Well-posedness for multi-point BVPs for fractional differential equations with Riesz-Caputo derivative

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Abstract

In this work, a class of nonlinear multi-point boundary value problems (BVPs) in the context of fractional differential equations involving the Riesz-Caputo derivative is proposed. The nonlinearity term f involves the left Caputo derivative. Under given some conditions, the existence and uniqueness of the solution are provided. Though we apply the standard tools of the fixed point theory to develop the existence and uniqueness criteria for the solutions of given problems, the obtained results are new in the given scenario. Finally, some examples are given to illustrate our main results.

Keywords: Well-posedness, Fixed point theorem, Riesz-Caputo derivative, Multi-point BVPs

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1 Introduction

In this paper, we investigate the existence and uniqueness of solutions to the following boundary value problems (BVPs) of fractional differential equations

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involving the Riesz-Caputo derivative and multi-point boundary conditions:

$$\begin{aligned} &{}_{0}^{RC}D_{1}^{\alpha}\omega(\tau) = f(\tau, \omega(\tau), {}_{0}^{C}D_{\tau}^{\beta}\omega(\tau)), \\ &\omega(0) = 0, \quad \omega(1) = \sum_{i=1}^{m} \beta_i \omega(\xi_i), \end{aligned} \tag{1}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $\beta_i > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 \leq \tau \leq 1$, ${}_{0}^{RC}D_{1}^{\alpha}$ is a Riesz-Caputo derivative, ${}_{0}^{C}D_{\tau}^{\beta}$ is the left Caputo derivative of order β and $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$. β_i and $\xi_i (i = 1, 2, ...m)$ satisfying the following condition:

$$\triangle := \sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1.$$

In recent years, with the development of science and technology, there are lots of works devoted to the study of fractional differential equations, see [1–4] and the references therein. Fractional differential equations with Riesz-Caputo derivative have been of great interest in recent years. This is because of both the intensive development of the theory of Riesz derivative itself and the applications of such construction in various scientific fields. There are a few papers to study that the fractional differential equations problems with the Riesz-Caputo derivative[5–10, 12, 13]. By means of new fractional Gronwall inequalities and some fixed point theorems, Chen et al. [8] studied the existence of solutions for the two-point BVPs involving the Riesz-Caputo derivative given by

$$\begin{aligned} &{}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) = f(\tau, \omega(\tau)), \ \tau \in [0, T], \ \alpha \in (0, 1], \\ &\omega(0) = \omega_0, \quad \omega(T) = \omega_T, \end{aligned}$$

where ${}_{0}^{RC}D_{T}^{\alpha}$ is a Riesz-Caputo derivative. In [10], the authors studied the existence of positive for the above BVPs by using Leray-Schauder theorem and Krasonselskii's fixed point theorem in a cone, where $T = 1$. In [7], by employing new fractional Gronwall inequalities and some fixed point theorems, the authors investigated the existence results of solutions for the two-point anti-periodic BVPs involving the Riesz-Caputo derivative given by

$$\begin{aligned} &{}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) = f(\tau, \omega(\tau)), \ \tau \in [0, T], \ \alpha \in (1, 2], \\ &\omega(0) + \omega(T) = 0, \quad \omega'(0) + \omega'(T) = 0, \end{aligned}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous with respect to τ and ω.

In [14], by means of a fixed point theorem on a cone, the authors investigated the existence of positive solutions for the following singular fractional BVP

$$\begin{aligned} &D_{0+}^{\alpha}\omega(\tau) + f(\tau, \omega(\tau), D_{0+}^{\beta}\omega(\tau)) = 0, \\ &\omega(0) = \omega(1) = 0, \end{aligned}$$

where $\alpha \in (1,2)$, $\alpha - \beta \geq 1$, $f : [0,1] \times (0,+\infty) \times \mathbb{R} \to (0,+\infty)$ satisfies Carathéodory type consitions. Here D^{α}_{0+} denotes the Riemann-Liouville fractional derivative, the nonlinear term $f(\tau, \omega, \nu)$ may be singular at $\omega = 0$.

In [15], Zhang et. al. studied the following fractional differential equation

$$\begin{aligned} &D^{\alpha}_{\tau}\omega(\tau) + f(\tau, \omega(\tau), D^{\beta}_{\tau}\omega(\tau)) = 0, \\ &D^{\beta}_{\tau}\omega(0) = 0, D^{\beta}_{\tau}\omega(1) = \int_0^1 g(\varsigma) D^{\beta}_{\tau}\omega(\varsigma) dA(\varsigma), \end{aligned}$$

where D^{α}_{τ} is Riemann-Lioville's fractional derivative, $0 < \beta \leq 1 < \alpha \leq 2$, $\alpha - \beta > 1$, A is a function of bounded variation and dA can be a signed measure, $f \in C((0,1) \times (0,+\infty) \times (0,+\infty), (0,+\infty))$, and $f(\tau, \omega, \nu)$ may be singular at both $\tau = 0, 1$ and $\omega = \nu = 0$.

To the author's knowledge, no one have considered the qualitative properties of solutions to multi-point BVPs of fractional differential equation involving the Riesz-Caputo derivative. In this paper, the purpose of this study is to establish some existence and uniqueness results for the problem (1) by using Krasnoselskii's fixed-point theorem, Schauder fixed point theorem, Leray-Schauder's degree theory and the Banach contraction principle. Though the tools used in this paper are standard, their application in the framework of the given problem is new. Furthermore, instead of $f(\tau, \omega(\tau))$, we consider the nonlinear term $f(\tau, \omega(\tau), {}^C_0D^{\beta}_{\tau}\omega(\tau))$, which leads to extra difficulties. Finally, the multi-point is involved in boundary conditions.

This paper is organized as follows. In Section 2, we introduce some basic definitions and preliminaries results. In Section 3, we prove the main results of this paper, which includes the existence and uniqueness of solutions to the problem (1). Some examples are given in Section 4.

2 Preliminaries

In this section, we sum up some definitions, lemmas and preliminary facts will be applied to this paper.

Definition 1 (see[11]) The fractional left, right and Riemann-Liouville fractional integral of order $n - 1 < \alpha \leq n$ are defined as

$$\begin{aligned} ({}_0I^{\alpha}_{\tau}\omega)(\tau) &= \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau - \varsigma)^{\alpha-1} \omega(\varsigma) d\varsigma, \\ ({}_{\tau}I^{\alpha}_{T}\omega)(\tau) &= \frac{1}{\Gamma(\alpha)} \int_{\tau}^{T} (\varsigma - \tau)^{\alpha-1} \omega(\varsigma) d\varsigma, \\ ({}_0I^{\alpha}_{T}\omega)(\tau) &= \frac{1}{\Gamma(\alpha)} \int_0^{T} \mid \tau - \varsigma \mid^{\alpha-1} \omega(\varsigma) d\varsigma, \end{aligned}$$

where $n \in \mathbb{N}$, $0 \leq \tau \leq T$, Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} \tau^{\alpha-1} e^{-\tau} d\tau$.

