PURELY PERIODIC ROSEN CONTINUED FRACTIONS

M. JELLALI

ABSTRACT. In this paper, we consider the two Hecke groups G_4 and G_6 generated by the transformations $\langle S, T\rangle$ defined by $S(z) = z + \lambda_m$ and $T(z) = -1/z$ where $\lambda_m = 2\cos(\pi/m)$ with $m \in \{4, 6\}$. We give a full characterization of purely periodic Rosen continued fractions over G_4 and G_6. Finally, we end by finding a family of examples of purely periodic Rosen expansions of period length two and some related examples.

1. INTRODUCTION

In 1954, a new class of continued fractions was introduced By D. Rosen [3] closely associated with the Hecke groups G_m. The Hecke groups are the set of linear fractional transformations $\langle S, T\rangle$ defined by $S(z) = z + \lambda_m$ and $T(z) = -1/z$ where $\lambda_m = 2\cos(\pi/m)$ with $m \geq 3$. For a fixed $m \geq 3$, a λ_m-continued fraction noted $\lambda_m cf$ is an expression of the form:

$$r_0\lambda_m + \cfrac{\epsilon_1}{r_1\lambda_m + \cfrac{\epsilon_2}{r_2\lambda_m + \ddots}} = [r_0\lambda_m, \epsilon_1/r_1\lambda_m, \epsilon_2/r_2\lambda_m, \cdots],$$

with $\epsilon_i = \pm 1, r_0 \in \mathbb{Z}$ and $r_i \in \mathbb{Z}^+$.

The $\lambda_m cf$ of any real number is given by a nearest multiple of λ_m- algorithm. More precisely, any real number α can be written as follows: $\alpha = r_0\lambda_m + \epsilon_1 R_1$ where r_0 is a nearest multiple of λ_m to α, $\epsilon_1 = \pm 1$, $0 \leq R_1 < \lambda_m/2$, and r_0 can be obtained by the so-called nearest integer function $\{\cdot\}$ as follows: $r_0 = \{\alpha/\lambda_m\}$, then apply the algorithm to $1/R_1$ and continue the expansion.

In [1], Galois proved that a real quadratic number $x > 1$ of the form $a + b\sqrt{\delta}$ where $a, b \in \mathbb{Z}$ and $\delta \in \mathbb{N}$ has purely periodic simple continued fraction expansion if and only if its conjugate $\overline{x} = a - b\sqrt{\delta} \in (-1, 0)$. So, it was natural to ask whether similar results can be proven with Rosen continued fractions over the two Hecke groups G_4 and G_6 whose underlying fields are the quadratic extensions of $\mathbb{Q}$, that are, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ respectively.

In [5], the authors showed that all units of $\mathbb{Z}(\sqrt{D})$ have purely periodic Rosen

---

Date: February 10, 2023.

1991 *Mathematics Subject Classification.* 11A55, 11J70, 11Y65.

Key words and phrases. Hecke groups, Rosen continued fractions, purely periodic elements.

10 Feb 2023 08:49:47 PST
220925-jellali Version 4 - Submitted to Rocky Mountain J. Math.

continued fractions where D is a squarefree integer. Note that the results of [5] concern only the units in the ring $\mathbb{Z}[\sqrt{D}]$. In this paper, we are interested especially with elements of $\mathbb{Z}[\lambda_m]; m \in \{4, 6\}$ whose Rosen continued fractions are purely periodic despite they are not necessary units.

The main purpose of this paper is to give a characterization of elements whose $\lambda_m cf$ is purely periodic over the two Hecke groups G_4 and G_6.
So, our paper is organized as follows. In Section 2, we define some basic notions and give some preliminary results. The main results of this paper, those of Section 3, are devoted to describe purely periodic Rosen continued fractions over G_4 and G_6. Finally, in Section 4, the length of such purely periodic elements is discussed with concrete examples.

