ON THE HILBERT-TYPE OPERATORS ACTING FROM FUNCTION SPACES INTO SEQUENCE SPACES

JIANJUN JIN

ABSTRACT. In this paper we introduce and study some Hilbert-type operators acting from the function spaces into the sequence spaces. We give some sufficient and necessary conditions for the boundedness and compactness of these Hilbert-type operators. Also, for some special cases, we obtain the sharp estimates for the norms of certain Hilbert-type operators.

1. INTRODUCTIONS AND MAIN RESULTS

In this paper, for two positive numbers A, B, we write $A \preceq B$, or $A \succeq B$, if there exists a positive constant C independent of the arguments such that $A \leq CB$, or $A \geq CB$, respectively. We will write $A \asymp B$ if both $A \preceq B$ and $A \succeq B$.

Let $p > 1$. We denote by q the conjugate index of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We denote the interval $(0, +\infty)$ by $\mathbb{R}_+$. Let $\mathcal{M}(\mathbb{R}_+)$ be the class of all measurable functions on $\mathbb{R}_+$. The usual Lesbegue space of measurable functions on $\mathbb{R}_+$, denoted by $L^p(\mathbb{R}_+)$, is defined as

$$L^p(\mathbb{R}_+) := \{f \in \mathcal{M}(\mathbb{R}_+) : \|f\|_p = (\int_{\mathbb{R}_+} |f(x)|^p \, dx)^{\frac{1}{p}} < +\infty\}.$$

We denote by l^p the Lesbegue space of infinite sequences, i.e.,

$$l^p := \{a = \{a_n\}_{n=1}^{\infty} : \|a\|_p = (\sum_{n=1}^{\infty} |a_n|^p)^{\frac{1}{p}} < +\infty\}.$$

The classical Hilbert operator H, induced by Hilbert kernel $\frac{1}{x+y}$, is defined as

$$H(f)(y) := \int_{\mathbb{R}_+} \frac{f(x)}{x+y} \, dx, \; f \in \mathcal{M}(\mathbb{R}_+), \; y \in \mathbb{R}_+.$$

The Hilbert series operator, denoted by $\widetilde{H}$, is similarly defined as

$$\widetilde{H}(a)(n) = \sum_{n=1}^{\infty} \frac{a_m}{m+n}, \; a = \{a_m\}_{m=1}^{\infty}, \; n \in \mathbb{N}.$$

It is well known that

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Proposition 1.1. *Let $p>1$. Then $H(\widetilde{H})$ is bounded on $L^p(\mathbb{R}_+)(l^p)$, respectively. Moreover, the norm $\|H\|(\|\widetilde{H}\|)$ of $H(\widetilde{H})$ is $\pi\csc\frac{\pi}{p}$, respectively, where*

$$\|H\|:=\sup_{f(\neq\theta)\in L^p(\mathbb{R}_+)}\frac{\|Hf\|_p}{\|f\|_p},\quad \|\widetilde{H}\|:=\sup_{a(\neq\theta)\in l^p}\frac{\|\widetilde{H}a\|_p}{\|a\|_p}.$$

Recently, many attentions have been paid to the study of the boundedness and estimates for the norms of the following type operator H_K, which is defined as

$$H_K(f)(n):=\int_{\mathbb{R}_+}f(x)K(x,n)\,dx,\ f\in\mathcal{M}(\mathbb{R}_+),\ n\in\mathbb{N},$$

where $K(x,y)$ is a non-negative function on $\mathbb{R}_+\times\mathbb{R}_+$. For more detailed introductions to this topic, see the book [6] of Yang and Lokenath, and the references cited therein. It should be pointed out that Hardy et. al. first considered this topic in [4] (Theorem 351).

In particular, take $K(x,y)=\frac{1}{x+y}$, we get the Hilbert-type operator $\widehat{H}$ as follows.

$$\widehat{H}(f)(n):=\int_{\mathbb{R}_+}\frac{f(x)}{x+n}dx,\ f\in\mathcal{M}(\mathbb{R}_+),n\in\mathbb{N}.$$

In this paper, we continue to study this topic. By introducing some parameters, we define the following Hilbert-type operator $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ as

$$\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}(f)(n):=n^{\frac{1}{p}[(\theta_2-1)+\beta\theta_2]}\int_{\mathbb{R}_+}\frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}f(x)}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{\lambda}}dx,\ f\in\mathcal{M}(\mathbb{R}_+),\ n\in\mathbb{N},$$

where $\lambda>0$, $0<\theta_1,\theta_2\le 1$, $-1<\alpha,\beta<p-1$.

When $\theta_1=\theta_2=\lambda=1$, $\alpha=\beta=0$, the operator $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ reduces to the operator $\widehat{H}$. We first study the boundedness of $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$. We will prove that

Theorem 1.2. *Let $p>1$, $\lambda>0$, $0<\theta_1,\theta_2\le 1$, $-1<\alpha,\beta<p-1$, and $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ be defined as obove. Then $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p if and only if $\lambda\ge 1+\frac{1}{p}(\beta-\alpha)$.*

Next, we consider the case when $\lambda=1+\frac{1}{p}(\beta-\alpha)$. In this case, we denote by $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$ the operator instead of $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$. That is

$$\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}(f)(n):=n^{\frac{1}{p}[(\theta_2-1)+\beta\theta_2]}\int_{\mathbb{R}_+}\frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}f(x)}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}dx,\ f\in\mathcal{M}(\mathbb{R}_+),\ n\in\mathbb{N}.$$

We denote by $\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\|$ the norm of $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$. We will show that

Theorem 1.3. *Let $p>1$, $0<\theta_1,\theta_2\le 1$, $-1<\alpha,\beta<p-1$, and $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$ be defined as above. Then $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p and*

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\|=\frac{1}{\theta_2^{\frac{1}{p}}\theta_1^{\frac{1}{q}}}B\Big(\frac{1+\beta}{p},\frac{p-1-\alpha}{p}\Big).\tag{1.1}$$

Here $B(\cdot,\cdot)$ is the well-known Beta function, which is defined as

$$B(u,v)=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}\,dt,\ u>0,v>0.$$

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It is known that

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

where $\Gamma(\cdot)$ is the Gamma function, defined as

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt,\ x>0.$$

For more infromations to these special functions, see [1].

