A low rank ODE's based technique for numerical approximation of lower bounds of structured singular value

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Abstract. The structured singular value is a well-known mathematical quantity which plays a vital role to investigate the stability, instability, robustness and performance of linear input and output systems in control. In this article, we present an iterative method for the approximation of the lower bounds of structured singular values. The iterative method is based on low rank ordinary differential equations and is restricted to perform when pure complex full block uncertainties are under consideration. The numerical experimentation's show the behavior of lower and upper bounds of structured singular values. Furthermore, we investigate and analyze graphically the behaviour of eigenvalues and singular values. Furthermore, The pseudo-spectrum of three dimensional real valued matrices is inspected with the help of Eigtool.

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1 Introduction

The μ-value [14] known as structured singular value is an important mathematical tool in control theory which allows to discuss problems arising in the stability analysis and synthesis of linear control systems and to quantify the stability of linear systems subject to structured perturbations. The structures addressed by the structured singular value are very general and allows to cover all types of parametric uncertainties that can be incorporated into the control system by using real and complex Linear Fractional Transformations LFT's. For more detail, we refer [2, 4, 8–10, 14, 16] and the references therein for applications of structured singular value.

The versatility of the structured singular value comes at an expense of being notoriously hard, in fact Non-deterministic Polynomial time, that is, NP hard [3] to compute. The numerical algorithms used in practice provides both upper and lower bounds of structured singular value. An upper bound of the structured singular value provides sufficient conditions to guarantee robust stability of feedback system. The structured singular value lower bounds yield the conditions which are enough to discuss the instability of feedback system.

The mussv function is avaliable in the Matlab Control Toolbox and approximates SSV bounds from above and below. This function uses famous techniques like diagonal balancing and Linear Matrix Inequlaity [5] to approximate the bounds of structured singular value from above. The generalization of power method [15] approximates structured singular value bounds from below.

In [19], the authors has presented a survey on the existing commercial software and their numerical performance for the computation of both upper and lower bounds structured singular values. A. Packard and Panday [20] has shown the fact that structured singular value could have the discontinuties when Δ is considered as a pure real perturbation. The problem related with the discontinuity of the approximation of structured singular values was analyzed and solved by Young et al. [21]. The main contribution was the addition of some suitable small complex quantities to the matrix Δ.

In [22], an optimization problem was formulated and solved for the approximation of lower bounds of structured singular values. In [23], Ge and Ghu formulated structured singular value problem as a constraint optimization (minimization) problem and then presented it for the approximation of structured singular value.

In [24], the structured singular value problem is formulated as a non-linear programming problem and furthermore F-modified Sub-Gradient (F-MSG) algorithm is presented for the computation of structured singular bounds from below.

In [25], the computation of μ-values for the companion matrices is presented. The comparison of lower bounds with the well-known MATLAB routine mussv is investigated and analyzed. Some new results concerning the stability analysis of non-linear differential equations are presented in [26]. The stability of boundedness of solutions to non-linear differential systems of second order is presented in [27]. The asymptotic behaviors of linear advanced systems of differential equations is analyzed and presented in [28]. The distinguishability of the

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descriptor systems with regular pencil is studied in [29].

In this article we present the numerical approximation and comparison of numerical results obtained for lower bounds of structured singular value. For experimentation we have downloaded the numerical data for inverted pendulum from $\langle http://w3.onera.fr/smac/smart\rangle$.

Overview of article. The section 2 contributes the framework related to our study. In an appropriate manner, it demonstrate how an approximation of structured singular value bounds can be approximated by low ranked ordinary differential equations based numerical method. In section 3, we present new results for the approximation of the lower bounds of structured singular values for pure full complex blocks. Our numerical method comprise an inner-outer algorithm. The outer algorithm approximates an uncertainty level ϵ while inner algorithm determines an extremizer of the structured spectral value set. Furthermore, in this section we interpret that how an inner-algorithm demonstrate the behavior of structured singular value bounds when pure complex structured uncertainties are under consideration for the approximation of structured singular values. The outer-algorithm assimilate fast Newton's method to fix the uncertainty level ϵ. Section 4 is devoted to spectra and pseudo-spectra for the family of matrices obtained from $\langle http://w3.onera.fr/smac/smart\rangle$. In section 5, we present the numerical experimentation. Finally, we conclude our paper in section 6.

2 Framework

Consider n-dimensional complex (or real) valued matrix $A \in \mathbb{K}^{n,n}$ with $\mathbb{K} = \mathbb{C}(\mathbb{R})$, $\mathbb{C}$ and $\mathbb{R}$ denotes complex and real valued matrices, respectively. The set of block diagonal matrices $\mathbb{BLK}$ are defined as

$$\mathbb{BLK} = \{diag(s_1 I_{r_1}, \ldots, s_N I_{r_S}, \Delta_1, \ldots, \Delta_F), s_i \in \mathbb{C}(\mathbb{R}), \Delta_j \in \mathbb{C}^{m_j,m_j}(\mathbb{R}^{m_j,m_j})\}. \tag{2.1}$$

The matrices I_{r_i} $\forall i = 1:S$ denotes an identity matrices. The matrices denoted with notation Δ_j $\forall j = 1:F$ serves as full complex or real matrices. The full blocks are real and complex in nature and could possess any structure. In literature, particularly [14] generally does not carry the full blocks which are real. In system theory, the number of full blocks emerging from uncertainties associated with frequency response are complex valued matrices rather than real or mixed real and complex matrices.

Our methodology contemplate not only square matrices but rectangular matrices too. In literature, the accessible numerical methods frequently does not cope with rectangular matrices. Furthermore, the order in which number of scalar blocks and full blocks pop up in the set of block diagonal matrices does not matter. For instance, the number of scalar blocks can appear following subsequently the number of full complex blocks and vice-versa.