2. Preliminaries

Let $m = 4$ or 6, then $\lambda_m = \sqrt{2}$ or $\sqrt{3}$ respectively. From a nearest integer algorithm, we can conclude that every real number α can be written as follows

$$\alpha = r_0\lambda_m + \cfrac{\epsilon_1}{r_1\lambda_m + \cfrac{\epsilon_2}{r_2\lambda_m + \ddots}} = [r_0\lambda_m, \epsilon_1/r_1\lambda_m, \epsilon_2/r_2\lambda_m, \cdots],$$

with $\epsilon_i = \pm 1, r_0 \in \mathbb{Z}$ and $r_i \in \mathbb{Z}^+$.

We recall that the n^{th} convergent p_n/q_n, obtained by truncating the $\lambda_m cf$ of α after exactly n steps, is given by:

$$p_n/q_n = [r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_n/r_n\lambda_m],$$

where $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ are two sequences of real numbers defined by the following recurrence relations:

$$p_{-1} = 1; p_0 = 0; p_n = r_n\lambda_m p_{n-1} + \epsilon_n p_{n-2}; n \geq 1,$$

$$q_{-1} = 0; q_0 = 1; q_n = r_n\lambda_m q_{n-1} + \epsilon_n q_{n-2}; n \geq 1.$$

So, it follows that:

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}\epsilon_1\epsilon_2\cdots\epsilon_n.$$

We may also prove by simple induction that for every convergent p_n/q_n, exactly one of p_n, q_n is in $\mathbb{Z}$, the other is in $\lambda_m\mathbb{Z}$. More precisely, if $p_n \in \mathbb{Z}$, then $p_{n+1} \in \lambda\mathbb{Z}$, $q_n \in \lambda\mathbb{Z}$ and $q_{n+1} \in \mathbb{Z}$; $\forall n \geq 0$. Also from the definition of p_n and q_n, we may easily remark that $(q_n)_{n\geq 1}$ is an increasing sequence of real numbers and by a simple induction we may show that

$$p_n/p_{n-1} = [r_n\lambda_m, \epsilon_n/r_{n-1}\lambda_m, \ldots, \epsilon_1/r_0\lambda_m]. \qquad (*)$$

10 Feb 2023 08:49:47 PST
220925-jellali Version 4 - Submitted to Rocky Mountain J. Math.

By a periodic Rosen continued fraction, we mean any Rosen continued fraction which is eventually periodic, more precisely $\lambda_m cf(\alpha)$ is eventually periodic if and only if $\exists s \in \mathbb{N}$, such that:

$$\lambda_m fr(\alpha) = \left[r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_s/r_s\lambda_m, \epsilon_{s+1}/\overline{r_{s+1}\lambda_m, \ldots, \epsilon_n/r_n\lambda_m; \epsilon}\right],$$

where $\epsilon_i = \pm 1$. In this case, the expression $[r_{s+1}\lambda_m, \ldots, \epsilon_n/r_n\lambda_m; \epsilon]$ will be called the period of the $\lambda_m cf$ of α. And when the period begins with the first term r_0 of the Rosen continued fraction of α, we say that this Rosen continued fraction is purely periodic which will be noted as follows:

$$\alpha = \left[\overline{r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_s/r_s\lambda_m, \epsilon}\right].$$

For more details one can see [3], [4], [2] and [6]. Let us consider the following sets:

$$\mathbb{Q}(\lambda_m) = \{\frac{a + b\lambda_m}{c}; a, b \in \mathbb{Z} \text{ and } c \in \mathbb{Z}^*\}$$

and

$$\mathbb{Z}[\lambda_m] = \{a + b\lambda_m; a, b \in \mathbb{Z}\}.$$

For given $\alpha \in \mathbb{Z}[\lambda_m]$, $\alpha = a + b\lambda_m$ where $a, b \in \mathbb{Z}$, we denote by $\overline{\alpha} = a - b\lambda_m$ its conjugate.

Let $\mathbf{P}$ be the set of positive non-square rational integers and $\mathbf{S}$ be the set of square integers.