When $\theta_1=\theta_2=1$, we see from Theorem 1.2 that $\mathbf{H}^{\alpha,\beta}_{1,1,\lambda}$ is not bounded from $L^p(\mathbb{R}_+)$ into l^p if $\lambda<1+\frac{1}{p}(\beta-\alpha)$. On the other hand, we notice that, for $\lambda>0$,

$$\int_{[0,1)} t^{x+n-1}(1-t)^{\lambda-1}dt=\frac{\Gamma(x+n)\Gamma(\lambda)}{\Gamma(x+n+\lambda)}, x>0, n\geq 1.$$

Then, by using the fact

$$\Gamma(x)=\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}[1+r(x)],\ |r(x)|\leq e^{\frac{1}{12x}}-1,\ x>0, \tag{1.2}$$

we see that

$$\int_{[0,1)} t^{x+n-1}(1-t)^{\lambda-1}dt\asymp\frac{1}{(x+n)^\lambda}.$$

Hence, in order for $\mathbf{H}^{\alpha,\beta}_{1,1,\lambda}: L^p(\mathbb{R}_+)\to l^p$ to be bounded when $\lambda<1+\frac{1}{p}(\beta-\alpha)$, let μ be a positive Borel measure on $[0,1)$, we will consider the operator $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}$, which is defined as

$$\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}(f)(n):=n^{\frac{\beta}{p}}\int_{\mathbb{R}_+} x^{-\frac{\alpha}{p}}\mu_\lambda[x+n]f(x)dx,\ f\in\mathcal{M}(\mathbb{R}_+),\ n\in\mathbb{N}.$$

Where

$$\mu_\lambda[z]:=\int_{[0,1)} t^{z-1}(1-t)^{\lambda-1}d\mu(t),\ z,\lambda>0. \tag{1.3}$$

We then study the problem of characterizing measures μ such that $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}: L^p(\mathbb{R}_+)\to l^p$ is bounded. We provide a sufficient and necessary condition of μ for which $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p. We will prove the following

Theorem 1.4. *Let $p>1,\lambda>0,\ -1<\alpha,\beta<p-1$. Let μ be a positive Borel measure on $[0,1)$ such that $d\nu(t):=(1-t)^{\lambda-1}d\mu(t)$ is a finite measure on $[0,1)$, and $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}$ be defined as above. Then $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}: L^p(\mathbb{R}_+)\to l^p$ is bounded if and only if ν is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$.*

Here, for $s>0$, a positive Borel measure μ on $[0,1)$, we say μ is an s-Carleson measure if there is a constant $C>0$ such that

$$\mu([t,1))\leq C(1-t)^s$$

holds for all $t\in[0,1)$.

We also characterize measures μ such that $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}: L^p(\mathbb{R}_+)\to l^p$ is compact. We shall show that

Theorem 1.5. *Let $p>1,\lambda>0,\ -1<\alpha,\beta<p-1$. Let μ be a positive Borel measure on $[0,1)$ such that $d\nu(t):=(1-t)^{\lambda-1}d\mu(t)$ is a finite measure on $[0,1)$, and $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}$ be defined as above. Then $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}: L^p(\mathbb{R}_+)\to l^p$ is compact if and only if ν is a vanishing $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$.*

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Here, an s-Carleson measure μ on $[0,1)$ is said to be a vanishing s-Carleson measure, if it satisfies further that

$$\lim_{t\to 1^-}\frac{\mu([t,1))}{(1-t)^s}=0.$$

The paper is organized as follows. We will first prove Theorem 1.3 in the next section. The proof of Theorem 1.2 will be give in Section 3. We prove Theorem 1.4 and 1.5 in Section 4. Final remarks will be present in Section 5.

2. Proof of Theorem 1.3

For $p>1$, $0<\theta_1,\theta_2\le 1$, $-1<\alpha,\beta<p-1$, we set

$$w_1(n):=\int_0^\infty \frac{x^{\theta_1-1}}{(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}\cdot\frac{n^{\frac{\theta_2(1+\beta)}{p}}}{x^{\frac{\theta_1(1+\alpha)}{p}}}\,dx,\ n\in\mathbb{N};$$

$$w_2(x):=\sum_{n=1}^\infty \frac{n^{\theta_2-1}}{(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}\cdot\frac{x^{\frac{\theta_1(p-1-\alpha)}{p}}}{n^{\frac{\theta_2(p-1-\beta)}{p}}},\ x>0.$$

Then, by a change of variables, we have

$$\begin{aligned} w_1(n) &= \int_0^\infty \frac{s^{-\frac{1+\alpha}{p}}}{(s+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}\cdot n^{\frac{\theta_2(1+\beta)}{p}}\,dx \\ &= \frac{1}{\theta_1}\int_0^\infty \frac{t^{-\frac{1+\alpha}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt=\frac{1}{\theta_1}B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p}). \end{aligned} \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} w_2(x) &\le \int_0^\infty \frac{u^{\theta_2-1}}{(x^{\theta_1}+u^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}\cdot\frac{x^{\frac{\theta_1(p-1-\alpha)}{p}}}{u^{\frac{\theta_2(p-1-\beta)}{p}}}\,du \\ &= \frac{1}{\theta_2}\int_0^\infty \frac{t^{-\frac{p-1-\beta}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt=\frac{1}{\theta_2}B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p}). \end{aligned} \tag{2.2}$$

Here, we have used the change of variables $t=u^{\theta_2}/x^{\theta_1}$.

