upper bound which improves on both the upper bounds in (1.2)
17 and (1.3).

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19 **Theorem 2.10.** If $A \in \mathcal{B}(\mathcal{H})$, then

$$20 \quad w^2(A) \leq \frac{1}{4} \left\| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \right\| + \frac{1}{2} w \left(|A^*|^{2(1-\alpha)} |A|^{2\alpha} \right),$$

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22 for all $\alpha \in [0, 1]$.

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24 *Proof.* Let $x \in \mathcal{H}$ with $\|x\| = 1$, then by Lemma 2.8, we have

$$\begin{aligned} 25 \quad |\langle Ax, x \rangle|^2 &\leq \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} x, x \rangle \\ 26 &= \langle |A|^{2\alpha} x, x \rangle \langle x, |A^*|^{2(1-\alpha)} x \rangle \\ 27 &\leq \frac{1}{2} \| |A|^{2\alpha} x \| \| |A^*|^{2(1-\alpha)} x \| + \frac{1}{2} |\langle |A|^{2\alpha} x, |A^*|^{2(1-\alpha)} x \rangle| \quad (\text{by Lemma 2.9}) \\ 28 &= \frac{1}{2} \langle |A|^{4\alpha} x, x \rangle^{1/2} \langle |A^*|^{4(1-\alpha)} x, x \rangle^{1/2} + \frac{1}{2} |\langle |A|^{2\alpha} x, |A^*|^{2(1-\alpha)} x \rangle| \\ 29 &\leq \frac{1}{4} \left(\langle |A|^{4\alpha} x, x \rangle + \langle |A^*|^{4(1-\alpha)} x, x \rangle \right) + \frac{1}{2} |\langle |A|^{2\alpha} x, |A^*|^{2(1-\alpha)} x \rangle| \\ 30 &= \frac{1}{4} \langle (|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) x, x \rangle + \frac{1}{2} |\langle |A^*|^{2(1-\alpha)} |A|^{2\alpha} x, x \rangle| \\ 31 &\leq \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w \left(|A^*|^{2(1-\alpha)} |A|^{2\alpha} \right). \end{aligned}$$

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38 Therefore, taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$39 \quad w^2(A) \leq \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w \left(|A^*|^{2(1-\alpha)} |A|^{2\alpha} \right),$$

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42 as desired. □

In particular, considering $\alpha = \frac{1}{2}$ in Theorem 2.10, we get the following inequality [7, Cor. 2.6]

$$(2.3) \quad w^2(A) \leq \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} w(|A^*||A|).$$

Next, considering the minimum over $\alpha \in [0, 1]$ in Theorem 2.10, we get the following corollary.

Corollary 2.11. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^2(A) \leq \min_{0 \leq \alpha \leq 1} \left\{ \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w(|A^*|^{2(1-\alpha)} |A|^{2\alpha}) \right\}.$$

Clearly, we have

$$\min_{0 \leq \alpha \leq 1} \left\{ \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w(|A^*|^{2(1-\alpha)} |A|^{2\alpha}) \right\} \leq \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} w(|A^*||A|).$$

In order to appreciate the inequality in Corollary 2.11 we give the following example.

Example 2.12. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by simple calculation, we have

$$|A|^{4\alpha} + |A^*|^{4(1-\alpha)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + 16^{1-\alpha} & 0 \\ 0 & 0 & 16^\alpha \end{pmatrix} \text{ and } |A^*|^{2(1-\alpha)} |A|^{2\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4^{1-\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| = \begin{cases} 1 + 16^{1-\alpha} & \text{if } 0 \leq \alpha \leq r_0 \\ 16^\alpha & \text{if } r_0 \leq \alpha \leq 1, \end{cases}$$

where $16^{r_0} = \frac{1+\sqrt{65}}{2}$. Also,

$$w(|A^*|^{2(1-\alpha)} |A|^{2\alpha}) = \begin{cases} 4^{1-\alpha} & \text{if } 0 \leq \alpha < 1 \\ 1 & \text{if } \alpha = 1. \end{cases}$$