In [6], the authors defined the following set:

$$\mathbf{R}_m = \{\alpha \in \mathbb{R},\ \alpha^2 \in \mathbb{Z}[\lambda_m]; \alpha \text{ is hyperbolic fixed point of } G_m\},$$

where $m = 4$ or 6 and proved that hyperbolic fixed points of G_4 and G_6 are exactly periodic ones except for those elements having the period of $\overline{[2, 1 \cdots\cdots, 1]}$ corresponding to the cusps which are parabolic fixed points of Hecke groups G_4 and G_6 (see [6]). Schmidt and Sheingorn [6] showed that for each of G_4 and G_6, the periodic Rosen continued fraction expansions other than those of exactly one period are in 1-1 correspondence with the fixed points of hyperbolic elements of the group. In particular, the authors showed that

$$\mathbf{R}_m = (\mathbb{Z}^* + \lambda_m\mathbb{Z})\bigcup\{\lambda_m\sqrt{p} \ \mid\ p \in \mathbf{P}\}\bigcup\{\sqrt{p} \ \mid\ p \in (\mathbf{P} \ \setminus \lambda_m^2\mathbf{S})\}.$$

It follows from this that all elements of $\mathbf{R}_m$ have periodic $\lambda_m cf$. From this full characterization of the real numbers which have periodic Rosen continued fraction expansions given in [6], we aim to determine in the cases G_4 and G_6 exactly which values correspond to the purely periodic Rosen continued fraction expansions.

3. Main results

Recall that for $m=4, \lambda_m=\sqrt{2}$ and for $m=6, \lambda_m=\sqrt{3}$, so we will consider the two Hecke groups G_4 and G_6.

In [4], the authors showed that for every $m \geq 4$, if $\frac{\sqrt{D}}{C} > 2/\lambda_m$ with $C, D \in \mathbb{Z}[\lambda_m]$, then $\frac{\sqrt{D}}{C}$ cannot have purely periodic Rosen continued fraction expansion. Following the same technical argument as in [4], we state the following result.

Proposition 3.1. *Let $\alpha \in \mathbf{R}_m$ greater than $2/\lambda_m$. If*

$$\alpha \in \{\lambda_m\sqrt{p} \ \mid p \in \mathbf{P}\} \bigcup \{\sqrt{p} \ \mid \ p \in (\mathbf{P} \setminus \lambda_m^2 \mathbf{S})\},$$

then α is not purely periodic.

Proof. Suppose that the $\lambda_m cf(\alpha)$ is purely periodic, then

$$\begin{aligned} \alpha &= \left[\overline{r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_n/r_n\lambda_m, \epsilon}\right], \\ &= [r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_n/r_n\lambda_m, \epsilon/\alpha]. \end{aligned}$$

Let now $p_n/q_n = [r_0\lambda_m, \epsilon_1/r_1\lambda_m, \ldots, \epsilon_n/r_n\lambda_m]$ be the n^{th} convergent of α, it follows that:

$$\alpha = \frac{\alpha p_n + \epsilon p_{n-1}}{\alpha q_n + \epsilon q_{n-1}},$$

and then α verifies the following equation

$$\alpha^2 q_n + (\epsilon q_{n-1} - p_n)\alpha - \epsilon p_{n-1} = 0.$$

Since $\alpha \in \{\lambda_m\sqrt{p} \ \mid \ p \in \mathbf{P}\} \bigcup \{\sqrt{p} \ \mid \ p \in (\mathbf{P} \setminus \lambda_m^2 \mathbf{S})\}$, then $\alpha \notin \mathbb{Q}(\lambda_m)$ but $\alpha^2 \in \mathbb{Z}[\lambda_m]$, immediately we get

$$\begin{cases} \epsilon q_{n-1} = p_n, \\ \alpha^2 = \dfrac{\epsilon p_{n-1}}{q_n}. \end{cases}$$