We start to prove Theorem 1.3. For $f\in L^p(\mathbb{R}_+), n\in\mathbb{N}$, we write

$$\begin{aligned} & n^{\frac{1}{p}[(\theta_2-1)+\beta\theta_2]}\left|\int_{\mathbb{R}_+}\frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}f(x)}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}dx\right| \\ &\le n^{\frac{1}{p}[(\theta_2-1)+\beta\theta_2]}\int_{\mathbb{R}_+}\frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}|f(x)|}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}dx \\ &= \int_{\mathbb{R}_+}\left\{[V(x,n)]^{\frac{1}{p}}E_1(x,n)\cdot[V(x,n)]^{\frac{1}{q}}E_2(x,n)\right\}dx:=I(n), \end{aligned}$$

where

$$V(x,n)=\frac{1}{(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}};$$

$$E_1(x,n)=\frac{x^{\frac{\theta_1(1+\alpha)}{pq}-\frac{\theta_1\alpha}{p}}}{n^{\frac{\theta_2(p-1-\beta)}{p^2}}}\cdot n^{\frac{1}{p}(\theta_2-1)}\cdot|f(x)|;$$

$$E_2(x,n)=\frac{n^{\frac{\theta_2(p-1-\beta)}{p^2}+\frac{\theta_2\beta}{p}}}{x^{\frac{\theta_1(1+\alpha)}{pq}}}\cdot x^{\frac{1}{q}(\theta_1-1)}.$$

Applying the Hölder's inequality on $I(n)$, we get from (2.1) that

$$\begin{aligned} I(n) &\leq \left[\int_{\mathbb{R}_+} V(x,n)[E_1(x,n)]^p\,dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} V(x,n)[E_2(x,n)]^q\,dx\right]^{\frac{1}{q}} \\ &= [w_1(n)]^{\frac{1}{q}}\left[\int_{\mathbb{R}_+} V(x,n)[E_1(x,n)]^p\,dx\right]^{\frac{1}{p}}. \end{aligned}$$

It follows from (2.2) that

$$\begin{aligned} &\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}f\|_p=[\sum_{n=1}^{\infty}I^p(n)]^{\frac{1}{p}} \\ &\leq \frac{1}{\theta_1^{\frac{1}{q}}}[B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p})]^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\int_{\mathbb{R}_+} V(x,n)[E_1(x,n)]^p\,dx\right]^{\frac{1}{p}} \\ &= \frac{1}{\theta_1^{\frac{1}{q}}}[B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p})]^{\frac{1}{q}}\left[\int_{\mathbb{R}_+} w_2(x)|f(x)|^p\,dx\right]^{\frac{1}{p}} \\ &\leq \frac{1}{\theta_2^{\frac{1}{p}}\theta_1^{\frac{1}{q}}}B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p})\|f\|_p. \end{aligned}$$

This means that $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p, and

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\|\leq \frac{1}{\theta_2^{\frac{1}{p}}\theta_1^{\frac{1}{q}}}B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p}). \tag{2.3}$$

For $\varepsilon>0$, we define $\widetilde{f}(x)=0$, if $x\in(0,1)$; $\widetilde{f}(x)=\varepsilon^{\frac{1}{p}}x^{-\frac{1+\theta_1\varepsilon}{p}}$, if $x\in[1,\infty)$. Then, we easily see that $\|\widetilde{f}\|_p^p=\theta_1^{-1}$.

We write

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\widetilde{f}\|_p^p=\varepsilon\sum_{n=1}^{\infty}n^{(1+\beta)\theta_2-1}\cdot[J(n)]^p. \tag{2.4}$$

Here

$$J(n):=\int_{\mathbb{R}_+}\frac{x^{\frac{1}{q}[(\theta_1-1)-(q-1)\alpha\theta_1]}\cdot x^{-\frac{1+\varepsilon\theta_1}{p}}}{(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}\,dx.$$

On the other hand, a calculation yields that

$$J(n)=\frac{1}{\theta_1}n^{-\frac{\theta_2}{p}(1+\beta+\varepsilon)}\int_{\frac{1}{n^{\theta_2}}}^{\infty}\frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt. \tag{2.5}$$

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Also, when $\varepsilon < p-1-\alpha$, we have

$$\begin{aligned}
&\int_{\frac{1}{n^{\theta_2}}}^{\infty} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt \\
&= \int_0^{\infty} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt - \int_0^{\frac{1}{n^{\theta_2}}} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt \\
&= B(\frac{1+\beta}{p}+\frac{\varepsilon}{p}, \frac{p-1-\alpha}{p}-\frac{\varepsilon}{p}) - \int_0^{\frac{1}{n^{\theta_2}}} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt \\
&:= L(\varepsilon) - U(n). \qquad (2.6)
\end{aligned}$$

Combine (2.4), (2.5) and (2.6), we get that

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\widetilde{f}\|_p^p \;\geq\; \frac{\varepsilon}{\theta_1^p}\sum_{n=1}^{\infty} n^{-1-\varepsilon\theta_2}\cdot[L(\varepsilon)-U(n)]^p. \qquad (2.7)$$