Now,

$$\min_{0 \leq \alpha \leq r_0} \left\{ \frac{1 + 16^{1-\alpha}}{4} + \frac{4^{1-\alpha}}{2} \right\} = \left\{ \frac{33 + \sqrt{65}}{4(1 + \sqrt{65})} + \frac{4\sqrt{2}}{2\sqrt{1 + \sqrt{65}}} \right\} \approx 2.0724.$$

$$\min_{r_0 \leq \alpha \leq 1} \left\{ \frac{16^\alpha}{4} + \frac{4^{1-\alpha}}{2} \right\} = \left\{ \frac{1 + \sqrt{65}}{8} + \frac{4\sqrt{2}}{2\sqrt{1 + \sqrt{65}}} \right\} \approx 2.0724.$$

Thus,

$$\min_{0 \leq \alpha \leq 1} \left\{ \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w(|A^*|^{2(1-\alpha)} |A|^{2\alpha}) \right\} \approx 2.0724.$$

1 Also, we have

$$2 \quad \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} w(|A^*||A|) = \frac{9}{4} = 2.25.$$

3 Hence, for the matrix A ,

$$4 \quad \min_{0 \leq \alpha \leq 1} \left\{ \frac{1}{4} \| |A|^{4\alpha} + |A^*|^{4(1-\alpha)} \| + \frac{1}{2} w(|A^*|^{2(1-\alpha)} |A|^{2\alpha}) \right\} < \frac{1}{4} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} w(|A^*||A|).$$

5 Thus the inequality in Corollary 2.11 is a refinement of the inequality (2.3) and hence it also refines
6 the inequalities (1.2) and (1.3).

7 3. Numerical radius inequalities of product of operators

8 We begin this section with the following two lemmas, first one can be found in [19, p. 20] and second
9 one is obvious.

10 **Lemma 3.1.** Let $A \in \mathcal{B}(\mathcal{H})$ be positive and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then for all $r \geq 1$, we have

$$11 \quad \langle Ax, x \rangle^r \leq \langle A^r x, x \rangle.$$

12 **Lemma 3.2.** For all $a, b \in \mathbb{R}$, we have $|a + b| \leq \sqrt{2}|a + ib|$.

13 We now prove an improvement of (1.5).

14 **Theorem 3.3.** Let $A, B \in \mathcal{B}(\mathcal{H})$. Then, for all $r \geq 1$,

$$15 \quad w^{2r}(B^*A) \leq \frac{1}{2} w^2(|A|^{2r} + i|B|^{2r}).$$

16 *Proof.* Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have,

$$\begin{aligned} 17 \quad |\langle B^*Ax, x \rangle|^{2r} &= |\langle Ax, Bx \rangle|^{2r} \\ 18 &\leq \|Ax\|^{2r} \|Bx\|^{2r} \\ 19 &\leq \langle |A|^{2r}x, x \rangle \langle |B|^{2r}x, x \rangle \text{ (by Lemma 3.1)} \\ 20 &\leq \frac{1}{4} (\langle |A|^{2r}x, x \rangle + \langle |B|^{2r}x, x \rangle)^2 \\ 21 &\leq \frac{1}{2} |\langle |A|^{2r}x, x \rangle + i \langle |B|^{2r}x, x \rangle|^2 \text{ (by Lemma 3.2)} \\ 22 &= \frac{1}{2} |\langle (|A|^{2r} + i|B|^{2r})x, x \rangle|^2 \\ 23 &\leq \frac{1}{2} w^2(|A|^{2r} + i|B|^{2r}). \end{aligned}$$

24 Taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$25 \quad w^{2r}(B^*A) \leq \frac{1}{2} w^2(|A|^{2r} + i|B|^{2r}),$$

26 as required. □

1 It is easy to verify that, $w^2(|A|^{2r} + i|B|^{2r}) \leq \| |A|^{4r} + |B|^{4r} \|$. Thus, the inequality in Theorem 3.3
 2 refines the inequality (1.5) for $r \geq 2$.