So, $\alpha^2 = \frac{p_{n-1}}{q_n} < \frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-1}}{p_n}$ $(\epsilon = 1)$ and from $(*)$, we get $\frac{p_n}{p_{n-1}} \geq \lambda_m/2$, which implies that $\alpha^2 \leq 2/\lambda_m$. And since $2/\lambda_m > 0$, then $\alpha < 2/\lambda_m$ which is absurd . □

Theorem 3.2. *Let $\alpha \in \mathbf{R}_m$ greater than $2/\lambda_m$.*
α is purely periodic if and only if $\alpha \in \{\mathbb{Z}^* + \lambda_m\mathbb{Z}\}$ and $\overline{\alpha} \in (-\lambda_m/2, \lambda_m/2)$.

Proof. Suppose that α is purely periodic, then it is periodic. From the last proposition all elements of $\{\lambda_m\sqrt{p} \setminus p \in \mathbf{P}\} \bigcup \{\sqrt{p} \setminus p \in (\mathbf{P} \setminus \lambda_m^2\mathbf{S})\}$ cannot be purely periodic and since $\alpha \in \mathbf{R}_m$, it then follows that $\alpha \in \{\mathbb{Z}^* + \lambda_m\mathbb{Z}\}$.
Our next goal is to prove that $\overline{\alpha} \in (-\lambda_m/2, \lambda_m/2)$. Since α is purely periodic,

then it is a root of $f(x) = x^2 q_n + (\epsilon q_{n-1} - p_n)x - \epsilon p_{n-1} = 0$. Let now α^* be the second root of f. When checking for $f(\lambda_m/2)$ and $f(-\lambda_m/2)$, we remark that they are of opposite signs which implies that α^* is in $(-\lambda_m/2, \lambda_m/2)$. Since α is a root of f, it then follows that

$$\alpha^2 q_n + (\epsilon q_{n-1} - p_n)\alpha - \epsilon p_{n-1} = 0.$$

Two cases arise: ($q_n, p_{n-1} \in \mathbb{Z}$ and $q_{n-1}, p_n \in \lambda_m\mathbb{Z}$) or ($q_n, p_{n-1} \in \lambda_m\mathbb{Z}$ and $q_{n-1}, p_n \in \mathbb{Z}$). If $q_n, p_{n-1} \in \mathbb{Z}$ and $q_{n-1}, p_n \in \lambda_m\mathbb{Z}$, it then follows that:

$$\overline{\alpha}^2 q_n + (-\epsilon q_{n-1} + p_n)\overline{\alpha} - \epsilon p_{n-1} = 0.$$

Combining the last two equations, we get the following result:

$$(\alpha + \overline{\alpha})(q_n(\alpha - \overline{\alpha}) + \epsilon q_{n-1} - p_n) = 0.$$

So, $\alpha = -\overline{\alpha}$ (which is impossible since $\alpha \in \mathbb{Z}^* + \lambda_m\mathbb{Z}$) or $\overline{\alpha} = \dfrac{-p_n + \epsilon q_{n-1}}{q_n} + \alpha$.
Since α^* is the second root of f, it then follows that $\alpha + \alpha^* = \dfrac{p_n - \epsilon q_{n-1}}{q_n}$.
This proves that $\alpha^* = -\overline{\alpha}$. Finally, we infer from $\alpha^* \in (-\lambda_m/2, \lambda_m/2)$, that $\overline{\alpha} \in (-\lambda_m/2, \lambda_m/2)$. (Note that we find a similar result even if $q_n, p_{n-1} \in \lambda_m\mathbb{Z}$ and $q_{n-1}, p_n \in \mathbb{Z}$).
Conversely, let $\alpha \in \{\mathbb{Z}^* + \lambda_m\mathbb{Z}\}$ and $\overline{\alpha} \in (-\lambda_m/2, \lambda_m/2)$. We aim to prove that α is purely periodic. Since $\alpha \in \mathbf{R}_m$, then its $\lambda_m cf$ is periodic. By a nearest multiple of λ_m- algorithm, one can write

$$\alpha = \alpha_0 = r_0\lambda_m + \frac{\epsilon_1}{\alpha_1} > 2/\lambda_m \quad \text{where} \quad 0 < 1/\alpha_1 < \lambda_m/2; \; r_0 \in \mathbb{N}.$$