By using the Bernoulli's inequality(see [5]), we obtain that

$$[L(\varepsilon)-U(n)]^p \geq [L(\varepsilon)]^p \left[1-\frac{p}{L(\varepsilon)}\int_0^{\frac{1}{n^{\theta_2}}} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt\right]. \qquad (2.8)$$

We also note that

$$\varepsilon\sum_{n=1}^{\infty} n^{-1-\varepsilon\theta_2} = \frac{1}{\theta_2}[1+o(1)], \quad \varepsilon\to 0^+, \qquad (2.9)$$

and, for $\varepsilon < p-1-\nu$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1-\varepsilon\theta_2}\int_0^{\frac{1}{n^{\theta_2}}} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{1+\frac{1}{p}(\beta-\alpha)}}\,dt \\
&\leq \sum_{n=1}^{\infty} n^{-1-\varepsilon\theta_2}\int_0^{\frac{1}{n^{\theta_2}}} t^{-\frac{1+\alpha+\varepsilon}{p}}\,dt \\
&= \frac{1}{p-1-\alpha-\varepsilon}\sum_{n=1}^{\infty} n^{-1-\theta_2(\frac{p-1-\alpha}{p}+\frac{\varepsilon}{q})} = O(1),\ \varepsilon\to 0^+. \qquad (2.10)
\end{aligned}$$

It follows from (2.7)-(2.10) that

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\widetilde{f}\|_p^p \;\geq\; \frac{1}{\theta_2\theta_1^p}[1+o(1)]\cdot[L(\varepsilon)]^p\cdot[1-\varepsilon O(1)].$$

Hence, we get that

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\| \geq \frac{\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\widetilde{f}\|_p}{\|\widetilde{f}\|_p} \geq \frac{1}{\theta_2^{\frac{1}{p}}\theta_1^{\frac{1}{q}}}[1+o(1)]^{\frac{1}{p}}\cdot[L(\varepsilon)]\cdot[1-\varepsilon O(1)]^{\frac{1}{p}}.$$

Take $\varepsilon\to 0^+$, we see that

$$\|\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}\| \geq \frac{1}{\theta_2^{\frac{1}{p}}\theta_1^{\frac{1}{q}}}B(\frac{1+\beta}{p},\frac{p-1-\alpha}{p}). \qquad (2.11)$$

Combine (2.3) and (2.11), we see that (1.1) is true and this proves Theorem 1.3.

3. Proof of Theorem 1.2

We first prove the "if" part. If $\lambda \geq 1+\frac{1}{p}(\beta-\alpha)$, then, for $f \in \mathcal{M}(\mathbb{R}_+)$, $n \in \mathbb{N}$, it is easy to see that

$$\left|\int_{\mathbb{R}_+} \frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}f(x)}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{\lambda}}dx\right| \leq \int_{\mathbb{R}_+} \frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}|f(x)|}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{1+\frac{1}{p}(\beta-\alpha)}}dx.$$

Consequently, in view of the boundedness of $\widetilde{\mathbf{H}}^{\alpha,\beta}_{\theta_1,\theta_2}$ from $L^p(\mathbb{R}_+)$ into l^p, we conclude that $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p when $\lambda \geq 1+\frac{1}{p}(\beta-\alpha)$. The "if" part is proved.

Next, we prove the "only if" part. We will show that, if $\lambda < 1+\frac{1}{p}(\beta-\alpha)$, then $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ is not bounded from $L^p(\mathbb{R}_+)$ into l^p.

For $\varepsilon > 0$, we take, as is done in the proof of Theorem 1.3, $\widetilde{f}(x) = 0$, if $x \in (0,1)$; $\widetilde{f}(x) = \varepsilon^{\frac{1}{p}} x^{-\frac{1+\theta_1\varepsilon}{p}}$, if $x \in [1,\infty)$. Then, we have $\|\widetilde{f}\|_p^p = \theta_1^{-1}$.

Hence we obtain that

$$\begin{aligned}
\|\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}\widetilde{f}\|_p^p &= \sum_{n=1}^{\infty} n^{(\theta_2-1)+\beta\theta_2}\left[\int_1^{\infty} \frac{x^{\frac{1}{q}[(\theta_1-1)+\alpha\theta_1]}\cdot x^{-\frac{1+\theta_1\varepsilon}{p}}}{x^{\alpha\theta_1}(x^{\theta_1}+n^{\theta_2})^{\lambda}}\,dx\right]^p \\
&= \sum_{n=1}^{\infty} n^{(\theta_2-1)+\beta\theta_2}\left[\int_1^{\infty} \frac{s^{-\frac{1+\alpha+\varepsilon}{p}}}{(s+n^{\theta_2})^{\lambda}}\,ds\right]^p \\
&= \sum_{n=1}^{\infty} n^{-1-p\theta_2[\lambda-1-\frac{1}{p}(\beta-\alpha)+\frac{\varepsilon}{p}]}\left[\int_{\frac{1}{n^{\theta_2}}}^{\infty} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{\lambda}}\,dt\right]^p \\
&\geq \sum_{n=1}^{\infty} n^{-1-p\theta_2[\lambda-1-\frac{1}{p}(\beta-\alpha)+\frac{\varepsilon}{p}]}\left[\int_1^{\infty} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{\lambda}}\,dt\right]^p.
\end{aligned}$$