3 For an improvement of (1.6) we need the following lemma, that can be found in [1].

4 **Lemma 3.4.** *Let f be a non-negative increasing convex function on $[0, \infty)$ and let $A, B \in \mathcal{B}(\mathcal{H})$ be*
 5 *positive. Then*

$$6 \quad \left\| f\left(\frac{A+B}{2}\right) \right\| \leq \left\| \frac{f(A)+f(B)}{2} \right\|.$$

8 **Theorem 3.5.** *If $A, B \in \mathcal{B}(\mathcal{H})$, then for $r \geq 1$,*

$$9 \quad w^{2r}(B^*A) \leq \frac{1}{2} \left(\frac{\| |B|^2|A|^2 + |A|^2|B|^2 \|}{2} \right)^r + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \|.$$

10 *Proof.* Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then, we have

$$11 \quad \begin{aligned} 12 \quad |\langle B^*Ax, x \rangle|^2 &= |\langle Ax, Bx \rangle|^2 \\ 13 &\leq \|Ax\|^2 \|Bx\|^2 \\ 14 &\leq \langle |A|^2x, x \rangle \langle |B|^2x, x \rangle \\ 15 &\leq \frac{1}{4} (\langle |A|^2x, x \rangle + \langle |B|^2x, x \rangle)^2 \\ 16 &= \frac{1}{4} (\langle (|A|^2 + |B|^2)x, x \rangle)^2 \\ 17 &\leq \frac{1}{4} \langle (|A|^2 + |B|^2)^2x, x \rangle \\ 18 &= \frac{1}{4} \langle (|A|^4 + |B|^4 + |A|^2|B|^2 + |B|^2|A|^2)x, x \rangle \\ 19 &\leq \frac{1}{4} \| (|A|^4 + |B|^4 + |A|^2|B|^2 + |B|^2|A|^2) \| \\ 20 &\leq \frac{1}{4} \| |A|^4 + |B|^4 \| + \frac{1}{4} \| |A|^2|B|^2 + |B|^2|A|^2 \|. \end{aligned}$$

21 Thus, we have

$$22 \quad \begin{aligned} 23 \quad |\langle B^*Ax, x \rangle|^{2r} &\leq \left(\frac{1}{4} \| |A|^4 + |B|^4 \| + \frac{1}{4} \| |A|^2|B|^2 + |B|^2|A|^2 \| \right)^r \\ 24 &\leq \frac{1}{2} \left(\frac{\| |A|^4 + |B|^4 \|}{2} \right)^r + \frac{1}{2} \left(\frac{\| |A|^2|B|^2 + |B|^2|A|^2 \|}{2} \right)^r \\ 25 &\leq \frac{1}{4} \| |A|^{4r} + |B|^{4r} \| + \frac{1}{2} \left(\frac{\| |A|^2|B|^2 + |B|^2|A|^2 \|}{2} \right)^r \quad (\text{by Lemma 3.4}). \end{aligned}$$

26 Hence, taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$27 \quad w^{2r}(B^*A) \leq \frac{1}{2} \left(\frac{\| |B|^2|A|^2 + |A|^2|B|^2 \|}{2} \right)^r + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \|.$$

28 \square

1 **Remark 3.6.** By using convexity property of $f(t) = t^r, r \geq 1$, we have

$$\begin{aligned}
 & \frac{1}{2} \left(\frac{\| |B|^2 |A|^2 + |A|^2 |B|^2 \|}{2} \right)^r + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \| \\
 &= \frac{1}{2} w^r \left(\frac{|B|^2 |A|^2 + |A|^2 |B|^2}{2} \right) + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \| \\
 &\leq \frac{1}{2} \left(\frac{1}{2} w(|B|^2 |A|^2) + \frac{1}{2} w(|A|^2 |B|^2) \right)^r + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \| \\
 &\leq \frac{1}{4} (w^r(|B|^2 |A|^2) + w^r(|A|^2 |B|^2)) + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \| \\
 &= \frac{1}{2} w^r(|B|^2 |A|^2) + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \|.
 \end{aligned}$$

13 Thus, for all $r \geq 1$, we have

$$\begin{aligned}
 w^{2r}(B^*A) &\leq \frac{1}{2} \left(\frac{\| |B|^2 |A|^2 + |A|^2 |B|^2 \|}{2} \right)^r + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \| \\
 &\leq \frac{1}{2} w^r(|B|^2 |A|^2) + \frac{1}{4} \| |B|^{4r} + |A|^{4r} \|.
 \end{aligned}$$

18 Therefore, the inequality in Theorem 3.5 is a refinement of that in (1.6).

20 Finally, as a direct application of Theorem 2.3, we obtain the following inequality for the generalized commutators of operators. We use the idea used in the proof of [6, Th. 3.2].