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Definition 2 (see[11]) The classical Riesz-Caputo derivative of order $\alpha > 0$ is given by

$$\begin{aligned}{}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{T}\frac{\omega^{(n)}(\varsigma)}{|\tau-\varsigma|^{\alpha+1-n}}d\varsigma \\ &= \frac{1}{2}({}_{0}^{C}D_{\tau}^{\alpha} + (-1)^{n}{}_{\tau}^{C}D_{T}^{\alpha})\omega(\tau), \quad n\in\mathbb{N}, \quad 0\le\tau\le T,\end{aligned}$$

where ${}_{0}^{C}D_{\tau}^{\alpha}$ is the left hand side Caputo derivative, ${}_{\tau}^{C}D_{T}^{\alpha}$ is the right hand side Caputo derivative, which are respectively given by

$$\begin{aligned}{}_{0}^{C}D_{\tau}^{\alpha}\omega(\tau) &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{\tau}\frac{\omega^{(n)}(\varsigma)}{(\tau-\varsigma)^{\alpha+1-n}}d\varsigma, \quad n\in\mathbb{N}, \quad 0\le\tau\le T, \\ {}_{\tau}^{C}D_{T}^{\alpha}\omega(\tau) &= \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{\tau}^{T}\frac{\omega^{(n)}(\varsigma)}{(\varsigma-\tau)^{\alpha+1-n}}d\varsigma, \quad n\in\mathbb{N}, \quad 0\le\tau\le T.\end{aligned}$$

In addition, if $1 < \alpha \le 2$ and $\omega(\tau) \in AC^{n}[0,T]$, then

$${}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) = \frac{1}{2}({}_{0}^{C}D_{\tau}^{\alpha} - {}_{\tau}^{C}D_{T}^{\alpha})\omega(\tau).$$

Lemma 1 (see[11]) Let $n-1 < \alpha \le n,\ n\in\mathbb{N},\ \omega(\tau)\in AC^{n}[0,T]$, then

$${}_{0}I_{\tau}^{\alpha}{}_{0}^{C}D_{\tau}^{\alpha}\omega(\tau) = \omega(\tau) - \sum_{i=0}^{n-1}\frac{\omega^{(i)}(0)}{i!}(\tau-0)^{i}$$

and

$${}_{\tau}I_{T}^{\alpha}{}_{\tau}^{C}D_{T}^{\alpha}\omega(\tau) = (-1)^{n}\left(\omega(\tau) - \sum_{i=0}^{n-1}\frac{(-1)^{i}\omega^{(i)}(T)}{i!}(T-\tau)^{i}\right).$$

Thus, we have

$$\begin{aligned}{}_{0}I_{T}^{\alpha}{}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) &= \frac{1}{2}({}_{0}I_{\tau}^{\alpha}{}_{0}^{C}D_{\tau}^{\alpha} + {}_{\tau}I_{T}^{\alpha}{}_{0}^{C}D_{\tau}^{\alpha})\omega(\tau) + (-1)^{n}\frac{1}{2}({}_{0}I_{\tau}^{\alpha}{}_{\tau}^{C}D_{T}^{\alpha} + {}_{\tau}I_{T}^{\alpha}{}_{\tau}^{C}D_{T}^{\alpha})\omega(\tau) \\ &= \frac{1}{2}({}_{0}I_{\tau}^{\alpha}{}_{0}^{C}D_{\tau}^{\alpha} + (-1)^{n}{}_{\tau}I_{T}^{\alpha}{}_{\tau}^{C}D_{T}^{\alpha})\omega(\tau).\end{aligned}$$

In addition, if $1 < \alpha \le 2$ and $\omega(\tau) \in C^{1}[0,T]$, then

$${}_{0}I_{T}^{\alpha}{}_{0}^{RC}D_{T}^{\alpha}\omega(\tau) = \omega(\tau) - \frac{1}{2}(\omega(0)+\omega(T)) - \frac{1}{2}[\omega^{'}(0)\tau - \omega^{'}(T)(T-\tau)].$$

Lemma 2 Suppose that $\triangle := \sum_{i=1}^{m}\beta_i\xi_i^{\alpha-1} < 1$, $\beta_i > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $1 < \alpha \le 2$, $0 \le \tau \le 1$, then for $h \in L^{1}[0,1]$, the following BVP

$$\begin{cases} {}_{0}^{RC}D_{1}^{\alpha}\omega(\tau) = h(\tau), \quad \tau\in[0,1], \\ \omega(0) = 0, \quad \omega(1) = \sum_{i=1}^{m}\beta_i\omega(\xi_i) \end{cases} \tag{2}$$

has a unique solution

$$\omega(\tau) = \int_{0}^{1}G(\tau,\varsigma)h(\varsigma)d\varsigma + \frac{\tau}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_{0}^{1}g(\xi_i,\varsigma)h(s)d\varsigma,$$

where

$$\begin{aligned}G(\tau,\varsigma) &= g(\tau,\varsigma) - \frac{2\tau(1-\varsigma)^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}, \\ g(\tau,\varsigma) &= \frac{1}{\Gamma(\alpha)}\left[(1-\varsigma)^{\alpha-1} - (\alpha-1)(1-\varsigma)^{\alpha-2}(1-\tau) + |\tau-\varsigma|^{\alpha-1}\right].\end{aligned}$$

Proof From Lemma 1, we have

$$\begin{aligned}\omega(\tau) =&\frac{\omega(0)+\omega(1)}{2}+\frac{\omega^{'}(0)\tau-\omega^{'}(1)(1-\tau)}{2}+\int_0^{\tau}\frac{(\tau-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma\\&+\int_{\tau}^{1}\frac{(\varsigma-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma.\end{aligned}\tag{3}$$

Furthermore, we have

$$\omega^{'}(\tau)=\frac{\omega^{'}(0)+\omega^{'}(1)}{2}+\int_0^{\tau}\frac{(\tau-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}h(\varsigma)d\varsigma-\int_{\tau}^{1}\frac{(\varsigma-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\varsigma)d\varsigma.\tag{4}$$

Using the boundary condition $\omega(0)=0$, (3) and (4), we have

$$\begin{aligned}&\frac{1}{2}\omega(1)=\frac{1}{2}\omega^{'}(0)+\int_0^1\frac{(1-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma\\&\frac{1}{2}\omega^{'}(1)=\frac{1}{2}\omega^{'}(0)+\int_0^1\frac{(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}h(\varsigma)d\varsigma\\&\omega(\tau)=\frac{1}{2}\omega(1)+\frac{1}{2}(\omega^{'}(0)\tau-\omega^{'}(1)(1-\tau))+\int_0^1\frac{|\tau-\varsigma|^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma.\end{aligned}\tag{5}$$

From (5), we have

$$\begin{aligned}\omega(\tau)=&\omega^{'}(0)\tau+\int_0^1\frac{(1-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma-\int_0^1\frac{(1-\varsigma)^{\alpha-2}(1-\tau)}{\Gamma(\alpha-1)}h(\varsigma)d\varsigma\\&+\int_0^{\tau}\frac{(\tau-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma+\int_{\tau}^{1}\frac{(\varsigma-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\varsigma)d\varsigma\\=&\omega^{'}(0)\tau+\int_0^1 g(\tau,\varsigma)h(\varsigma)d\varsigma.\end{aligned}\tag{6}$$

By $\omega(1)=\sum_{i=1}^{m}\beta_i\omega(\xi_i)$, combining with (6), we obtain

$$\omega^{'}(0)=\frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 g(\xi_i,\varsigma)h(s)d\varsigma-\int_0^1\frac{2(1-\varsigma)^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}h(\varsigma)d\varsigma.\tag{7}$$

Substituting (7) into (6), we obtain

$$\omega(\tau)=\int_0^1 G(\tau,\varsigma)h(\varsigma)d\varsigma+\frac{\tau}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 g(\xi_i,\varsigma)h(s)d\varsigma.$$