We suppose $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda} : L^p(\mathbb{R}_+) \to l^p$ is bounded. Then there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
C_1 &\geq \frac{\|\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}\widetilde{f}\|_p^p}{\|\widetilde{f}\|_p^p} \\
&\geq \theta_1 \sum_{n=1}^{\infty} n^{-1+p\theta_2[1-\lambda+\frac{1}{p}(\beta-\alpha)-\frac{\varepsilon}{p}]}\left[\int_1^{\infty} \frac{t^{-\frac{1+\alpha+\varepsilon}{p}}}{(1+t)^{\lambda}}\,dt\right]^p. \qquad (3.1)
\end{aligned}$$

But, if $\lambda < 1+\frac{1}{p}(\beta-\alpha)$, then, when $\varepsilon < p[(1-\lambda)+\frac{1}{p}(\beta-\alpha)]$, we see from $p\theta_2[1-\lambda+\frac{1}{p}(\beta-\alpha)-\frac{\varepsilon}{p}] := \delta > 0$ that

$$\sum_{n=1}^{\infty} n^{-1+p\theta_2[1-\lambda+\frac{1}{p}(\beta-\alpha)-\frac{\varepsilon}{p}]} = \sum_{n=1}^{\infty} n^{-1+\delta} = +\infty.$$

This means that (3.1) is a contradiction. This proves that $\mathbf{H}^{\alpha,\beta}_{\theta_1,\theta_2,\lambda}$ is not bounded from $L^p(\mathbb{R}_+)$ into l^p if $\lambda < 1+\frac{1}{p}(\beta-\alpha)$. The theorem is proved.

4. Proofs of Theorem 1.4 and 1.5

In the proofs of Theorem 1.4 and 1.5, we need the following

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Lemma 4.1. *Let $\lambda > 0, -1 < \alpha, \beta < p-1$. Let μ be a positive Borel measure on $[0,1)$ and $\mu_\lambda[z]$ be defined as in (1.3) for $z > 0$. Set $d\nu(t) = (1-t)^{\lambda-1}d\mu(t)$. If ν is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$, then*

$$\mu_\lambda[x+n] \preceq \frac{1}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}} \tag{4.1}$$

holds for all $x > 0, n \in \mathbb{N}$. Furthermore, if ν is a vanishing $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$, then, for any $x > 0$,

$$\mu_\lambda[x+n] = o\left(\frac{1}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}}\right), \quad n \to \infty. \tag{4.2}$$

Proof. We first consider the case when $x > 0, n \geq 2$, we get from integration by parts that

$$\begin{aligned}
\mu_\lambda[x+n] &= \int_0^1 t^{x+n-1} d\nu(t) \\
&= \nu([0,1)) - (x+n-1)\int_0^1 t^{x+n-1}\nu([0,t))dt \\
&= (x+n-1)\int_0^1 t^{x+n-2}\nu([t,1))dt.
\end{aligned}$$

If ν is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$, then we see that there is a constant $C_2 > 0$ such that

$$\nu([t,1)) \leq C_2(1-t)^{1+\frac{1}{p}(\beta-\alpha)}$$

holds for all $t \in [0,1)$. It follows that

$$\begin{aligned}
\mu_\lambda[x+n] &\leq C_2(x+n-1)\int_0^1 t^{x+n-2}(1-t)^{1+\frac{1}{p}(\beta-\alpha)}dt \\
&= C_2\frac{(x+n-1)\Gamma(x+n-1)\Gamma(2+\frac{1}{p}(\beta-\alpha))}{\Gamma(x+n+1+\frac{1}{p}(\beta-\alpha))}.
\end{aligned}$$

By using (1.2) again, we obtain that

$$\frac{(x+n-1)\Gamma(x+n-1)\Gamma(2+\frac{1}{p}(\beta-\alpha))}{\Gamma(x+n+1+\frac{1}{p}(\beta-\alpha))} \asymp \frac{1}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}}.$$

Next we consider the case when $x > 0, n = 1$. When $x \geq 1, n = 1$, by repeating the arguments above, we easily see that (4.1) also holds.

When $x \in (0,1), n = 1$, we see from, ν is a finite measure on $[0,1)$, that

$$\mu_\lambda[x+1] = \int_0^1 t^x d\nu(t) \leq \nu([0,1)).$$

This implies that

$$\mu_\lambda[x+1] \preceq \frac{1}{(x+1)^{1+\frac{1}{p}(\beta-\alpha)}}$$

holds for all $x \in (0,1)$. It follows that

$$\mu_\lambda[x+n] \preceq \frac{1}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}}$$

holds for all $x > 0, n \in \mathbb{N}$.

Similarly, if ν is a vanishing $[1 + \frac{1}{p}(\beta - \alpha)]$-Carleson measure on $[0, 1)$, by minor modifications of above arguments, we can show that (4.2) holds. The lemma is proved. □

We start to prove Theorem 1.4.

Proof of "if" part of Theorem 1.4. By Lemma 4.1 and checking the proof of Theorem 1.3, we see that $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}$ is bounded from $L^p(\mathbb{R}_+)$ into l^p, if $d\nu(t) = (1-t)^{\lambda-1}d\mu(t)$ is a $[1 + \frac{1}{p}(\beta - \alpha)]$-Carleson measure on $[0, 1)$. The "if" part of Theorem 1.4 is proved. □

Proof of "only if" part of Theorem 1.4. In our proof, we need the following well-known estimate, see [7], Page 54. Let $0 < w < 1$. For any $c > 0$, we have

$$\sum_{n=1}^{\infty} n^{c-1} w^{2n} \asymp \frac{1}{(1 - w^2)^c}. \tag{4.3}$$

For any $0 < w < 1$, we define

$$f_w(x) := \begin{cases} 0, & \text{if } x \in (0, 1], \\ (1 - w^2)^{\frac{1}{p}} w^{\frac{2(k-1)}{p}}, & \text{if } x \in (k, k + 1], k \in \mathbb{N}. \end{cases} \tag{4.4}$$

Then we easily see that

$$\|f_w\|_p^p = (1 - w^2) \sum_{k=1}^{\infty} w^{2(k-1)} = 1.$$

Also, we set $g_w := \{g_n\}_{n=1}^{\infty}$ with $g_n = (1 - w^2)^{\frac{1}{q}} w^{\frac{2(n-1)}{q}}$. Then we have $\|g_w\|_q^q = 1$.