Let now $\alpha_n = r_n\lambda_m + \dfrac{\epsilon_{n+1}}{\alpha_{n+1}}, n \geq 1$. So, $\dfrac{\epsilon_{n+1}}{\alpha_{n+1}} = \alpha_n - r_n\lambda_m$ which leads to:

$$\frac{\epsilon_{n+1}}{\overline{\alpha_{n+1}}} = \overline{\alpha_n} + r_n\lambda_m. \qquad (**)$$

Let us prove that $\overline{\alpha_n} \in (-\lambda_m/2, \lambda_m/2)$. By induction, we have $\overline{\alpha_0} = \overline{\alpha} \in (-\lambda_m/2, \lambda_m/2)$. Suppose that $\overline{\alpha_n} \in (-\lambda_m/2, \lambda_m/2)$, then from the fact that $\dfrac{\epsilon_{n+1}}{\overline{\alpha_{n+1}}} = \overline{\alpha_n} + r_n\lambda_m$, we get $\overline{\alpha_{n+1}} \in (-\lambda_m/2, \lambda_m/2)$.
It follows from $(**)$, that r_n is a nearest multiple of λ_m to $\dfrac{\epsilon_{n+1}}{\overline{\alpha_{n+1}}}$, which is the key remark to pursue our proof.
α is periodic, then $\exists i, j \in \mathbb{N}$ such that $j > i$ and $\alpha_i = \alpha_j$, which implies that $\overline{\alpha_i} = \overline{\alpha_j}$ and $\dfrac{\epsilon_i}{\overline{\alpha_i}} = \dfrac{\epsilon_j}{\overline{\alpha_j}}$ where $\epsilon_i = \epsilon_j = \epsilon$. It then follows that $r_{i-1} = r_{j-1}$ and $r_{i-1} + \dfrac{\epsilon}{\overline{\alpha_i}} = r_{j-1} + \dfrac{\epsilon}{\overline{\alpha_j}}$, which leads to $\alpha_{i-1} = \alpha_{j-1}$. Continuing to reduce indices in this way, we eventually obtain $\alpha_{j-i} = \alpha_0$. Hence α is purely periodic. $\square$

4. Period length of some elements in G_4 and G_6

We end this work with an interesting family of examples of purely periodic expansions of period length two and some related examples.
The next few results follows work in [5] but with non units. Recall that we mean by units in the ring of integers $\mathbb{Z}[\lambda_m]$, all elements $\omega = a + b\lambda_m$ where $a, b \in \mathbb{Z}$ and verifying $N(\omega) = \omega\overline{\omega} = \mp 1$.
Without loss of generalities, let $\lambda_m = \lambda_4 = \sqrt{2}$, completely analogous results hold for $\lambda_m = \lambda_6 = \sqrt{3}$.
Let $\omega = 2+\sqrt{2} = a_1 + b_1\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Note that $\omega > 2/\lambda_4$, $\overline{\omega} \in (-\lambda_m/2, \lambda_m/2)$ and for $n > 1$, by a simple induction, we show that $\omega^{n+1} = a_{n+1} + b_{n+1}\sqrt{2}$ where $a_{n+1} = 2(a_n + b_n)$, $b_{n+1} = a_n + 2b_n$ and $\overline{\omega^n} \in (-\lambda_m/2, \lambda_m/2)\,; \forall n \geq 1$.
Note that $(b_n)_{n>1}$ is a strictly increasing sequence. By induction again, one can prove that $N(\omega^n) = a_n^2 - 2b_n^2 = 2^n$ and $b_n \geq 2^n; \forall n > 1$.
Let $\omega^n = a_n + b_n\sqrt{2}$, we have to find a nearest multiple of $\sqrt{2}$ to ω^n by technical approximations as follows :

$$|\frac{a_n^2}{b_n^2} - 2| = |\frac{a_n^2 - 2b_n^2}{b_n^2}| = \frac{2^n}{b_n^2} \leq \frac{1}{2^n}.$$