The proof is completed. □

3 Main results

Let $X=C([0,1])$ be a Banach space with the maximum norm $\|\omega\|_X=\max_{\tau\in[0,1]}\|\omega(\tau)\|$, and the Banach space $Y=\{\omega:\omega\in C[0,1],\ {}_0^C D_\tau^\sigma\omega\in C[0,1], 0<\sigma<1\}$ with the norm

$$\|\omega\|_Y=\max_{\tau\in[0,1]}|\omega(\tau)|+\max_{\tau\in[0,1]}|{}_0^C D_\tau^\sigma\omega(\tau)|.$$

Denote

$$\mu=\left|\int_0^1\left\{\frac{(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{2(1-\varsigma)^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\right\}\varphi(\varsigma)d\varsigma\right|+\max_{\tau\in[0,1]}\left|\int_0^1\frac{|\tau-\varsigma|^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\varsigma)d\varsigma\right|,$$

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$$\nu = \max_{\tau\in[0,1]} \int_0^1 \mid G(\tau,\varsigma)\varphi(\varsigma) \mid d\varsigma + \frac{\Gamma(2-\beta)+1}{(1-\Delta)\Gamma(2-\beta)} \sum_{i=1}^m \beta_i \int_0^1 \mid g(\xi_i,\varsigma)\varphi(\varsigma) \mid d\varsigma,$$

$$\chi = \frac{1}{\Gamma(\alpha+1)} + \frac{|(1-\Delta)2^{1-\alpha}+2|}{(1-\Delta)\Gamma(\alpha+1)} + \frac{1}{\Gamma(2-\beta)}\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)} + \frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\right) + \rho.$$

where ρ is defined in (H4). In order to obtain our main results, we give some conditions on the function f:

(H1) $f : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous;

(H2) There exists a nonnegative real valued function $\varphi \in L[0,1]$ such that

$$|f(\tau,u,v)| \le \varphi(\tau) + k_1|u| + k_2|v|,$$

where $k_1, k_2 \ge 0$ are constants and $k_1 + k_2 < \chi^{-1}$;

(H3) There exist two constants $l_1,\ l_2 > 0$ such that

$$|f(\tau,u_1,v_1) - f(\tau,u_2,v_2)| \le l_1|u_1-u_2| + l_2|v_1-v_2|$$

for all $\tau\in[0,1]$ and all $u_1,u_2,v_1,v_2\in\mathbb{R}$;

(H4) The constant

$$\rho = \frac{(1+\Gamma(2-\beta))}{(1-\Delta)\Gamma(\alpha+1)\Gamma(2-\beta)} \sum_{i=1}^m \beta_i(|1-(1-\xi_i)\alpha| + \xi_i^\alpha + (1-\xi_i)^\alpha)$$

and $l_1 + l_2 < \rho^{-1}$.

The BVPs (1) can be converted into a fixed point problem $T\omega = \omega$, where the operator $T : Y \to Y$ is presented by

$$\begin{aligned}(T\omega)(\tau) = &\int_0^1 G(\tau,\varsigma) f(\varsigma,\omega(\varsigma), {}^C_0D^\beta_\tau\omega(\varsigma))d\varsigma \\ &+ \frac{\tau}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 g(\xi_i,\varsigma) f(\varsigma,\omega(\varsigma), {}^C_0D^\beta_\tau\omega(\varsigma))d\varsigma.\end{aligned} \tag{8}$$

Now, we present the first result of this paper by applying Krasnoselskii fixed point theorem.

Theorem 1 *Suppose that (H1), (H2), (H3) and (H4) hold. Then the fractional BVP (1) has at least one solution in Y.*

Proof Consider a ball

$$\Omega_{r_1} := \{\omega\in Y : \|\omega\|_Y \le r_1, \tau\in[0,1]\}.$$

where

$$r_1 \ge \frac{\nu + 4\mu(\Gamma(2-\beta))^{-1}}{1-(k_1+k_2)\chi}.$$

Obviously, Ω_{r_1} is a closed, convex and bounded set.

Next, we subdivided the operator T into two operator T_1, T_2 : $\Omega_{r_1} \to R$ as follows:

$$(T_1\omega)(\tau) = \int_0^1 G(\tau,\varsigma)f(\varsigma,\omega(\varsigma),{}_0^CD_\tau^\beta\omega(\varsigma))d\varsigma$$
$$(T_2\omega)(\tau) = \frac{\tau}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 g(\xi_i,\varsigma)f(\varsigma,\omega(\varsigma),{}_0^CD_\tau^\beta\omega(\varsigma))d\varsigma.$$

If $\omega \in \Omega_{r_1}$, by the condition (H2), then we have that

$$0 \le |\omega(\tau)| \le \max_{\tau\in[0,1]}|\omega(\tau)| \le \|\omega\|_Y \le r_1,$$
$$0 \le |{}_0^CD_\tau^\beta\omega(\tau)| \le \max_{\tau\in[0,1]}|{}_0^CD_\tau^\beta\omega(\tau)| \le \|\omega\|_Y \le r_1.$$

Hence,

$$f(\tau,\omega(\tau),{}_0^CD_\tau^\beta\omega(\tau)) \le \varphi(\tau) + (k_1+k_2)r_1. \tag{9}$$

The proof is divided into several steps.

Step 1. $T_1\omega + T_2\omega \in \Omega_{r_1}$. For any $\omega \in \Omega_{r_1}$, we have from (9)

$$\begin{aligned}|(T_1\omega)(\tau)| =& \left|\int_0^1 G(\tau,\varsigma)f(\varsigma,\omega(\varsigma),{}_0^CD_\tau^\beta\omega(\varsigma))d\varsigma\right| \\ \le& \int_0^1 |G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma + (k_1+k_2)r_1\left|\int_0^1 G(\tau,\varsigma)d\varsigma\right|.\end{aligned}$$

For notational convenience, we denote by

$$\begin{aligned}&{}_0I_\tau^\alpha(1) = \int_0^1 \frac{(1-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma, \quad {}_0I_1^\alpha(1) = \int_0^1 \frac{|\tau-\varsigma|^{\alpha-1}}{\Gamma(\alpha)}d\varsigma, \\ &{}_0I_\tau^\alpha(\tau) = \int_0^1 \frac{\tau(1-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma.\end{aligned} \tag{10}$$

From (6)-(7), we have

$$\begin{aligned}\left|\int_0^1 G(\tau,\varsigma)d\varsigma\right| =& \left[{}_0I_\tau^\alpha(1) - (1-\tau){}_0I_\tau^{\alpha-1}(1) + {}_0I_1^\alpha(1) - \frac{2}{1-\Delta}{}_0I_\tau^\alpha(\tau)\right] \\ \le& \left[|{}_0I_\tau^\alpha(1) - (1-\tau){}_0I_\tau^{\alpha-1}(1)| + |{}_0I_1^\alpha(1) - \frac{2}{1-\Delta}{}_0I_\tau^\alpha(\tau)|\right] \\ \le& \left|\frac{1}{\Gamma(\alpha+1)} + \frac{\tau-1}{\Gamma(\alpha)}\right| + \left|\frac{\tau^\alpha+(1-\tau)^\alpha}{\Gamma(\alpha+1)} + \frac{2\tau}{(1-\Delta)\Gamma(\alpha+1)}\right| \\ \le& \frac{1}{\Gamma(\alpha+1)} + \frac{(1-\Delta)2^{1-\alpha}+2}{(1-\Delta)\Gamma(\alpha+1)}.\end{aligned}$$