We begin with the definition of structured singular value for a given n-dimensional matrix A with respect to set of block diagonal matrices $\mathbb{BLK}$ as:

Definition 2.1. [14] For a given A, the n-dimensional real and complex valued matrix A and the set of block diagonal matrices $\mathbb{BLK}$, structured singular value is defined as

$$\mu_{\mathbb{BLK}}(A) := \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \mathbb{BLK}, \det(I - A\Delta) = 0\}}. \tag{2.2}$$

In Definition 2.1, $\|\cdot\|_2$ represents the matrix l_2-norm, $det(\cdot)$ represents the determinant of perturbed matrix $(I - A\Delta)$. In particular, $\mu_{\mathbb{BLK}}(A) = 0$ if $\det(I - A\Delta) \neq 0, \forall\, \Delta \in \mathbb{BLK}$.

Definition 2.1 returns precisely the largest singular value for the given matrix A and set of block diagonal matrices $\mathbb{BLK} = \mathbb{K}^{n\times n}$. But, if $\mathbb{BLK}$ is considerd as a genaric set having both real and complex uncertainties then structured singular values yields only an upper bound of $\mu_{\mathbb{BLK}}(A)$. This can be further cultivated by manipulating various properties of $\mu_{\mathbb{BLK}}$ [1].

We denote with $\mathbb{BLK}^*$, the set of block diagonal matrices instead of $\mathbb{BLK}$ when pure complex uncertainties are treated. For uncertainty Δ belonging to set of pure complex uncertainties, that is, $\Delta \in \mathbb{BLK}^*$ entails that $e^{i\varphi}\Delta \in \mathbb{BLK}^*$ for any $\varphi \in \mathbb{R}$. Consecutively, there exists $\Delta \in \mathbb{BLK}^*$ so that $\rho(A\Delta) = 1$ if and only if there is $\Delta' \in \mathbb{BLK}^*$ which possesses the same norm, such that $A\Delta'$ has the maximum eigenvalue equals 1. In turn, we have that $\det(I - A\Delta') = 0$. This suggests the following reformulated definition of structured singular value for pure complex uncertainties.

Definition 2.2. The spectral radius of a matrix A is denoted by $\rho(\cdot)$ and is defined as

$$\rho(A) := max\{|\lambda(A)| : \lambda \in \Lambda(A)\}.$$

In Definition 2.2, $\Lambda(\cdot)$ is known as the spectrum of a matrix.

Definition 2.3. For a given n-dimensional real or complex values matrix A and $\mathbb{BLK}^*$, set of block diagonal matrices, the structured singular value is defined as

$$\mu_{\mathbb{BLK}^*}(A) := \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \mathbb{BLK}^*,\ \rho(A\Delta) = 1\}}. \tag{2.3}$$

In Definition 2.3, the quantity $\rho(A\Delta)$ represents the spectral radius, that is, the largest eigenvalue in modulus of a matrix $(A\Delta)$. For λ to be any non-zero largest eigenvalue of matrix A, the matrix $\Delta = \lambda^{-1}I$ satisfies the constraints of an optimization problem which pop up in definition of structured singular value and hence inaugurate structured singular value lower bound for purely complex uncertainties. We examine that $\mu_{\mathbb{BLK}^*} = \rho(A)$ for $\mathbb{BLK}^* = \{sI : s \in \mathbb{C}\}$ which is the fact that the spectral radius and the largest singular value appears as special cases of structured singular values.

2.1 A reformulation of structured singular value

For n-dimensional complex valued matrix $A \in \mathbb{C}^{n\times n}$, the structured epsilon spectral value set (containing singular values) with respect to an uncertainty level ϵ is defined as

$$\Lambda_\epsilon^{\mathbb{BLK}}(A) := \{\sigma \in \Lambda(\epsilon A\Delta) : \Delta \in \mathbb{BLK},\ \sigma_{max} \leq 1\}. \tag{2.4}$$

In Equ. (2.4), the operator $\Lambda(\cdot)$ stands for the spectrum of a matrix $(\epsilon A\Delta)$ and σ_{max} denotes the largest singular value. For a family of pure complex uncertainties $\mathbb{BLK}^*$, the structured epsilon spectral value set $\Lambda_{\epsilon}^{\mathbb{BLK}^*}(A)$ is naturally a disk centered at origion 0. The structured epsilon spectral value set for pure complex uncertainties as defined as

$$\Sigma_{\epsilon}^{\mathbb{BLK}^*}(A) = \{\xi = 1 - \sigma : \sigma \in \Lambda_{\epsilon}^{\mathbb{BLK}}(A)\}, \tag{2.5}$$

encourage us to write structured singular value defined in Equ. (2) as follows

$$\mu_{\mathbb{BLK}}(A) = \frac{1}{\text{arg min}\{0 \in \Sigma_{\epsilon}^{\mathbb{BLK}}(A)\}}. \tag{2.6}$$

The Equ. (2.3) allow us to alternatively write down structured singular values as

$$\mu_{\mathbb{BLK}^*}(A) = \frac{1}{\text{arg min}\{\sigma_{max} = 1\}}, \tag{2.7}$$

when only purely complex uncertainties set $\mathbb{BLK}^*$ is under consideration. In above equation, the quantity $\sigma \in \Lambda_{\epsilon}^{\mathbb{BLK}^*}(A)$ and we may write $\Lambda_{\epsilon}^{\mathbb{BLK}^*}(A) \subset D$, where D represents an open complex unit disk iff $\mu_{\mathbb{BLK}^*}(A) < 1/\epsilon$.