In view of the boundedness of $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu} : L^p(\mathbb{R}_+) \to l^p$, then we get from the duality that

$$\begin{aligned} 1 &\succeq \sum_{n=1}^{\infty} g_n \cdot \widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu}(f_w)(n) \\ &= (1 - w^2)^{\frac{1}{q}} \sum_{n=1}^{\infty} n^{\frac{\beta}{p}} w^{\frac{2(n-1)}{q}} \cdot \int_{\mathbb{R}_+} \left[x^{-\frac{\alpha}{p}} f_w(x) \int_0^1 t^{x+n-1} d\nu(t) \right] dx \\ &\geq (1 - w^2)^{\frac{1}{q}} \sum_{n=1}^{\infty} n^{\frac{\beta}{p}} w^{\frac{2(n-1)}{q}} \cdot \int_{\mathbb{R}_+} \left[x^{-\frac{\alpha}{p}} f_w(x) \int_w^1 t^{x+n-1} d\nu(t) \right] dx. \end{aligned} \tag{4.5}$$

We notice that

$$t^x \geq t^{k+1}, \text{when } x \in (k, k + 1] \tag{4.6}$$

holds for all $k \in \mathbb{N}, t \in (0, 1)$, and

$$x^{-\frac{\alpha}{p}} \asymp k^{-\frac{\alpha}{p}} \tag{4.7}$$

holds for $x \in (k, k + 1]$, $k \in \mathbb{N}$.

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It follows that

$$\begin{aligned} 1 &\succeq (1-w^2)^{\frac{1}{q}} \sum_{n=1}^{\infty} n^{\frac{\beta}{p}} w^{\frac{2(n-1)}{q}} \cdot \sum_{k=1}^{\infty} \left[k^{-\frac{\alpha}{p}} (1-w^2)^{\frac{1}{p}} w^{\frac{2(k-1)}{p}} \int_w^1 t^{k+n} d\nu(t) \right] \\ &\geq (1-w^2)\nu([w,1)) \sum_{n=1}^{\infty} n^{\frac{\beta}{p}} w^{\frac{2(n-1)}{q}} \cdot \sum_{k=1}^{\infty} \left[k^{-\frac{\alpha}{p}} w^{\frac{2(k-1)}{p}} w^{k+n} \right] \\ &\geq (1-w^2)\nu([w,1)) \left[\sum_{n=1}^{\infty} n^{\frac{\beta}{p}} w^{(\frac{2}{q}+1)n} \right] \cdot \left[\sum_{k=1}^{\infty} k^{-\frac{\alpha}{p}} w^{(\frac{2}{p}+1)k} \right] \end{aligned}$$

Thus, by (4.3), we have

$$1 \succeq (1-w^2)\nu([w,1)) \cdot \frac{1}{(1-w^2)^{1+\frac{\beta}{p}}} \cdot \frac{1}{(1-w^2)^{1-\frac{\alpha}{p}}}$$

This implies that

$$\nu([w,1)) \preceq (1-w^2)^{1+\frac{1}{p}(\beta-\alpha)}$$

for all $0 < w < 1$. It follows that ν is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$ and the "only if" part is proved. □

We next prove Theorem 1.5. We first show the "if" part of Theorem 1.5. Let $\mathfrak{N} \in \mathbb{N}$. We define the operator $\mathbf{H}^{[\mathfrak{N}]}$ as, for $f \in \mathcal{M}(\mathbb{R}_+)$,

$$\mathbf{H}^{[\mathfrak{N}]}(f)(n) := n^{\frac{\beta}{p}} \int_{\mathbb{R}_+} x^{-\frac{\alpha}{p}} \mu_\lambda[x+n] f(x) dx,$$

when $n \leq \mathfrak{N}$, and $\mathbf{H}^{[\mathfrak{N}]}(f)(n) := 0$, when $n \geq \mathfrak{N}+1$.

We see that $\mathbf{H}^{[\mathfrak{N}]}$ is a finite rank operator and hence it is compact from $L^p(\mathbb{R}_+)$ into l^p.

By Lemma 4.1, we see that, for any $\epsilon > 0$, there is an $\widehat{\mathbf{N}} \in \mathbb{N}$ such that

$$\mu_\lambda[x+n] \preceq \frac{\epsilon}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}}$$

holds for all $x > 0, n > \widehat{\mathbf{N}}$.

Then, we see from

$$\|(\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} - \mathbf{H}^{[\mathfrak{N}]})f\|_p^p = \sum_{n=\mathfrak{N}+1}^{\infty} n^{\frac{\beta}{p}} \int_{\mathbb{R}_+} x^{-\frac{\alpha}{p}} \mu_\lambda[x+n] f(x) dx.$$

that,

$$\|(\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} - \mathbf{H}^{[\mathfrak{N}]})f\|_p^p \preceq \epsilon^p \sum_{n=\mathfrak{N}+1}^{\infty} n^{\frac{\beta}{p}} \int_{\mathbb{R}_+} \left| \frac{f(x)}{(x+n)^{1+\frac{1}{p}(\beta-\alpha)}} dx \right|^p,$$

when $\mathfrak{N} > \widehat{\mathbf{N}}$.