Thus,

$$|(T_1\omega)(\tau)| \le \max_{\tau\in[0,1]}\int_0^1 |G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma + (k_1+k_2)r_1\left(\frac{1}{\Gamma(\alpha+1)} + \frac{(1-\Delta)2^{1-\alpha}+2}{(1-\Delta)\Gamma(\alpha+1)}\right).$$

Furthermore, we have from (9)-(10)

$$
\begin{aligned}
&|(T_1\omega)'(\tau)| \\
=&\Big| -\int_0^1 \frac{2(1-\varsigma)^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)} f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma))d\varsigma + \int_0^1 \frac{(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)} f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma)) \\
&+ \int_0^\tau \frac{(\tau-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)} f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma)) - \int_\tau^1 \frac{(\varsigma-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma))\Big| \\
\le&\Big| \int_0^1 \frac{(1-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\varsigma)d\varsigma - \int_0^1 \frac{2(1-\varsigma)^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\varphi(\varsigma)d\varsigma\Big| + \Big|\int_0^\tau \frac{(\tau-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\varsigma)d\varsigma \\
&+ \int_\tau^1 \frac{(\varsigma-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\varsigma)d\varsigma\Big| + (k_1+k_2)r_1(|{}_0I_\tau^{\alpha-1}(1) - \frac{2}{1-\Delta}{}_0I_\tau^{\alpha}(1)| + {}_0I_1^{\alpha-1}(1)) \\
\le&\mu + (k_1+k_2)r_1\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-1)}\left(\frac{\tau^{\alpha-1}}{\alpha} + \frac{(1-\tau)^{\alpha-1}}{\alpha}\right)\right) \\
\le&\mu + (k_1+k_2)r_1\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)} + \frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\right),
\end{aligned}
$$

where use the following inequality

$$
\begin{aligned}
{}_0I_1^{\alpha-1}(1) =& \int_0^\tau \frac{(\tau-\varsigma)^{\alpha-2}}{\Gamma(\alpha-1)}d\varsigma + \int_\tau^1 \frac{(\varsigma-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\varsigma \\
=&\frac{1}{\Gamma(\alpha-1)}\left(\frac{\tau^{\alpha-1}}{\alpha} + \frac{(1-\tau)^{\alpha-1}}{\alpha}\right) \\
\le&\frac{2^{2-\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Therefore, we know that

$$
\begin{aligned}
\|T_1\omega\|_X \le& \max_{\tau\in[0,1]}\int_0^1 |G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma + (k_1+k_2)r_1 \\
&\times\left(\frac{1}{\Gamma(\alpha+1)} + \frac{(1-\Delta)2^{1-\alpha}+2}{(1-\Delta)\Gamma(\alpha+1)}\right), \qquad (11) \\
\|(T_1\omega)'\|_X \le&\mu + (k_1+k_2)r_1\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)} + \frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|(T_2\omega)(\tau)| =& \left|\frac{\tau}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 g(\xi_i,\varsigma)f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta\omega(\varsigma))d\varsigma\right| \\
\leq& \frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 \mid g(\xi_i,\varsigma)\varphi(\varsigma)\mid d\varsigma \\
&+\frac{(k_1+k_2)r_1}{1-\Delta}\sum_{i=1}^{m}\beta_i\left|\int_0^1 g(\xi_i,\varsigma)d\varsigma\right| \\
\leq& \frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta}\sum_{i=1}^{m}\beta_i\Bigg[\Bigg|{}_0I_\tau^\alpha(1) \\
&-(1-\xi_i){}_0I_\tau^{\alpha-1}(1)\Bigg|+\int_0^{\xi_i}\frac{(\xi_i-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma+\int_{\xi_i}^1\frac{(\varsigma-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma\Bigg] \\
\leq& \frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta} \\
&\times\sum_{i=1}^{m}\beta_i\left(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\left|\int_0^1 g(\xi_i,\varsigma)d\varsigma\right| =& \left|{}_0I_\tau^\alpha(1)-(1-\xi_i){}_0I_\tau^{\alpha-1}(1)+\int_0^{\xi_i}\frac{(\xi_i-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma+\int_{\xi_i}^1\frac{(\varsigma-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma\right| \\
\leq& |{}_0I_\tau^\alpha(1)-(1-\xi_i){}_0I_\tau^{\alpha-1}(1)|+\int_0^{\xi_i}\frac{(\xi_i-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma+\int_{\xi_i}^1\frac{(\varsigma-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma\Bigg| \\
\leq& \frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|(T_2\omega)'(\tau)| =& \left|\frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 g(\xi_i,\varsigma)f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta\omega(\varsigma))d\varsigma\right| \\
\leq& \frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta}\sum_{i=1}^{m}\beta_i\left|\int_0^1 g(\xi_i,\varsigma)d\varsigma\right| \\
\leq& \frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta} \\
&\times\sum_{i=1}^{m}\beta_i\left(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\right).
\end{aligned}
$$

Therefore, we know that

$$\begin{aligned}\|(T_2\omega)\|_X \leq&\frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1|g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta}\\&\times\sum_{i=1}^{m}\beta_i\Big(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\Big),\\\|(T_2\omega)'\|_X \leq&\frac{1}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1|g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)r_1}{1-\Delta}\\&\times\sum_{i=1}^{m}\beta_i\Big(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\Big).\end{aligned}$$

Furthermore, from Definition 2, we have

$$\begin{aligned}|({}_0^C D_\tau^\beta T_1\omega)(\tau)| &\leq \frac{1}{\Gamma(1-\beta)}\int_0^\tau(\tau-\varsigma)^{-\beta}|(T_1\omega)'(\varsigma)d\varsigma|\\&\leq\frac{\|(T_1\omega)'\|_X}{\Gamma(2-\beta)},\end{aligned}$$

and

$$\begin{aligned}|({}_0^C D_\tau^\beta T_2\omega)(\tau)| &\leq \frac{1}{\Gamma(1-\beta)}\int_0^\tau(\tau-\varsigma)^{-\beta}|(T_2\omega)'(\varsigma)d\varsigma|\\&\leq\frac{\|(T_2\omega)'\|_X}{(1-\beta)\Gamma(1-\beta)},\end{aligned}$$

which means that

$$\|({}_0^C D_\tau^\beta T_1\omega)\|_X\leq\frac{\|(T_1\omega)'\|_X}{\Gamma(2-\beta)},\quad \|({}_0^C D_\tau^\beta T_2\omega)\|_X\leq\frac{\|(T_2\omega)'\|_X}{\Gamma(2-\beta)}. \tag{12}$$

Therefore,

$$\begin{aligned}\|T\omega\|_Y =&\|T\omega\|_X+\|{}_0^C D_\tau^\beta T\omega\|_X\\\leq&\|T_1\omega\|_X+\|T_2\omega\|_X+\|{}_0^C D_\tau^\beta T_1\omega\|_X+\|{}_0^C D_\tau^\beta T_2\omega\|_X\\\leq&\nu+\frac{\mu}{\Gamma(2-\beta)}+(k_1+k_2)r_1\chi\\\leq&r_1,\end{aligned}$$

which yields that $T_1\omega+T_2\omega\in\Omega_{r_1}$.

Step 2. The operator T_1 compact and continuous.