2.2 An optimization problem

Consider an optimization problem which is basically a minimization problem

$$\xi(\epsilon) = \arg\min |\xi|, \tag{2.8}$$

where $\xi \in \Sigma_{\epsilon}^{\mathbb{BLK}}(A)$ for some established small parameter $\epsilon > 0$. From above arguments structured singular value $\mu_{\mathbb{BLK}}(A)$ is nothing but the reciprocal of the minimum value of ϵ which yields $\xi(\epsilon) = 0$. This recommend us a two-level algorithm to solve the minimization problem. First, we pursue to solve minimization problem indicated in Equ. (2.8) to approximate the minimum value of an eigenvalue. The outer algorithm to deal with the minimization problem include the computation and then variation of a small parameter ϵ. This employ the knowledge of the computation of an exact derivative $\Delta(\epsilon)$ w.r.t ϵ.

3 The Results

3.1 The inner-algorithm

This section is devoted to determine the solution of optimization problem as presented in Equ. (2.8). For this purpose, we need the computation of an some admissible perturbation Δ for an approximation to $\mu_{\mathbb{BLK}^*}(A)$ with respect to the set of purely complex uncertainties of the form

$$\mathbb{BLK}^* = \{diag(s_1 I_{r_1}, ..., s_N I_{r_N}; \Delta_1, ..., \Delta_F) : s_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j, m_j}\}. \tag{3.1}$$

For any given $A \in \mathbb{C}^{m,n}$, the maximum characterization of the largest singular value $\sigma_1(A)$ is a key step. Let
$$s =: \{A \in \mathbb{C}^{m,n} : \sigma_1(A) \leq 1\}.$$

Assumption 3.1. We assume that for a given $A \in \mathbb{C}^{m,n}$, $|a_{ij}| \leq 1$ for any i and j. All the elements a_{ij} in the i-th row and j-th column are zero if for some i and j, $|a_{ij}| = 1$.

The following Theorem 3.1 show that the largest singular value, that is, $\sigma_1(A)$ for given $A \in s$ is equivalent to maximum value of $\langle A, B \rangle$ with $B \in s$.

Theorem 3.1. *For $A, B \in \mathbb{C}^{m,n}$ in s, the largest singular value $\sigma_1(A)$ is given by*

$$\sigma_1(A) = max\{\langle A, B \rangle = Trace(A^* B)\},$$

where $*$ is the complex conjugate transpose of A and $\langle \cdot, \cdot \rangle$ denotes the inner-product between two matrices.

Proof. We begin with the singular value decomposition of A, that is, $A = U\Sigma V^t$. Therefore, we have that,
$$max\{\langle A, B \rangle : B \in s\} = max\{\langle U\Sigma V^t, B \rangle : B \in s\}$$
$$= max\{\langle \Sigma, U^t B V \rangle : B \in s\} = max\{\langle \Sigma, M \rangle : M \in s\},$$
with $M = U^t BV$. Furthermore, $\langle \Sigma, M \rangle = \sum_{r=1}^{q} \sigma_r(A) m_{rr}$, where m_{rr} denotes the diagonal entries of M and $\sum_{r=1}^{q} \sigma_r(A)$ denotes the sum of first q largest singular values of A. From assumption (3.1.), $|m_{rr}| \leq 1$ and the fact that $\sum_{r=1}^{q} \sigma_r(A) \leq \sigma_1(A)$ yields that $\langle \Sigma, M \rangle = \sigma_1(A)$, which completes proof. □

The following definition allow us the computation of local extremizer of structured spectral value sets defined in Equ. (2.4) and Equ. (2.5), respectively.

Definition 3.2. A local extremizer of a matrix $\Delta \in \mathbb{BLK}^*$ is a matrix valued function such that $\sigma_{\max}(\epsilon A \Delta) \leq 1$ and $(\epsilon A \Delta)$ possesses a maximum singular value which locally maximizes the singular value belonging the set $\Lambda_{\epsilon}^{\mathbb{BLK}^*}(A)$. We give following assumption for the set of block diagonal matrices $\mathbb{BLK}^*$.

Assumption 3.2. We assume that the set of block diagonal matrices contains only a number of full pure complex blocks instead of containing both complex scalar blocks and full complex blocks. That is,
$$\mathbb{BLK}^* = \{diag(\Delta_1, ..., \Delta_F) : \Delta_j \in \mathbb{C}^{m_j, m_j}, \ \forall j = 1 : F\}. \tag{3.2}$$

The following Theorem 3.3 allow us the computation of an important characterization of the local extremizer Δ_{local} corresponding to $\Lambda_{\epsilon}^{\mathbb{BLK}^*}(A)$.

Theorem 3.3. *Let*

$$\Delta_{local} = \{diag(\Delta_1, ..., \Delta_F)\}, \ \ \sigma_{\max}(\Delta_{local}) = 1,$$

be a local extremizer of $\Lambda_{\epsilon}^{\mathbb{BLK}^*}(A)$*. Assume that perturbed matrix* $(\epsilon A \Delta_{local})$ *has a largest singular value* σ*, with the right and left singular-vectors* u *and* v *partitioned as according to size and shape of* Δ_{local}*. Let* $z = Au$ *such that the size of components* u_k *and* z_k *of singular-vectors equals the size of some block in* Δ_{local}*. Furthermore, consider the non-degeneracy condition for the full pure complex blocks as*

$$\|u_r\|_2 \cdot \|v_r\|_2 \neq 0 \quad \forall\, r = 1, \ldots, F. \tag{3.3}$$

Then

$$\sigma_{\max}(\Delta_r) = 1 \quad \forall r = 1, \ldots, F.$$

Proof. We give prove by contradiction. Consider a matrix valued function $\Delta(t)$ having a block diagonal structure