Consequently, by checking the proof of Theorem 1.3, we see that, for any $\epsilon > 0$, it holds that

$$\|(\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} - \mathbf{H}^{[\mathfrak{N}]})f\|_p \preceq \epsilon \|f\|_p$$

for all $\mathfrak{N} > \widehat{\mathbf{N}}$. It follows that $\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta}$ is compact from $L^p(\mathbb{R}_+)$ to l^p. This proves the "if" part.

Finally, we show the "only if" part. For $0 < w < 1$. We take f_w as be defined in (4.4). It is easy to check that f_w is convergent weakly to 0 in $L^p(\mathbb{R}_+)$.

Since $\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta}$ is compact from $L^p(\mathbb{R}_+)$ to l^p, we get

$$\lim_{w\to 1^-} \|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p = 0. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned}\|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p^p &= \sum_{n=1}^{\infty} n^{\beta}\left\{\int_{\mathbb{R}_+}\left[x^{-\frac{\alpha}{p}} f_w(x)\int_0^1 t^{x+n-1}d\nu(t)\right]dx\right\}^p\\ &\geq \sum_{n=1}^{\infty} n^{\beta}\left\{\int_{\mathbb{R}_+}\left[x^{-\frac{\alpha}{p}} f_w(x)\int_w^1 t^{x+n-1}d\nu(t)\right]dx\right\}^p.\end{aligned}$$

It follows from (4.6) and (4.7) that

$$\|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p^p \succeq (1-w^2)\sum_{n=1}^{\infty} n^{\beta}\left\{\sum_{k=1}^{\infty}\left[k^{-\frac{\alpha}{p}} w^{\frac{2(k-1)}{p}}\int_w^1 t^{k+n}d\nu(t)\right]\right\}^p$$

Consequently, we obtain that

$$\|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p^p \succeq (1-w^2)[\nu([w,1))]^p\left[\sum_{n=1}^{\infty} n^{\beta} w^{pn}\right]\left[\sum_{k=1}^{\infty} k^{-\frac{\alpha}{p}} w^{\frac{2(k-1)}{p}+k}\right]^p$$

Then, by again (4.3), we get that

$$\|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p^p \succeq (1-w^2)[\nu([w,1))]^p\cdot\frac{1}{(1-w^2)^{1+\beta}}\cdot\frac{1}{(1-w^2)^{p-\alpha}}.$$

This means that

$$\nu([w,1)) \preceq \|\widehat{\mathbf{H}}_{\lambda,\mu}^{\alpha,\beta} f_w)\|_p (1-w^2)^{1+\frac{1}{p}(\beta-\alpha)}.$$

It follows from (4.8) that ν is a vanishing $[1+\frac{1}{p}(\beta-\alpha)$-Carleson measure on $[0,1)$. This proves the "only if" part of Theorem 1.5 and the proof of Theorem 1.5 is completed.

5. Final Remarks

Remark 5.1. We first remark that the assumptions $-1<\alpha,\beta<p-1$ in Theorem 1.2 and 1.3 are both necessary.

We consider the case when $\theta_1=\theta_2=\lambda=1, \alpha=\beta:=\gamma$. That is to say, we will consider the operator

$$\mathbf{H}_{1,1,1}^{\gamma,\gamma}(f)(n) = n^{\frac{\gamma}{p}}\int_{\mathbb{R}_+}\frac{x^{-\frac{\gamma}{p}}}{x+n} f(x)\,dx, f\in\mathcal{M}(\mathbb{R}_+), n\in\mathbb{N}.$$

We will write $\widehat{\mathbf{H}}_{\gamma}$ to denote $\mathbf{H}_{1,1,1}^{\gamma,\gamma}$. We shall show that

Proposition 5.2. *$\widehat{\mathbf{H}}_{\gamma}$ is not bounded from $L^p(\mathbb{R}_+)$ into l^p, if $\gamma\leq -1$, or $\gamma\geq p-1$.*

Proof. For $\varepsilon>0$, we take

$$\widehat{f}(x) := \begin{cases} 0, & \text{if } x\in(0,1],\\ \varepsilon^{\frac{1}{p}} x^{-\frac{1+\varepsilon}{p}}, & \text{if } x\in[1,\infty).\end{cases}$$

Then, we easily see that $\|\widehat{f}\|_p^p = 1$, and

$$\|\widehat{\mathbf{H}}_\gamma \widehat{f}\|_p^p = \sum_{n=1}^{\infty} n^{\gamma} \left[\int_1^{\infty} \frac{x^{-\frac{1+\gamma+\varepsilon}{p}}}{x+n}\, dx\right]^p.$$

(1) If $\gamma < -1$, when $\varepsilon < -(\gamma+1)$, we see from $1+\gamma+\varepsilon < 0$ that, for a fixed $n \geq 1$, it holds that

$$\int_1^{\infty} \frac{x^{-\frac{1+\gamma+\varepsilon}{p}}}{x+n}\, dx \geq \int_n^{\infty} \frac{x^{-\frac{1+\gamma+\varepsilon}{p}}}{x+n}\, dx \geq \frac{1}{2}\int_n^{\infty} x^{-1-\frac{1+\gamma+\varepsilon}{p}}\, dx = +\infty.$$

This means that $\widehat{\mathbf{H}}_\gamma$ is not bounded from $L^p(\mathbb{R}_+)$ into l^p in this case.