From condition (H1), the operator T_1 is continuous. According to Step 1, we have from (11)-(12)

$$\begin{aligned}\|T_1\omega\|_X\leq&\max_{\tau\in[0,1]}\int_0^1|G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma+(k_1+k_2)r_1\\&\times\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{|(1-\Delta)2^{1-\alpha}+2|}{(1-\Delta)\Gamma(\alpha+1)}\Big),\\\|({}_0^C D_\tau^\beta T_1\omega)\|_X\leq&\frac{\|(T_1\omega)'\|_X}{\Gamma(2-\beta)}.\end{aligned}$$

Thus, for $\forall \omega \in \Omega_{r_1}$, we have

$$\begin{aligned}
\|T_1\omega\|_Y =& \|T_1\omega\|_X + \|({}_0^C D_\tau^\beta T_1\omega)(\tau)\|_X \\
&\le \max_{\tau\in[0,1]} \int_0^1 |G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma + (k_1+k_2)r_1 \\
&\times \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|(1-\Delta)2^{1-\alpha}+2|}{(1-\Delta)\Gamma(\alpha+1)}\right) \\
&+ \frac{\mu}{\Gamma(2-\beta)} + (k_1+k_2)r_1\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)} + \frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\right),
\end{aligned}$$

which means that T_1 is uniformly bounded on Ω_{r_1}.

Next we prove the compactness of the operator T_1.

For any $0 \le \tau_1 < \tau_2 \le 1, \omega \in \Omega_{r_1}$, let $M = \max_{\tau\in[0,1],\omega\in\Omega_{r_1}} f(\tau,\omega(\tau), {}_0^C D_\tau^\beta \omega(\varsigma))$ $+1$, we have

$$\begin{aligned}
|(T_1\omega)(\tau_1) - (T_1\omega)(\tau_2)| \le& \int_0^1 |G(\tau_1,\varsigma) - G(\tau_2,\varsigma)| \mid f(\varsigma,\omega(\varsigma), {}_0^C D_\tau^\beta \omega(\varsigma)) \mid d\varsigma \\
\le& M\int_0^1 |G(\tau_1,\varsigma) - G(\tau_2,\varsigma)|d\varsigma \\
\le& M\Bigg\{\int_0^1 \frac{2(\tau_2-\tau_1)(1-\varsigma)^{\alpha-1}}{(1-\nabla)\Gamma(\alpha)} + \frac{(1-\varsigma)^{\alpha-2}(\tau_2-\tau_1)}{\Gamma(\alpha-1)}d\varsigma \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_0^1 |\tau_2-\varsigma|d\varsigma - \frac{1}{\Gamma(\alpha)}\int_0^1 |\tau_1-\varsigma|d\varsigma\right|\Bigg\} \\
\le& M\{I_1 + I_2\},
\end{aligned}$$

where

$$\begin{aligned}
I_1 =& \int_0^1 \frac{2(\tau_2-\tau_1)(1-\varsigma)^{\alpha-1}}{(1-\nabla)\Gamma(\alpha)} + \frac{(1-\varsigma)^{\alpha-2}(\tau_2-\tau_1)}{\Gamma(\alpha-1)}d\varsigma, \\
I_2 =& \left|\frac{1}{\Gamma(\alpha)}\int_0^1 |\tau_2-\varsigma|^{\alpha-1}d\varsigma - \frac{1}{\Gamma(\alpha)}\int_0^1 |\tau_1-\varsigma|^{\alpha-1}d\varsigma\right|.
\end{aligned}$$

Obviously, it is easy to see that $I_1 \to 0$ as $\tau_2 \to \tau_1$.

On the other hand, we have

$$\begin{aligned}
I_2 =& \Bigg|\int_0^{\tau_1} \frac{(\tau_1-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma - \int_0^{\tau_2} \frac{(\tau_2-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma \\
&+ \int_{\tau_1}^1 \frac{(\varsigma-\tau_1)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma - \int_{\tau_2}^1 \frac{(\varsigma-\tau_2)^{\alpha-1}}{\Gamma(\alpha)}d\varsigma\Bigg| \\
\le& \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} [(\tau_2-\varsigma)^{\alpha-1} - (\tau_1-\varsigma)^{\alpha-1}]d\varsigma + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-\varsigma)^{\alpha-1}d\varsigma \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\tau_2}^1 [(\varsigma-\tau_1)^{\alpha-1} - (\varsigma-\tau_2)^{\alpha-1}]d\varsigma + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\varsigma-\tau_2)^{\alpha-1}d\varsigma \\
=& \frac{1}{\Gamma(\alpha+1)}[\tau_2^\alpha - \tau_1^\alpha + (1-\tau_1)^\alpha - (1-\tau_2)^\alpha - (\tau_2-\tau_1)^\alpha - (\tau_1-\tau_2)^\alpha]
\end{aligned}$$

tending to 0 as $\tau_2 \to \tau_1$.

Furthermore, we obtain

$$\begin{aligned}&|{}_0^C D_\tau^\beta (T_1\omega)(\tau_1) - {}_0^C D_\tau^\beta (T_1\omega)(\tau_2)|\\&= \left|\frac{1}{\Gamma(1-\beta)}\int_0^{\tau_1}(\tau_1-\varsigma)^{-\beta}(T_1\omega)'(\varsigma)d\varsigma - \frac{1}{\Gamma(1-\beta)}\int_0^{\tau_2}(\tau_2-\varsigma)^{-\beta}(T_1\omega)'(\varsigma)d\varsigma\right|\\&\le \frac{\|(T_1\omega)'\|_X}{\Gamma(1-\beta)}\left|\int_0^{\tau_1}(\tau_1-\varsigma)^{-\beta}d\varsigma - \int_0^{\tau_2}(\tau_2-\varsigma)^{-\beta}d\varsigma\right|\\&\le \frac{\|(T_1\omega)'\|_X}{\Gamma(2-\beta)}|\tau_1^{1-\beta} - \tau_2^{1-\beta}|\end{aligned}$$

tending to 0 as $\tau_2 \to \tau_1$. So, T_1 is relatively compact on Ω_{r_1}. Hence, T_1 is compact on Ω_{r_1} by the Arzela-Ascoli Theorem.

Step 3. The operator T_2 is a contraction mapping.

For any $\omega,\ \nu \in \Omega_{r_1}$, $\tau \in [0,1]$, from (H3), we have

$$\begin{aligned}&|(T_2\omega)(\tau) - (T_2\nu)(\tau)|\\&\le \frac{\tau}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 |g(\xi_i,\varsigma)||f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma)) - f(\varsigma,\nu(\varsigma),{}_0^C D_\tau^\beta \nu(\varsigma))d\varsigma\\&\le \frac{1}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 |g(\xi_i,\varsigma)|(l_1+l_2)\|\omega-\nu\|_Y d\varsigma\\&\le \frac{l_1+l_2}{(1-\Delta)\Gamma(\alpha+1)}\sum_{i=1}^m \beta_i(|1-(1-\xi_i)\alpha| + \xi_i^\alpha + (1-\xi_i)^\alpha)\|\omega-\nu\|_Y.\end{aligned}$$