$$\Delta(t) = \{diag\left(\Delta_1(t), ..., \Delta_r(t) + t z_{s+r}(t) x_{s+r}^*(t), ..., \Delta_F(t)\right)\},$$

such that at $t = 0$, $\Delta(0)$ and Δ_{local} exactly matches and $\sigma_{\max}$ of $\Delta(t)$ is bounded above by 1 while considering sufficiently small value of t. The time-derivative of $\sigma(t)$ at $t = 0$ corresponding to matrix valued function $(\epsilon A \Delta(t))$ is computed as

$$\begin{aligned}
\frac{d}{dt}\left(\sigma(t)\right) &= u^*(t) \frac{d}{dt}\left(\epsilon A \Delta(t)\right) v(t) \\
&= \epsilon u^*(t) A \frac{d}{dt}\left(\Delta(t)\right) v(t) \\
&= \epsilon \left(z^*(t) \frac{d}{dt}\left(\Delta(t)\right) v(t) \right) \\
&= \epsilon \left(\|z_{s+r}\|^2 \|u_{s+r}\|^2 \right) > 0.
\end{aligned}$$

Here the quantity $z^*(t) = u^*(t) A$. Thus, the condition $\frac{d}{dt}\left(\sigma(t)\right) > 0$ at $t = 0$ is a contradiction to the optimal condition of σ and hence $\sigma_{\max}(\Delta_r) = 1$, $\forall\ r = 1 : F$. □

3.1.1 A gradient system of ordinary differential equations

In order to compute a local maximizer for the singular value σ with $\sigma \in \Lambda_{\epsilon}^{\mathbb{BLK}*}(A)$, we formulate a matrix valued function $\Delta(t)$ so that the largest singular value $\sigma(t)$ of matrix valued function $(\epsilon A \Delta(t))$ possesses a maximal local growth. We then determine a gradient system of ordinary differential equations satisfied by admissible uncertainty $\Delta(t)$. We refer to [17] for a complete detail about the construction of gradient system of ordinary differential equations.

Orthogonal projection onto $\mathbb{BLK}^*$**.** The following Lemma 3.4 helps to compute the orthogonal projection of a complex valued matrix P into the full complex blocks belonging the set of block diagonal matrices.

Lemma 3.4. *Let $P \in \mathbb{C}^{n,n}$ and let*

$$P \otimes I_{\mathbb{BLK}^*} = diag\left(P_1, P_2, ..., P_{s+r}\right)$$

represents the block diagonal matrix obtained by entry-wise multiplication of matrix P with block diagonal matrix $I_{\mathbb{BLK}^*}$. Then the orthogonal projection of matrix P onto $\mathbb{BLK}^*$ is given by

$$P_{\mathbb{BLK}^*} = diag\left(\Theta_1, ..., \Theta_F\right)$$

with $\Theta_1 = P_1, ..., \Theta_F = P_{s+r}$.

Proof. The proof is similar to proof of Lemma 3.5 in [17]. Additionally, we consider the number of full complex blocks and neglecting the number of repeated scalar complex blocks from the set of block diagonal matrices. □

The optimization problem. Next, we construct an optimization problem whose solution allow us to formulate a gradient system of ordinary differential equations. For an optimization problem, we consider that $\sigma(t)$ is the continuous branch of singular values of perturbed matrix $(\epsilon A \Delta(t))$ and $u(t),\ v(t)$ acts as it's right and left singular vectors, respectively such that $\|u(t)\| = \|v(t)\| = 1$ for all $t \in \mathbb{R}$. From Lemma 3.4 and the fact that

$$\frac{d}{dt}\left(\sigma(t)\right) = \epsilon u^*(t) A \frac{d}{dt}\left(\Delta(t)\right) v(t),$$

we have that

$$\begin{aligned} \frac{d}{dt}\left(\sigma^2(t)\right) = 2\sigma(t)\dot{\sigma}(t) = 2\sigma(t)\left(u^*(t)\frac{d}{dt}\left(\epsilon A \Delta(t)\right) v(t)\right) \\ = 2\epsilon\sigma(t)\left(z^*(t)\frac{d}{dt}\left(\Delta(t)\right) v(t)\right), \end{aligned}$$

where vector $z^*(t) = u^*(t)A$.

For $\Delta(t) \in BLK^* = diag\left(\Delta_1, ..., \Delta_F\right)$, the following optimization problem determines a direction $\frac{d}{dt}\left(\Delta(t)\right) = Z$ to maximizes the singular values $\sigma(t)$ over full complex block.

$$\widehat{Z} = \arg\ \min\{u^*(t) Z v(t)\}$$

$$Subject\ \ to$$

$$\langle \Delta_1(t), ..., \Delta_F,\ \Phi_1, ..., \Phi_F \rangle = 0.$$

The solution to above optimization problem is the direction $Z = diag\left(\phi_1, ..., \phi_F\right)$.

The system of ordinary differential equations. The solution Z to optimization problem and the orthogonal projection $P_{\mathbb{BLK}^*}$ allow us to write the gradient system of ordinary differential equations as follows

$$\frac{d}{dt}\left(\Delta(t)\right) = M_1 P_{\mathbb{BLK}^*}\left(z(t)u^*(t)\right) - M_2\Delta(t),$$

where $M_1, M_2 \in BLK^*$ are the diagonal matrices and $\Delta(t)$ being the initial value matrix.

The solution to above gradient system of ordinary differential equations provides an admissible perturbation $\Delta(t)$ as a part of inner-algorithm. Next, we aim to give an outer-algorithm for the computation of an admissible perturbation level ϵ. The outer-algorithm involves the computation of perturbation level while making use of Newton method.

3.2 The Outer-algorithm

In this section, we present the computation of an admissible perturbation level ϵ with the help of outer-algorithm. We again consider the case for set of block diagonal matrices $\mathbb{BLK}^*$ which contains only pure complex full blocks and the number of repeated complex scalar blocks are omitted.