Then, we can conclude that

$$\{\frac{a_n + b_n\sqrt{2}}{\sqrt{2}}\} \approx \{\frac{b_n\sqrt{2} + b_n\sqrt{2}}{\sqrt{2}}\} = 2b_n.$$

Thanks to our main theorem, all elements of the set $\{\omega^n,\ \forall n \geq 1\}$ are purely periodic.

Proposition 4.1. *Let $\omega^n = a_n + b_n\sqrt{2}$ (as defined above). If $2^{n-1} \mid b_n$, then $\forall n \geq 1$, the length of the period of ω^n is exactly 2.*

Proof.

$$\begin{aligned}
\omega^n &= a_n + b_n\sqrt{2}, \\
&= 2b_n\sqrt{2} + a_n - b_n\sqrt{2}, \\
&= 2b_n\sqrt{2} + \cfrac{1}{\cfrac{a_n + b_n\sqrt{2}}{2^n}}, \\
&= 2b_n\sqrt{2} + \cfrac{1}{(2b_n/2^n)\sqrt{2} + \cfrac{a_n - b_n\sqrt{2}}{2^n}}, \\
&= 2b_n\sqrt{2} + \cfrac{1}{(2b_n/2^n)\sqrt{2} + \cfrac{1}{\omega^n}}, \\
&= \left[\overline{2b_n\sqrt{2}, \frac{1}{(2b_n/2^n)\sqrt{2}}}\right].
\end{aligned}$$

□

Examples

$\omega = 2+\sqrt{2} = \left[\overline{2\sqrt{2}, 1/\sqrt{2}}\right]$.

$\omega^2 = 6+4\sqrt{2} = \left[\overline{8\sqrt{2}, 1/2\sqrt{2}}\right]$.

$\omega^4 = 68+48\sqrt{2} = \left[\overline{96\sqrt{2}, 1/6\sqrt{2}}\right]$.

Note that if b_n is not a multiple of 2^{n-1}, then the length of the period of ω^n is strictly greater than 2. Cite for example:

$\omega^3 = 20+14\sqrt{2} = \left[\overline{28\sqrt{2}, 1/4\sqrt{2}, -1/\sqrt{2}, 1/14\sqrt{2}, -1/\sqrt{2}, 1/3\sqrt{2}}\right]$.

References

[1] E.Galois, Sur la théorie des nombres. Annales des mathématiques pures et appliqées, vol. 19, (1828), 294-301.

[2] E.Hanson, A.Merberg, C.Towse and E.Yudovina, Generalized continued fractions and orbits under the action of Hecke triangle groups, ACTA ARITHMETICA, (2008), 134.4 .

[3] D. Rosen,A class of continued fractions associated with certain properly discontinuous groups,Duke Math.J.21(1954),549-563.

[4] D. Rosen and T.Schmidt, Hecke Groups And Continued Fractions, Bull. Austral. Math. Soc, Vol 46 (1992) , 459-474.

[5] R. Sahin, O. Koruoglu, S. Ikikardes and I. Naci Cangul, The connections between continued fractions representations of units and certain Hecke groups, Bulletin of the Malaysian Mathematical Society Series 2 , (2010).

[6] T.Schmidt and M.Sheingorn, Length Spectra Of The Hecke Triangle Groups, Mathematische Zeitschrift, Vol 220, (1995), 369-397.

[1]Faculté des Sciences de Sfax, BP 1171, Sfax 3000, Tunisie.
E-mail address: manel.jellali@fss.usf.tn