On the other hand,

$$(T_2\omega)'(\tau) = \frac{1}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 g(\xi_i,\varsigma) f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma))d\varsigma,$$

and

$$\begin{aligned}&|(T_2\omega)'(\tau) - (T_2\nu)'(\tau)|\\&= \left|\frac{1}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 g(\xi_i,\varsigma)(f(\varsigma,\omega(\varsigma),{}_0^C D_\tau^\beta \omega(\varsigma)) - f(\varsigma,\nu(\varsigma),{}_0^C D_\tau^\beta \nu(\varsigma)))d\varsigma\right|\\&\le \frac{1}{1-\Delta}\sum_{i=1}^m \beta_i \int_0^1 |g(\xi_i,\varsigma)|(l_1+l_2)\|\omega-\nu\|_Y d\varsigma\\&\le \frac{l_1+l_2}{(1-\Delta)\Gamma(\alpha+1)}\sum_{i=1}^m \beta_i(|1-(1-\xi_i)\alpha| + \xi_i^\alpha + (1-\xi_i)^\alpha)\|\omega-\nu\|_Y.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&|{}_0^C D_\tau^\beta (T_2\omega)(\tau) - {}_0^C D_\tau^\beta (T_2\nu)(\tau)|\\&= \left|\frac{1}{\Gamma(1-\beta)}\int_0^{\tau}(\tau-\varsigma)^{-\beta}(T_2\omega)'(\varsigma)d\varsigma - \frac{1}{\Gamma(1-\beta)}\int_0^{\tau}(\tau-\varsigma)^{-\beta}(T_2\nu)'(\varsigma)d\varsigma\right|\\&\le \frac{\|(T_2\omega)' - (T_2\nu)'\|_X}{\Gamma(1-\beta)}\int_0^{\tau}(\tau-\varsigma)^{-\beta}d\varsigma\\&\le \frac{(l_1+l_2)\|\omega-\nu\|_Y}{(1-\Delta)\Gamma(\alpha+1)\Gamma(2-\beta)}\sum_{i=1}^m \beta_i(|1-(1-\xi_i)\alpha| + \xi_i^\alpha + (1-\xi_i)^\alpha).\end{aligned}$$

Thus, it follows that

$$\|T_2\omega - T_2\nu\|_Y \leq (l_1 + l_2)\rho\|\omega - \nu\|_Y, \quad \text{and} \quad (l_1 + l_2)\rho < 1.$$

This means that T_2 is a contraction.

It follows Krasnoselskii fixed point theorem that the BVP(1) has at least one solution $\omega \in Y$. □

We change the condition (H2) to the following conditions:

(H2)′ There exists a nonnegative real-valued functions $\varphi \in L[0,1]$ such that

$$|f(\tau, u, v)| \leq \varphi(\tau) + k_1|u|^{\delta_1} + k_2|v|^{\delta_2},$$

where $k_1, k_2 \geq 0$ are constants and $\delta_1, \delta_2 \in (0,1)$; or

(H2)′′ $|f(\tau, u, v)| \leq \varphi(\tau) + k_1|u|^{\delta_1} + k_2|v|^{\delta_2}$, where $k_1, k_2 \geq 0$ are constants and $\delta_1, \delta_2 \in (1, +\infty)$;

Remark 1 In Theorem 1, the function f is required to satisfy the conditions (H2) and (H3). If (H2)′ or (H2)′′ is satisfied, the function f generally does not meet the condition (H3). Thus, if the conditions (H1)-(H2)′ or (H1)-(H2)′′ are satisfied, we apply Schauder fixed theorem to obtain the existence result of to (1). Meanwhile, if the conditions (H1)-(H2) are satisfied, we can also obtain the existence of the solution of (1) through the Schauder fixed theorem.

Theorem 2 *Suppose that (H1)-(H2)′ hold. Then the fractional BVP (1) has at least one solution in Y.*

Proof Define

$$\Omega_{r_2} := \{\omega \in Y : \|\omega\|_Y \leq r_2, \tau \in [0,1]\}.$$

where

$$r_2 \geq \max\left\{4\nu, \frac{4\mu}{\Gamma(2-\beta)}, (4k_1\chi)^{\frac{1}{1-\delta_1}}, (4k_2\chi)^{\frac{1}{1-\delta_2}}\right\}.$$

Obviously, Ω_{r_2} is a closed, convex and bounded set. Consider the operator T defined in (8) on Ω_{r_2}. Similar to the Step 1 in the proof process of Theorem 1, we know that $T(\Omega_{r_2}) \subset \Omega_{r_2}$, i.e., $T(\Omega_{r_2})$ is a uniformly bounded set.

Next, we will show that T is completely continuous.

In view of the continuity of f and G, the operator T is continuous.

Let τ_1, $\tau_2 \in [0,1]$ and $\omega \in \Omega_{r_2}$, then we have

$$\begin{aligned}
&|(T\omega)(\tau_1) - (T\omega)(\tau_2)| \\
&\leq \frac{|\tau_1 - \tau_2|}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i, \varsigma)f(\varsigma, \omega(\varsigma), {}_0^C D_\tau^\beta \omega(\varsigma))|d\varsigma + |(T_1\omega)(\tau_1) - (T_1\omega)(\tau_2)| \\
&\leq \frac{M|\tau_1 - \tau_2|}{1-\Delta}\sum_{i=1}^{m}\beta_i\int_0^1 |g(\xi_i, \varsigma)|d\varsigma + |(T_1\omega)(\tau_1) - (T_1\omega)(\tau_2)|
\end{aligned}$$

tending to 0 as $\tau_1 \to \tau_2$. That is, T_1 is equicontinuous.

$$\begin{aligned}
&|{}_0^C D_\tau^\beta (T\omega)(\tau_1) - {}_0^C D_\tau^\beta (T\omega)(\tau_2)| \\
&\leq \left| \frac{1}{\Gamma(1-\beta)} \int_0^{\tau_1} (\tau_1 - \varsigma)^{-\beta} (T\omega)'(\varsigma) d\varsigma - \frac{1}{\Gamma(1-\beta)} \int_0^{\tau_2} (\tau_2 - \varsigma)^{-\beta} (T\omega)'(\varsigma) d\varsigma \right| \\
&\leq \frac{\|(T_2\omega)\|_X' + \|(T_1\omega)\|_X'}{\Gamma(2-\beta)} |\tau_1^{1-\beta} - \tau_2^{1-\beta}| \\
&\to 0 \text{ as } \tau_1 \to \tau_2.
\end{aligned}$$

Therefore, we have $\|(T\omega)(\tau_1) - (T\omega)(\tau_2)\|_Y \to 0$ as $\tau_1 \to \tau_2$ for $\omega \in \Omega_{r_2}$. According to the Arzela-Ascoli theorem, we claim that T is completely continuous. Thus, the Schauder fixed point theorem implies the existence of a solution in Ω_{r_2} for the BVPs (1). □

Theorem 3 *Suppose that (H1)-(H2)$''$ hold. Then the fractional BVP (1) has at least one solution in Y.*

Proof The proof is similar to that of Theorem 2, so it is omitted. □

Next, we apply Leray-Schauder's degree theory to obtain the existence result of solution to (1).

Theorem 4 *Suppose that assumptions (H1) and (H2) hold. Then the fractional BVP (1) has at least one solution on Y.*

Proof Introduce a suitable ball $\Omega_{r_3} \subset Y$ as

$$\Omega_{r_3} := \{\omega \in Y : \|\omega\|_Y \leq r_3, \tau \in [0,1]\},$$

where r_3 is a positive constant and will be given later.

Obviously, Ω_{r_3} is a closed, convex and bounded set.