The following theorem 3.5 gives and explicit expression for the computation of the time-derivative for the continuous branch of the singular values $\sigma(\epsilon)$ for the matrix values function $\epsilon A\Delta$.

Theorem 3.5. *Let $\Delta(\epsilon) \in \mathbb{BLK}^* = diag\,(\Delta_1(\epsilon), ..., \Delta_F(\epsilon))$ and let $\sigma(\epsilon)$ be a smooth and continuous branch of singular values of matrices values function $(\epsilon A\Delta(\epsilon))$. Let $u(\epsilon)$ and $v(\epsilon)$ be the corresponding right and left singular vectors. Let $z(\epsilon) = Au(\epsilon)$, then*

$$\frac{d}{d(\epsilon)}\left(\sigma(\epsilon)\right) = \Sigma_{j=1}^{F}\left(\|z_{r+j}\|\|v_{r+j}\|\right) > 0.$$

Proof. We observe that

$$\frac{d}{d(\epsilon)}\left(\sigma(\epsilon)\right) = u^*(\epsilon)\left(\frac{d}{d(\epsilon)}\left(\epsilon A\Delta(\epsilon)\right)\right)v(\epsilon)$$

$$= u^*(\epsilon)\left(A\Delta(\epsilon) + A\epsilon\frac{d}{d(\epsilon)}(\Delta(\epsilon))\right)v(\epsilon)$$

$$= \langle u(\epsilon)v^*(\epsilon),\ A\Delta(\epsilon) + \epsilon A\frac{d}{d(\epsilon)}(\epsilon)\rangle. \tag{3.4}$$

In Equ. (3.4) the quantity $u^*(\epsilon)A\frac{d}{d(\epsilon)}v(\epsilon) = 0$. In turn, this shows that

$$\frac{d}{d(\epsilon)} = \langle u(\epsilon)v^*(\epsilon), A\Delta(\epsilon)\rangle$$

$$= \langle P_{\mathbb{BLK}^*}\left(z(\epsilon)u^*(\epsilon)\right),\ \Delta(\epsilon)\rangle,$$

with $\Delta(\epsilon) = M(\epsilon)P_{\mathbb{BLK}^*}\left(z(\epsilon)u^*(\epsilon)\right)$, where $M(\epsilon)$ has a block diagonal structure such that $\Delta(\epsilon)$ has a unit largest singular value.

Finally, Theorem 3.5 allow us to write the expression for the admissible perturbation level ϵ with Newton iteration as

$$\epsilon_{(m+1)} = \epsilon_{(m)} - \frac{\sigma_{(m)} - 1}{d(\sigma_{(m)})},$$

with $\sigma_{(m)} = \sigma(\epsilon_{(m)})$ and $d(\sigma_{(m)})$ is the time-derivative of $\sigma(\epsilon)$ at $\epsilon = \epsilon_{(m)}$. □

4 Spectra and Pseudo-Spectra

This section contributes the computation and behaviour of spectra and pseudo-spectra for the family of matrices corresponding to inverted pendulum.
Let H be a Hilbert space and denote the bounded operator on H by $B(H)$. Consider $A \in B(H)$. For $\epsilon > 0$, a small parameter, the pseudospectrum of $A \in B(H)$ is a set given by

$$\sigma_\epsilon(A) = \sigma(A) \bigcup \{z \in \mathbb{C} \backslash \sigma(A) \mid \|R_A(z)\| > \frac{1}{\epsilon}\}.$$

The resolvent $R_A(z) = (A - zI)^{-1}, z \notin \sigma(A)$ and I is the identity matrix. The resolvent is an analytic function with values belonging to $B(H)$.

Following theorem 4.1 gives an important conditions for $\sigma_\epsilon(A)$.

Theorem 4.1. *Let $A \in B(H)$ and $\epsilon > 0$, a small parameter. Then following statements are equivalent.*
(i) $z \in \sigma_\epsilon(A)$ where $z \in \mathbb{C} \backslash \sigma(A)$.
(ii) $z \in \sigma(A)$ where $\sigma(A)$ is the set of eigenvalues of $A \in B(H)$ or $\exists u \in H$ with u having unit-norm, that is, $\|u\| = 1$ such that $\|(A - zI)u\| < \epsilon, \epsilon > 0$.

Proof. First we show that $(i) \implies (ii)$. Consider that $z \in \sigma_\epsilon(A)$ while $z \notin \sigma(A)$, the set of eigenvalues of A. Then $\exists \vartheta \in H$ such that $\|R_A(z)\vartheta\| > \frac{1}{\epsilon}\|\vartheta\|$. Consider that $u = (A - zI)\vartheta$, then $\|(A - zI)u\| < \epsilon\|u\|$. Furthermore, $\|(A - zI)u\| < \epsilon$ for $\|u\| = 1$ and hence (ii) follows.
Next we show that $(ii) \implies (i)$. Let $z \notin \sigma(A)$ and take $u \in H$ with $\|u\| = 1$ and $\|R_A(z)u\| < \epsilon$, $\epsilon > 0$. We consider a rank-1 matrix B with $\|B\| < \epsilon$ and define

$$B\vartheta = -\langle u, \vartheta \rangle R_A(z)u.$$

From this, we conclude that $(R_A(z) + B)u = 0$ and $z \in \sigma(A + B)$.
To prove that $z \in \sigma_\epsilon(A)$, we make use of contradiction approach. Assume that $z \in \sigma(A + B)$ for $B \in B(H)$ with $\|B\| < \epsilon$ and furthermore we consider that $z \notin \sigma(A)$ and $\|(A - zI)\| \leq \frac{1}{\epsilon}$. We have that

$$A + B - zI = (I + B(A - zI))(A - zI).$$

Thus, our assumption conclude that $\|B(A - zI)\| < \epsilon \cdot \frac{1}{\epsilon} = 1$ and hence $(I + B(A - zI)^{-1})$ exists. Since, $(A - zI)^{-1}$ exists, so it follows that the matrix $(A + B - zI)^{-1}$ exists too and this contradicts that $z \in \sigma(A + B)$. Thus, finally we conclude that $z \in \sigma_\epsilon(A), \epsilon > 0$.