22 **Theorem 3.7.** Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned}
 & w(AXB \pm BYA) \\
 &\leq 2\sqrt{2} \|B\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2}{4}}.
 \end{aligned}$$

27 *Proof.* Let $x \in \mathcal{H}$ with $\|x\| = 1$. If $\|X\| \leq 1$ and $\|Y\| \leq 1$, then by Cauchy Schwarz inequality, we get

$$\begin{aligned}
 |\langle (AX \pm YA)x, x \rangle| &\leq |\langle Xx, A^*x \rangle| + |\langle Ax, Y^*x \rangle| \\
 &\leq \|A^*x\| + \|Ax\| \\
 &\leq \sqrt{2} (\|A^*x\|^2 + \|Ax\|^2)^{\frac{1}{2}} \\
 &\leq \sqrt{2} \|AA^* + A^*A\|^{\frac{1}{2}}.
 \end{aligned}$$

34 Taking supremum over $\|x\| = 1$, we get

$$(3.1) \quad w(AX \pm YA) \leq \sqrt{2} \|AA^* + A^*A\|^{\frac{1}{2}},$$

37 when $\|X\| \leq 1$ and $\|Y\| \leq 1$. Now we consider the general case, that is, $X, Y \in \mathcal{B}(\mathcal{H})$ are arbitrary. If $X = Y = 0$, then (3.1) holds trivially. If $\max\{\|X\|, \|Y\|\} \neq 0$, then $\left\| \frac{X}{\max\{\|X\|, \|Y\|\}} \right\| \leq 1$ and

40 $\left\| \frac{Y}{\max\{\|X\|, \|Y\|\}} \right\| \leq 1$, and so it follows from the inequality (3.1) that

$$(3.2) \quad w(AX \pm YA) \leq \sqrt{2} \max\{\|X\|, \|Y\|\} \|AA^* + A^*A\|^{\frac{1}{2}}.$$

1 Replacing X by XB and Y by BY in the inequality (3.2) we get,

$$2 \quad w(AXB \pm BYA) \leq \sqrt{2} \max \{ \|XB\|, \|BY\| \} \|AA^* + A^*A\|^{\frac{1}{2}}.$$

3
4 This implies that

$$5 \quad w(AXB \pm BYA) \leq \sqrt{2} \|B\| \max \{ \|X\|, \|Y\| \} \|AA^* + A^*A\|^{\frac{1}{2}}.$$

6
7 Therefore, by using the inequality in Theorem 2.3, we have

$$8 \quad w(AXB \pm BYA) \\ 9 \leq 2\sqrt{2} \|B\| \max \{ \|X\|, \|Y\| \} \sqrt{w^2(A) - \frac{|\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|}{4}},$$

10
11 as desired. □

12
13 Considering $X = Y = I$ (the identity operator) in Theorem 3.7 we get the following corollary.

14
15 **Corollary 3.8.** *If $A, B \in \mathcal{B}(\mathcal{H})$, then*

$$16 \quad w(AB \pm BA) \leq 2\sqrt{2} \|B\| \sqrt{w^2(A) - \frac{|\|\Re(A) + \Im(A)\|^2 - \|\Re(A) - \Im(A)\|^2|}{4}}.$$

17
18
19 **Remark 3.9.** Let $A, B \in \mathcal{B}(\mathcal{H})$. (i) Fong and Holbrook [14] proved that

$$20 \quad (3.3) \quad w(AB + BA) \leq 2\sqrt{2} \|B\| w(A).$$

21
22 Clearly, the inequality in Corollary 3.8 is stronger than the inequality (3.3).