Consider the operator T defined in (8) on Ω_{r_3}. We will prove that $T : \bar{\Omega}_{r_3} \to Y$ satisfies the condition

$$0 \notin (I - \vartheta T)(\partial\Omega_{r_3}), \quad \forall \omega \in \partial\Omega_{r_3}, \quad \forall \vartheta \in [0,1], \tag{13}$$

where I is the identity operator. Introduce the homotopy

$$h_\vartheta(\omega) = H(\vartheta, \omega) = \omega - \vartheta T\omega. \tag{14}$$

Next, we prove that h_ϑ is completely continuous.

If $\omega \in \Omega_{r_3}$, then we have that

$$\begin{aligned}
&0 \leq |\omega(\tau)| \leq \max_{\tau\in[0,1]} |\omega(\tau)| \leq \|\omega\|_Y \leq r_3, \\
&0 \leq |{}_0^C D_\tau^\beta \omega(\tau)| \leq \max_{\tau\in[0,1]} |{}_0^C D_\tau^\beta \omega(\tau)| \leq \|\omega\|_Y \leq r_3.
\end{aligned}$$

Hence,

$$f(\tau, \omega(\tau), {}_0^C D_\tau^\beta \omega(\tau)) \leq \varphi(\tau) + (k_1 + k_2) r_3.$$

As argued Theorem 2, T is continuous, uniformly bounded and equicontinuous. Therefore, according to the Arzela-Ascoli theorem, from (14), we know that h_ϑ is completely continuous. If (13) is satisfied, then the Leray-Schauder degrees are well defined. From the homotopy invariance and normalization degree, it follows that

$$\begin{aligned}\deg(h_\vartheta,\Omega_{r_3},0)=&\deg((I-\vartheta T),\Omega_{r_3},0)=\deg(h_1,\Omega_{r_3},0)\\ =&\deg(h_0,\Omega_{r_3},0)=\deg(I,\Omega_{r_3},0)=1\neq 0,\end{aligned}$$

since $0\in\Omega_{r_3}$. From the nonzero property of the Leray-Schauder degree, we get $h_1(\omega)=\omega-T\omega=0$ for at least one $\omega\in\Omega_{r_3}$. To give the value of r_3, we suppose that $\omega(\tau)=\vartheta T\omega(\tau)$ for some $\vartheta\in[0,1]$ and for all $\tau\in[0,1]$. Thus,

$$\begin{aligned}&|\omega(\tau)|=|\vartheta(T\omega)(\tau)|\leq\|T\omega\|_X\leq\|T_1\omega\|_X+\|T_2\omega\|_X\\ \leq&\max_{\tau\in[0,1]}\int_0^1|G(\tau,\varsigma)\varphi(\varsigma)|d\varsigma+(k_1+k_2)\|\omega\|_Y\left(\frac{1}{\Gamma(\alpha+1)}+\frac{|(1-\Delta)2^{1-\alpha}+2|}{(1-\Delta)\Gamma(\alpha+1)}\right)\\ &+\frac{1}{1-\Delta}\sum_{i=1}^m\beta_i\int_0^1|g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)\|\omega\|_Y}{1-\Delta}\\ &\times\sum_{i=1}^m\beta_i\left(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\right),\end{aligned}$$

and

$$\begin{aligned}&|{}_0^CD_\tau^\beta\omega(\tau)|=|\vartheta{}_0^CD_\tau^\beta T\omega(\tau)|\leq\|{}_0^CD_\tau^\beta T\omega\|_X\leq\|{}_0^CD_\tau^\beta T_1\omega\|_X+\|{}_0^CD_\tau^\beta T_2\omega\|_X\\ \leq&\frac{\mu}{\Gamma(2-\beta)}+\frac{(k_1+k_2)\|\omega\|_Y}{\Gamma(2-\beta)}\left(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)}+\frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\right)\\ &+\frac{1}{\Gamma(2-\beta)(1-\Delta)}\sum_{i=1}^m\beta_i\int_0^1|g(\xi_i,\varsigma)\varphi(\varsigma)|d\varsigma+\frac{(k_1+k_2)\|\omega\|_Y}{\Gamma(2-\beta)(1-\Delta)}\\ &\times\sum_{i=1}^m\beta_i\left(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\|\omega\|_Y=&\|\omega\|_X+\|{}_0^CD_\tau^\beta\omega\|_X=\vartheta\|T\omega\|_Y=\vartheta\|T\omega\|_X+\vartheta\|{}_0^CD_\tau^\beta T\omega\|_X\\ \leq&\nu+\frac{\mu}{\Gamma(2-\beta)}+(k_1+k_2)\chi\|\omega\|_Y,\end{aligned}$$

which means that

$$\|\omega\|_Y\leq(1-(k_1+k_2)\chi)^{-1}\left(\nu+\frac{\mu}{\Gamma(2-\beta)}\right).$$

Let $r_3=(1-(k_1+k_2)\chi)^{-1}\left(\nu+\frac{\mu}{\Gamma(2-\beta)}\right)+1$, the inequality (13) is satisfied. □

Theorem 5 *Suppose that (H1) and (H3) hold. If $l_1+l_2<\chi^{-1}$, then the BVP (1) has a unique solution.*

Proof By condition (H3), we obtain following estimate:

$$\begin{aligned}
&|(T\omega)(\tau)-(T\nu)(\tau)|\\
\leq&\int_0^1|G(\tau,\varsigma)||f(\varsigma,\omega(\varsigma),{}_0^CD_\tau^\beta\omega(\varsigma))-f(\varsigma,\nu(\varsigma),{}_0^CD_\tau^\beta\nu(\varsigma))|d\varsigma\\
&+\frac{1}{1-\Delta}\sum_{i=1}^m\beta_i\int_0^1|g(\xi_i,\varsigma)||f(\varsigma,\omega(\varsigma),{}_0^CD_\tau^\beta\omega(\varsigma))-f(\varsigma,\nu(\varsigma),{}_0^CD_\tau^\beta\nu(\varsigma))|d\varsigma\\
\leq&(l_1+l_2)\|\omega-\nu\|_Y\Big(\int_0^1|G(\tau,s)|d\varsigma+\frac{1}{1-\Delta}\sum_{i=1}^m\beta_i\int_0^1|g(\xi_i,\varsigma)|d\varsigma\Big)\\
\leq&\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{|(1-\Delta)2^{1-\alpha}+2|}{(1-\Delta)\Gamma(\alpha+1)}+\frac{1}{1-\Delta}\sum_{i=1}^m\beta_i\Big(\frac{|1-(1-\xi_i)\alpha|}{\Gamma(\alpha+1)}\\
&+\frac{1}{\Gamma(\alpha+1)}(\xi_i^\alpha+(1-\xi_i)^\alpha)\Big)(l_1+l_2)\|\omega-\nu\|_Y,
\end{aligned}$$

and

$$\begin{aligned}
|(T\omega)'(\tau)-(T\nu)'(\tau)|=&|(T_1\omega)'(\tau)-(T_1\nu)'(\tau)+(T_2\omega)'(\tau)-(T_2\nu)'(\tau)|\\
\leq&|(T_1\omega)'(\tau)-(T_1\nu)'(\tau)|+|(T_2\omega)'(\tau)-(T_2\nu)'(\tau)|\\
\leq&\Big(\frac{2-\alpha(1-\Delta)}{(1-\Delta)\Gamma(\alpha+1)}+\frac{2^{2-\alpha}}{\Gamma(\alpha+1)}\Big)(l_1+l_2)\|\omega-\nu\|_Y\\
&+\frac{l_1+l_2}{(1-\Delta)\Gamma(\alpha+1)}\sum_{i=1}^m\beta_i(|1-(1-\xi_i)\alpha|+\xi_i^\alpha+(1-\xi_i)^\alpha)\|\omega-\nu\|_Y,
\end{aligned}$$

$$\begin{aligned}
|{}_0^CD_\tau^\beta(T\omega)(\tau)-{}_0^CD_\tau^\beta(T\nu)(\tau)|\leq&\frac{1}{\Gamma(1-\beta)}\int_0^\tau(\tau-\varsigma)^{-\beta}|(T\omega)'(\varsigma)-(T\nu)'(\varsigma)|d\varsigma\\
\leq&\frac{1}{\Gamma(2-\beta)}\|(T\omega)'-(T\nu)'\|_X.
\end{aligned}$$