□

The following result holds true in finite dimensional case.

Theorem 4.2. *Let H be a finite dimensional Hilbert space and let $A \in B(H)$. Take $\epsilon > 0$, a small parameter, then $z \in \sigma_\epsilon A$ iff $\sigma_{\min}(A - zI) < \epsilon$, where $\sigma_{\min}(\cdot)$ denotes the smallest singular values of a matrix.*

This result provides a method to plot the pseudospectrum of a given matrix $A \in B(H)$. For this, we do need to select the grid of points in the complex plane and then we compute $\sigma_{\min}(A - zI)$ at each grid points. Finally, plotting the level curves for such points gives us beautiful portraits of pseudospectrum of $A \in B(H)$.

5 Numerical Experimentation

This section contributes the resemblance of lower bounds of SSV approximated by well-known Matlab function mussv and the algorithm [17] for a family of matrices obtained for inverted pendulum. Theses matrices are downloaded from $\langle http://w3.onera.fr/smac/smart \rangle$. Each admissible uncertainty computed by numerical algorithm [17] posses a unit l_2-norm while the same is true for it's each block. On the other hand it's possible that the admissible uncertainty computed by mussv possesses a unit l_2 norm but the same is not true for all blocks constituting it. By this mussv looses an optimality condition which says that each admissible uncertainty must have a unit l_2-norm.
Example 1. In the very first example we take a three dimensional matrix A_1 which is complex in nature.

$$A_1 = \begin{bmatrix} -0.0423 + 0.1622i & -0.0031 + 0.0118i & 0.0858 - 0.1579i \\ -0.0921 + 0.3534i & -0.0067 + 0.0257i & 0.6626 - 0.3441i \\ 0.0094 - 0.0360i & 0.0007 - 0.0026i & -0.0675 + 0.0351i \end{bmatrix}.$$

Consider a set of block diagonal uncertainties. Each block is complex scalar multiple of an identity matrix.

$$\mathbb{BLK} = \{diag(s_1 I_1, s_2 I_1, s_3 I_1) : s_1, s_2, s_3 \in \mathbb{C}\}.$$

An admissible uncertainty $\widehat{\Delta}$ is attain with mussv as

$$\widehat{\Delta} = \begin{bmatrix} -0.8446 - 4.0155i & 0 & 0 \\ 0 & -1.4537 - 3.8373i & 0 \\ 0 & 0 & -3.0990 - 2.6896i \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 4.1034$. An upper bound of SSV is obtained as $\mu_{PD}^{upper} = 0.2437$ while a same numerical value is achieved for lower bounds, that is, $\mu_{PD}^{lower} = 0.2437$.

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The algorithm [17] computes an admissible uncertainty $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.2018 - 0.9794i & 0 & 0 \\ 0 & -0.3620 - 0.9322i & 0 \\ 0 & 0 & -0.7601 - 0.6498i \end{bmatrix},$$

and $\epsilon^* = 4.1035$ while $\|\Delta^*\|_2 = 1$. The lower bound of SSV is attained as $\mu_{New}^{lower} = 0.2437$.

In Table 1, we make the comaprision of lower and upper bounds of structured singular values approximated by mussv and algorithm [17] for complex valued matrix A_1. The very first column of Table 1 show the dimension of the matrix A_1 under consideration. The second column show the set of block diagonal uncertainties denoted by BLK w.r.t which SSV is approximated. The third, fourth and fifth columns of Table 1 presents an upper and lower bounds approximated by μ_u^{mussv} and the lower bounds approximated by μ_l^{New} respectively.

$$A_1 = \begin{bmatrix} -0.0423 + 0.1622i & -0.0031 + 0.0118i & 0.0858 - 0.1579i \\ -0.0921 + 0.3534i & -0.0067 + 0.0257i & 0.6626 - 0.3441i \\ 0.0094 - 0.0360i & 0.0007 - 0.0026i & -0.0675 + 0.0351i \end{bmatrix},$$

and

n	BLK	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
03	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	0.8457	0.8457	0.8455
03	$\{diag(\Delta_1, \delta_2 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1 \in \mathbb{C}\}$	0.4730	0.4730	0.4729
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}\}$	0.2344	0.2342	0.2344
03	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\}$	0.8664	0.8664	0.8664

Example 2. In this example we consider a three dimensional complex valued matrix A_2.

$$A_2 = \begin{bmatrix} 0.3409 + 0.2623i & 0.0248 + 0.0191i & -0.1918 - 0.3915i \\ 0.6482 + 0.4989i & 0.0472 + 0.0363i & 0.0505 - 0.7445i \\ -0.2643 - 0.2034i & -0.0192 - 0.0148i & -0.0206 + 0.3036i \end{bmatrix},$$

and a set of block diagonal uncertainties as

$$\mathbb{BLK} = \{diag(s_1 I_1, s_2 I_1, s_3 I_1) : s_1, s_2, s_3 \in \mathbb{C}\}.$$

The mussv function computes an admissible uncertainty $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 1.0511 - 0.6612i & 0 & 0 \\ 0 & 0.8971 - 0.8586i & 0 \\ 0 & 0 & 0.1386 - 1.2341i \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 1.2418$. An upper bound of SSV is approximated as $\mu_{PD}^{upper} = 0.8053$ and the result holds for a lower bound.