(2) If $\gamma = -1$ or $\gamma \geq p-1$, we have

$$\begin{aligned}\|\widehat{\mathbf{H}}_\gamma \widehat{f}\|_p^p &= \sum_{n=1}^{\infty} n^{-\gamma} \left[\int_1^{\infty} \frac{x^{-\frac{1+\gamma+\varepsilon}{p}}}{x+n}\, dx\right]^p = \sum_{n=1}^{\infty} n^{-1-\varepsilon} \left[\int_{\frac{1}{n}}^{\infty} \frac{t^{-\frac{1+\gamma+\varepsilon}{p}}}{1+t}\, dt\right]^p \\ &\geq \sum_{n=1}^{\infty} n^{-1-\varepsilon} \left[\int_1^{\infty} \frac{t^{-\frac{1+\gamma+\varepsilon}{p}}}{1+t}\, dt\right]^p\end{aligned}$$

We note that

$$\sum_{n=1}^{\infty} n^{-1-\varepsilon} = \frac{1}{\varepsilon}[1+o(1)],\ \varepsilon \to 0^+.$$

Therefore, we conclude that

$$\|\widehat{\mathbf{H}}_\gamma \widehat{f}\|_p^p \geq \frac{1}{\varepsilon}[1+o(1)] \left[\int_1^{\infty} \frac{t^{-\frac{1+\gamma+\varepsilon}{p}}}{1+t}\, dt\right]^p.$$

Take $\varepsilon \to 0^+$, we get that $\|\widehat{\mathbf{H}}_\gamma \widehat{f}\|_p^p \to +\infty$. This implies that $\widehat{\mathbf{H}}_\gamma : L^p(\mathbb{R}_+) \to l^p$ is not bounded when $\gamma = -1$ or $\gamma \geq p-1$. The proposition is proved. □

Remark 5.3. Take $\lambda = 1$ in Theorem 1.4, we obtain that

Corollary 5.4. *Let $p > 1$, $-1 < \alpha, \beta < p-1$. Let μ be a finite positive Borel measure on $[0,1)$. Then $\widehat{\mathbf{H}}_\mu^{\alpha,\beta} : L^p(\mathbb{R}_+) \to l^p$ is bounded if and only if μ is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$. Here*

$$\widehat{\mathbf{H}}_\mu^{\alpha,\beta}(f)(n) := n^{\frac{\beta}{p}} \int_{\mathbb{R}_+} x^{-\frac{\alpha}{p}} \mu[x+n] f(x)dx,\ f \in \mathcal{M}(\mathbb{R}_+),\ n \in \mathbb{N},$$

and

$$\mu[z] := \int_{[0,1)} t^{z-1} d\mu(t),\ z > 0.$$

We notice that Proposition 2.1 in the recent work [2] of Bao et. al. implies that

Proposition 5.5. *Let $r \in \mathbb{R}, s > 0$ with $s > |r|$. Let μ, ν be two finite positive Borel measures on $[0,1)$ with $d\nu(t) = (1-t)^r d\mu(t)$. Then ν is a $(s+r)$-Carleson measure on $[0,1)$ if and only if μ is a s-Carleson measure on $[0,1)$.*

On the other hand, for $p > 1$, $\lambda > 0$, $-1 < \alpha, \beta < p-1$, if $\alpha \leq \beta$ and $0 < \lambda < 2 + \frac{1}{p}(\beta-\alpha)$, or $\alpha > \beta$ and $-\frac{1}{p}(\beta-\alpha) < \lambda < 2 + \frac{1}{p}(\beta-\alpha)$, then it holds that $|\lambda - 1| < 1 + \frac{1}{p}(\beta-\alpha)$.

Hence, from Proposition 5.5 and Theorem 1.4, we see that

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Corollary 5.6. *Let $p > 1$, $\lambda > 0$, $-1 < \alpha, \beta < p-1$. Let μ, ν be two finite positive Borel measures on $[0,1)$ with $d\nu(t) = (1-t)^{\lambda-1}d\mu(t)$.*

If $\alpha \leq \beta$ and $0 < \lambda < 2+\frac{1}{p}(\beta-\alpha)$, or $\alpha > \beta$ and $-\frac{1}{p}(\beta-\alpha) < \lambda < 2+\frac{1}{p}(\beta-\alpha)$. Then $\widehat{\mathbf{H}}^{\alpha,\beta}_{\lambda,\mu} : L^p(\mathbb{R}_+) \to l^p$ is bounded if and only if one of the following two conditions is satisfied.

(1) μ is a $[2+\frac{1}{p}(\beta-\alpha)-\lambda]$-Carleson measure on $[0,1)$.

(2) ν is a $[1+\frac{1}{p}(\beta-\alpha)]$-Carleson measure on $[0,1)$.

Remark 5.7. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the class of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{r\in(0,1)} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\}^{\frac{1}{p}},\ 0 < r < 1.$$

It is well known that a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ belongs to $H^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$. See [3] for more introductions to the theory of Hardy spaces.

We define

$$\mathcal{H}(f)(z) := \sum_{n=0}^{\infty}\left[\int_{\mathbb{R}_+} \frac{f(x)}{x+n+1}\,dx\right] z^n,\ f \in \mathcal{M}(\mathbb{R}_+), z \in \mathbb{C}.$$

Then, we see from Theorem 1.2 that $\mathcal{H}$ is bounded from $L^2(\mathbb{R}_+)$ into $H^2(\mathbb{D})$.

It is natural to consider the following

Question 5.8. *Whether $\mathcal{H}$ is bounded from $L^p(\mathbb{R}_+)$ into $H^p(\mathbb{D})$ for any $p > 1$?*

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School of Mathematics Sciences, Hefei University of Technology, Xuancheng Campus, Xuancheng 242000, P.R.China

Email address: jinjjhb@163.com, jin@hfut.edu.cn