Thus, we obtain that

$$\|(T\omega)'(\tau)-(T\nu)'(\tau)\|_Y<(l_1+l_2)\chi\|\omega-\nu\|_Y \text{ and } (l_1+l_2)\chi<1,$$

which means that T is a contraction. Therefore, the BVP (1) has a unique solution. □

4 Examples

Example 1 Consider the following BVP

$$\begin{aligned}
&{}_0^{RC}D_1^{\frac{3}{2}}\omega(\tau)=f(\tau,\omega(\tau),{}_0^CD_\tau^{\frac{1}{2}}\omega(\tau)),\ \tau\in[0,1],\ \alpha\in(1,2],\\
&\omega(0)=0,\quad \omega(1)=\frac{1}{4}\omega\left(\frac{1}{2}\right)+\frac{3}{8}\omega\left(\frac{3}{4}\right)+\frac{5}{16}\omega\left(\frac{7}{8}\right).
\end{aligned}\tag{15}$$

Taking

$$\begin{aligned}
&\beta_i=(2i-1)\left(\frac{1}{2}\right)^{i+1},\quad \xi_i=1-\left(\frac{1}{2}\right)^i,\quad i=1,2,3,\\
&f(\tau,\omega,\nu)=2\tau^2\Big(\sin^2\left(\frac{\pi}{200}\omega+\frac{1}{3}\right)+\frac{\pi}{100}\nu+1\Big).
\end{aligned}$$

By computation, we deduced that

$$f(\tau,\omega,\nu)\le 4\tau^2+1+\frac{\pi}{50}|\nu|,$$
$$|f(\tau,\omega_1,\nu_1)-f(\tau,\omega_2,\nu_2)|\le\frac{\pi}{100}|\omega_1-\omega_2|+\frac{\pi}{50}|\nu_1-\nu_2|.$$

Let $\varphi(\tau)=4\tau^2+1$, $k_1=0$, $k_2=l_2=\frac{\pi}{50}$, $l_1=\frac{\pi}{100}$. Furthermore,

$$\Delta=\sum_{i=1}^{3}\beta_i\xi_i^{\alpha-1}\approx 0.5774<1,\ \ \rho\approx 4.8782,$$
$$l_1+l_2=\frac{3\pi}{100},\ \ (l_1+l_2)\rho\approx 0.4598<1,$$
$$k_1+k_2=\frac{\pi}{50},\ \ \chi\approx 15.0825,\ \ (k_1+k_2)\chi\approx 0.9477<1.$$

Hence, the conditions (H1)-(H4) are satisfied. By Theorem 1, the BVPs (1) has a solution.

Example 2 Consider the following BVP

$$\begin{aligned}&{}_0^{RC}D_1^{\alpha}\omega(\tau)=f(\tau,\omega(\tau),{}_0^{C}D_\tau^{\beta}\omega(\tau)),\ \tau\in[0,1],\ \alpha\in(1,2],\\&\omega(0)=0,\quad \omega(1)=\frac{1}{2}\omega\left(\frac{1}{4}\right)+\frac{1}{4}\omega\left(\frac{1}{2}\right)+\frac{1}{4}\omega\left(\frac{3}{4}\right),\end{aligned}\tag{16}$$

where $0<\beta\le 1$. Taking

$$\beta_i=\frac{(i-1)!}{2^i},\quad \xi_i=\frac{i}{4},\quad i=1,2,3,$$
$$f(\tau,\omega,\nu)=\frac{\lambda_1\tau^{v}e^{\Delta\tau}}{1+\tau^2}+\frac{\lambda_2\sin\pi\tau}{\sqrt{\pi+|\omega|}}|\omega|^{\delta_1}+\frac{\lambda_3e^{-v\tau}}{\sqrt{4+|\nu|}}|\nu|^{\delta_2},$$

where $v,\ \lambda_i(i=1,2,3)>0$. By computation, we deduced that

$$\Delta=\sum_{i=1}^{3}\beta_i\xi_i^{\alpha-1}=\sum_{i=1}^{3}\frac{(i-1)!}{2^i}\left(\frac{i}{4}\right)^{\alpha-1}<\left(\frac{3}{4}\right)^{\alpha-1}<1,$$
$$f(\tau,\omega,\nu)\le\varphi(\tau)+k_1|\omega|^{\delta_1}+k_2|\nu|^{\delta_2},$$

where $\varphi(\tau)=\frac{\lambda_1\tau^{v}e^{\Delta\tau}}{1+\tau^2}$, $k_1=\frac{\lambda_2}{\sqrt{\pi}}$, $k_2=\frac{\lambda_3}{2}$. For $0<\delta_1,\ \delta_2<1$, the condition $(H2)^{'}$ holds and for $\delta_1,\ \delta_2>1$, the condition $(H2)^{''}$ holds. Hence, from Theorem 2 and 3, the BVPs (1) has a solution.

Example 3 Consider the following BVP

$$\begin{aligned}&{}_0^{RC}D_1^{\frac{3}{2}}\omega(\tau)=f(\tau,\omega(\tau),{}_0^{C}D_\tau^{\frac{1}{2}}\omega(\tau)),\ \tau\in[0,1],\ \alpha\in(1,2],\\&\omega(0)=0,\quad \omega(1)=\frac{5}{7}\omega\left(\frac{2}{5}\right)+\frac{2}{3}\omega\left(\frac{3}{5}\right)+\frac{8}{21}\omega\left(\frac{4}{5}\right).\end{aligned}\tag{17}$$

Taking

$$\beta_1=\frac{5}{7},\ \beta_2=\frac{2}{3},\ \beta_3=\frac{8}{21},\ \xi_i=\frac{i+1}{5},\ i=1,2,3,$$
$$f(\tau,\omega,\nu)=\frac{e^{-\Delta\tau}(\omega+\nu)}{(30\sqrt{\pi}+25e^{-\Delta\tau})(1+\omega+\nu)}.$$

For $\omega_1,\ \nu_1,\ \omega_2,\ \nu_2 \in \mathbb{R}$, by computation,

$$|f(\tau,\omega_1\nu_1) - f(\tau,\omega_2\nu_2)| < \frac{1}{30\sqrt{\pi}+25}(|\omega_1-\omega_2| + |\nu_1-\nu_2|),$$

$l_1 = l_2 = \frac{1}{30\sqrt{\pi}+25}$. Hence, the condition (H3) is satisfied. Furthermore,

$$l_1 + l_2 = \frac{2}{30\sqrt{\pi}+25}, \quad \chi \approx 17.0292,\ (l_1+l_2)\chi \approx 0.4357 < 1.$$

Thus Theorem 5 guarantees the uniqueness of a solution for the BVPs (1).

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