The algorithm [17] attains an admissible uncertainty $\epsilon^* \Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0.8422 - 0.5391i & 0 & 0 \\ 0 & 0.7201 - 0.6939i & 0 \\ 0 & 0 & 0.1233 - 0.9924i \end{bmatrix},$$

and $\epsilon^* = 1.2419$ while $\|\Delta^*\|_2 = 1$. The approximated lower bound of SSV is attain as $\mu_{New}^{lower} = 0.8052$.

In Table 2, we make the comaprision of lower and upper bounds of structured singular values approximated by mussv and algorithm [17] for a three dimensional complex valued matrix A_2. The very first column of Table 2 show the dimension of the matrix A_2 under consideration. The second column show the set of block diagonal uncertainties denoted by BLK w.r.t which SSV is approximated. The third, fourth and fifth columns of Table 2 presents an upper and lower bounds approximated by μ_u^{mussv} and the lower bounds approximated by μ_l^{New} respectively.

$$A_2 = \begin{bmatrix} 0.3409 + 0.2623i & 0.0248 + 0.0191i & -0.1918 - 0.3915i \\ 0.6482 + 0.4989i & 0.0472 + 0.0363i & 0.0505 - 0.7445i \\ -0.2643 - 0.2034i & -0.0192 - 0.0148i & -0.0206 + 0.3036i \end{bmatrix},$$

and

n	BLK	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
03	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	1.2598	1.2598	1.2597
03	$\{diag(\Delta_1, \delta_2 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1 \in \mathbb{C}\}$	1.2319	1.2318	1.2317
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}\}$	0.7416	0.7411	0.7411
03	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\}$	0.8664	0.8664	0.8664