23 (ii) Hirzallah and Kittaneh [13] improved on the inequality (3.3) to prove that

$$24 \quad (3.4) \quad w(AB \pm BA) \leq 2\sqrt{2} \|B\| \sqrt{w^2(A) - \frac{|\|\Re(A)\|^2 - \|\Im(A)\|^2|}{2}}.$$

25
26 Considering the same examples as in Remark 2.2(iii), we see that the inequalities in Corollary 3.8 and
27 (3.4) are not comparable, in general.

28 (iii) It follows from the Corollary 3.8 that if $w(AB \pm BA) = 2\sqrt{2} \|B\| w(A)$, then

$$29 \quad \|\Re(A) + \Im(A)\| = \|\Re(A) - \Im(A)\|.$$

30
31 **Data availability statement.**

32 Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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35 References

- 36 1. J. Aujla and F. Silva, Weak majorization inequalities and convex functions, *Linear Algebra Appl.* 369 (2003) 217-233.
37 2. S. Bag, P. Bhunia and K. Paul, Bounds of numerical radius of bounded linear operator using t -Aluthge transform,
38 *Math. Inequal. Appl.* 23(3) (2020) 991-1004.
39 3. P. Bhunia, S.S. Dragomir, M.S. Moslehian and K. Paul, *Lectures on numerical radius inequalities*, Infosys Science
40 Foundation Series in Mathematical Sciences, Springer, Cham, (2022) xii+209 pp. ISBN: 978-3-031-13669-6; 978-3-
41 031-13670-2
42 4. P. Bhunia and K. Paul, Refinements of norm and numerical radius inequalities, *Rocky Mountain J. Math.* 51 (2021), no.
6, 1953-1965.

- 1 5. P. Bhunia, K. Paul and R.K. Nayak, Sharp inequalities for the numerical radius of Hilbert space operators and operator
2 matrices, Math. Inequal. Appl. 24(1) (2021) 167-183.
- 3 6. P. Bhunia and K. Paul, Furtherance of Numerical radius inequalities of Hilbert space operators, Arch. Math. (Basel) 117
4 (2021), no. 5, 537–546.
- 5 7. P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math. 167
6 (2021), Paper No. 102959, 11 pp.
- 7 8. P. Bhunia and K. Paul, Development of inequalities and characterization of equality conditions for the numerical radius,
8 Linear Algebra Appl. 630 (2021) 306-315.
- 9 9. P. Bhunia and K. Paul, Some improvement of numerical radius inequalities of operators and operator matrices,
10 Linear Multilinear Algebra 70 (2022), no. 10, 1995-2013.
- 11 10. P. Bhunia, S. Bag and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials,
12 Linear Algebra Appl. 573 (2019) 166-177.
- 13 11. M.L. Buzano, Generalizzazione della diseguaglianza di Cauchy–Schwarz, Rend. Sem. Mat. Univ. e Politech. Trimo 31
14 (1971/73) 405-409.
- 15 12. S.S. Dragomir, Power inequalities for the numerical radius of a product of two operators in Hilbert spaces, Sarajevo J.
16 Math. 5(18) (2009) 269-278.
- 17 13. O. Hirzallah and F. Kittaneh, Numerical radius inequalities for several operators, Math. Scand. 114(1) (2014) 110-119.
- 18 14. C.-K. Fong and J.A.R. Holbrook, Unitarily invariant operator norms, Canad. J. Math. 35 (1983) 274-299.
- 19 15. Z. Heydarbeygi, M. Sababheh and H.R. Moradi, A convex treatment of numerical radius inequalities, Czechoslovak
20 Math. J. 72(147) (2022), no. 2, 601-614.
- 21 16. F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math. 168(1) (2005) 73-80.
- 22 17. F. Kittaneh, Numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix,
23 Studia Math. 158(1) (2003) 11-17.
- 24 18. T. Kato, Notes on some inequalities for linear operators, Math. Ann. 125 (1952) 208-212.
- 25 19. B. Simon, Trace ideals and their applications. Cambridge University Press, Cambridge (1979).
- 26 20. T. Yamazaki, On upper and lower bounds for the numerical radius and an equality condition, Studia Math. 178(1) (2007)
27 83-89.

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