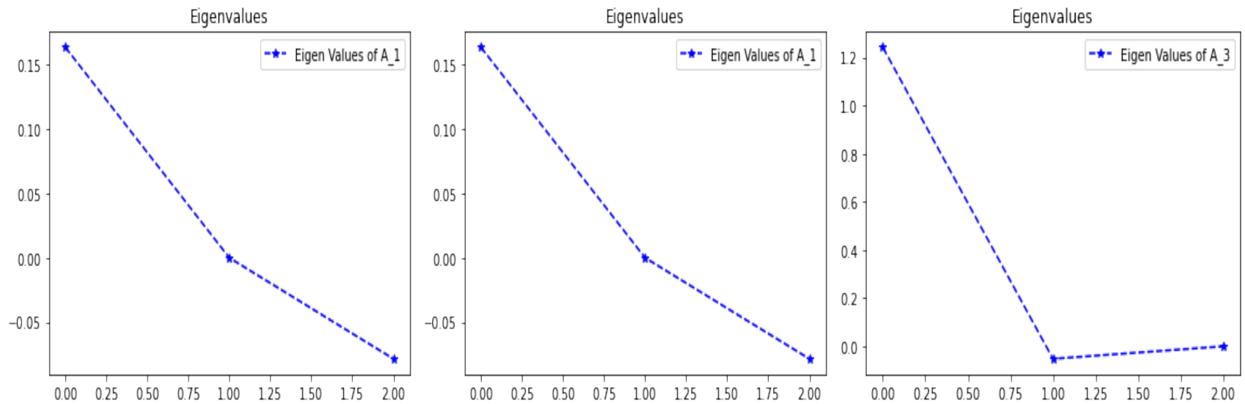

Figure 1: Behaviour of eigenvalues

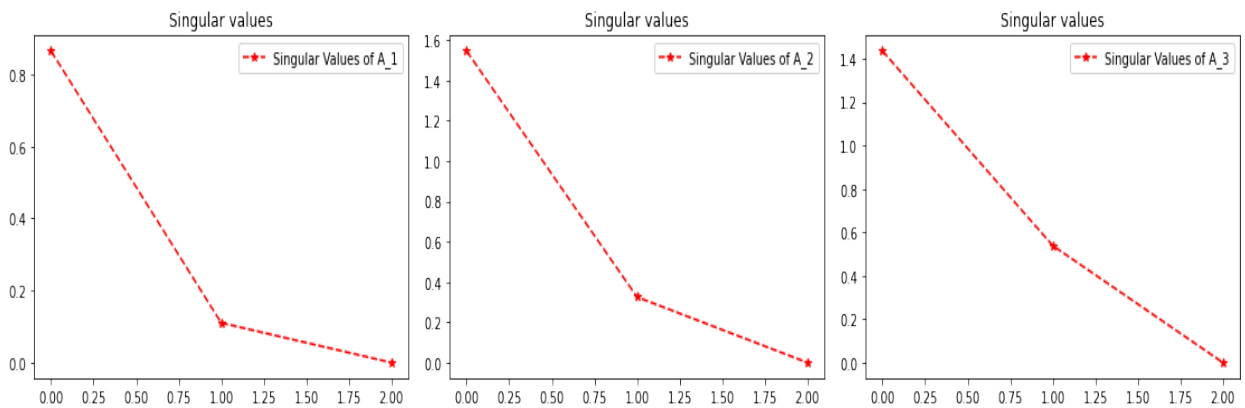

Figure 2: Behaviour of singular values

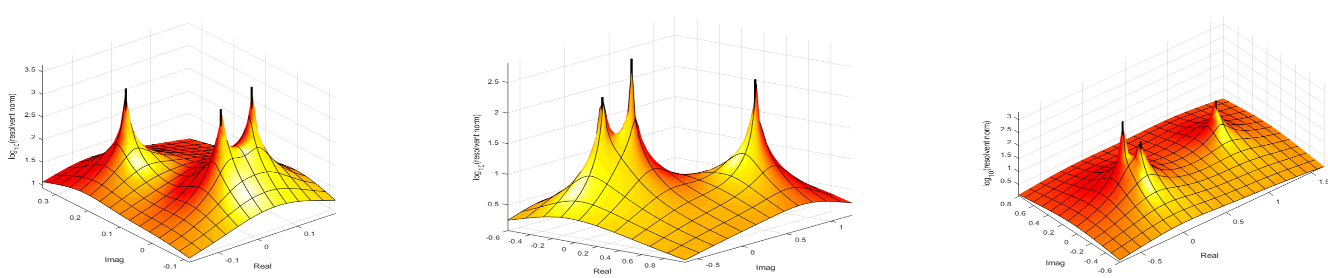

Figure 3: Pseudo-Spectrum of Matrices

In Figure 1, we present the spectrum or the set of eigenvalues corresponding to matrices A_1, A_2, A_3. From the graphical interpretation of the spectrum it's evident that some of the spectrum values are negative while other one are positive. The negative real part of the spectrum values reveals that feedback linear system under consideration is stable. The positive real part of the spectrum values suggests the insatiability.
Since, the computation of the structured singular values demands the computation of singular values. Thus, an enforcement to computation of singular values for each matrix. Figure 2 show the graphical interpretation of the singular values. From the geometrical interpretation it is evident that all singular values corresponding to A_1, A_2, A_3 are non-negative and most of them are strictly positive.
In Figure 3, we present the behaviour of the pseudo-spectrum for matrices A_1, A_2, A_3 with help of Eigtool [18]. The idea of the computation of pseudo-spectrum becomes valid if the matrices becomes non-normal where all eigen vectors losses orthogonality.

Example 3. Consider a three dimensional matrix A_3 which is complex in nature.

$$A_3 = \begin{bmatrix} 0.5330 - 0.0663i & 0.0388 - 0.0048i & -0.4365 - 0.2467i \\ 0.8365 - 0.1040i & 0.0609 - 0.0076i & -0.3423 - 0.3872i \\ -0.7675 + 0.0955i & -0.0559 + 0.0069i & 0.3141 + 0.3552i \end{bmatrix}.$$

We choose a set of block diagonal uncertainties as

$$\mathbb{BLK} = \{diag(s_1 I_1, s_2 I_1, s_3 I_1) : s_1, s_2, s_3 \in \mathbb{C}\}.$$

The MATLAB function mussv approximates an admissible uncertainty $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0.8289 + 0.1729i & 0 & 0 \\ 0 & 0.8455 + 0.0466i & 0 \\ 0 & 0 & 0.6226 - 0.5740i \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 0.8468$. The same upper and lower bounds of SSV are approximated having numerical value equals 1.1810.

The algorithm [17] attains the uncertainty $\epsilon^* \Delta^*$ with a unit l_2-norm as

$$\Delta^* = \begin{bmatrix} 0.9810 + 0.1941i & 0 & 0 \\ 0 & 0.9985 + 0.0556i & 0 \\ 0 & 0 & 0.7429 - 0.6694i \end{bmatrix},$$

and $\epsilon^* = 0.8468$ while $\|\Delta^*\|_2 = 1$. Lower bound of SSV is approximated as $\mu_{New}^{lower} = 1.1809$.

In Table 3, we make the comaprision of lower and upper bounds of structured singular values approximated by mussv and algorithm [17] for complex valued matrix A_3. The very first column of Table 3 show the dimension of the matrix A_3 under consideration. The second column show the set of block diagonal uncertainties denoted by BLK w.r.t which SSV is approximated. The third, fourth and fifth columns of Table 3 presents an upper and lower bounds approximated by μ_u^{mussv} and the lower bounds approximated by μ_l^{New} respectively.

$$A_3 = \begin{bmatrix} 0.5330 - 0.0663i & 0.0388 - 0.0048i & -0.4365 - 0.2467i \\ 0.8365 - 0.1040i & 0.0609 - 0.0076i & -0.3423 - 0.3872i \\ -0.7675 + 0.0955i & -0.0559 + 0.0069i & 0.3141 + 0.3552i \end{bmatrix},$$

and

n	BLK	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
03	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	1.3802	1.3802	1.3801
03	$\{diag(\Delta_1, \delta_2 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1 \in \mathbb{C}\}$	1.5224	1.5224	1.5223
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}\}$	1.0802	1.0802	1.0802
03	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\}$	1.5238	1.5238	1.5237
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{R}^{3,3}\}$	1.5238	1.5238	1.5237
03	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{2,2}\}$	1.5238	1.5238	1.5237
03	$\{diag(\Delta_1, \delta_1 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_2 \in \mathbb{R}\}$	1.5238	1.5238	1.5237
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1 \in \mathbb{R}, \delta_2, \delta_3 \in \mathbb{C}\}$	1.5238	1.5238	1.5237
03	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_3 \in \mathbb{C}, \delta_2, \in \mathbb{R}\}$	1.5238	1.5238	1.5237

6 Conclusion

In this article, we have presented the approximation of bounds of structured singular values for a family of matrices obtained for inverted problem. The obtained numerical results for the bounds of structured singular values guarantee the instability of the system corresponding to inverted pendulum. Furthermore, the attained numerical results show connection of the lower bounds of structured singular values with those approximated with mussv function which is easily available via Matlab control toolbox and alogorithm [17]. The behaviour of pseudo-spectrum is analyzed with the Eigtool. Our study on the computation of lower bounds of structured singular values for inverted pendulum leads us a direction for the computation of upper bounds. The computation of upper bounds is a part of our future research.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Author's contributions
M.U.R introduced the problem and J.A validated the results. M.U.R wrote the manuscript. All authors read and approved the final manuscript.

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