Existence of solutions for 2D nonlinear fractional Volterra integral equations in Banach Space

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Abstract

In this article, we study the existence of solutions for 2D fractional Volterra integral equations, using Petryshyn's fixed point theorem and measure of non-compactness in Banach Space $C([0,a]\times[0,b],\mathbb{R})$. Our results involve particular results gained from earlier studies under some weaker conditions. We also present some examples of the equation to confirm the efficiency of our results.

Keywords. Fixed point theorem, Measure of non-compactness (MNC), Fractional integral equation (FIE).

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1 Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary order. The FDEs have focused considerable interest because of their advantages in complex modeling phenomena. These equations capture nonlocal relations in time and space with power-law memory kernels. Due to the broader applications of FDEs in science and engineering, work in this area has grown significantly worldwide [1, 29, 30]. The differential equations (DEs) involving Riemann-Liouville (R-L) differential operators of fractional order appear to be essential in modeling various physical phenomena [2, 4, 19].

As it is known, integral equations are one of the effective tools in describing many problems and phenomena in various fields of natural sciences. Especially functional integral and

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integro-differential equations are widely used in the fields of physics, mechanics, population dynamics and epidemiology, (see[6, 5, 14, 20]).

On the other hand, in recent years, the usefulness of the concept of MNC has come to the fore while giving results about the existence and behavior of solutions of nonlinear integral equations (see[9, 10, 11, 12, 13, 15, 28]). By using the fixed point theorems associated with the MNC, results can be given both about the existence of the solution of the equation under consideration and about the behaviour of the solution (see[7, 16, 17, 26, 27]).

In addition, two-dimensional integral equations have also been interesting by many authors and many studies have been presented in this field. For example Das et al. [8] studied the solvability for 2D equation

$$z(s,t) = L(s,t) + G\left(s, t, z(s,t), \int_0^s \int_0^t g(s,t,u,\tau,z(u,\tau))d\tau du\right) \tag{1}$$

for $(s,t) \in [0,1] \times [0,1]$.
Maleknejad et al. [21] discussed the existence result for the following 2D fractional Volterra integral equation

$$z(s,t) = L(s,t) + \frac{1}{\Gamma\beta_1 \Gamma\beta_2} \int_0^s \int_0^t \frac{g(s,t,u,\tau,z(u,\tau))}{(s-u)^{1-\beta_1}(t-\tau)^{1-\beta_2}} d\tau du \tag{2}$$

for $(s,t) \in [0,1] \times [0,1]$.
Maleknejad et al. [21] discussed the existence result for the following 2D fractional Fredholm integral equation

$$z(s,t) = L(s,t) + \frac{1}{\Gamma\beta_1 \Gamma\beta_2} \int_0^a \int_0^b \frac{g(s,t,u,\tau,z(u,\tau))}{(s-u)^{1-\beta_1}(t-\tau)^{1-\beta_2}} d\tau du \tag{3}$$

for $(s,t) \in [0,1] \times [0,1]$.
Mishra et al. [22] discussed the existence result for the following 2D fractional Volterra integral equation

$$z(s,t) = L(s,t) + \frac{h(s,t,z(s,t))}{\Gamma\beta_1 \Gamma\beta_2} \int_0^s \int_0^t \frac{g(s,t,u,\tau,z(u,\tau))}{(s-u)^{1-\beta_1}(t-\tau)^{1-\beta_2}} d\tau du \tag{4}$$

for $(s,t) \in [0,\infty) \times [0,a]$.
Further, famous 2D integral equations of Fredholm and Hammerstein type [24] have the form

$$z(s,t) = L(s,t) + \int_0^1 \int_0^1 f(s,t,u,\tau,z(u,\tau))d\tau du, \tag{5}$$

and

$$z(s,t) = L(s,t) + \int_0^s \int_0^t g_1(s,t,u,\tau) g_2(u,\tau,z(u,\tau))d\tau du. \tag{6}$$

In this paper, we examine the existence of the solution of following nonlinear 2D FIE (Fractional Integral Equation) in the space $C(D,\mathbb{R})$

$$z(s,t)$$
$$= F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, z(p_1(s,t)),...,z(p_m(s,t))\right)$$
$$\times G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g(s,t,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s)-u)^{1-\beta_1}(\phi_2(t)-\tau)^{1-\beta_2}}d\tau du, z(q_1(s,t)),...,z(q_n(s,t))\right), \tag{7}$$

where $D=[0,a]\times[0,b]$. We use Petryshyn's fixed point theorem associated with the MNC as the main tool.

It should see that Eq. (7) is more general than equations (1)-(6). For example, if

$$q(s,t)=\delta_2(s,t)=(s,t), F(s,t,z_1,z_2)=1, \beta_1=\beta_2=1, \varphi_2(s)=s, \phi_2(t)=t, m=n=1,$$

and

$$F(s,t,z_1,z_2)=1, G(s,t,z_1,z_2)=z_1, \varphi_2(s)=s, \phi_2(t)=t, m=n=1,$$

then Eq. (7) can be converted to Eq. (1) and Eq.(2) .

If $F(s,t,z_1,z_2)=1$, $G(s,t,z_1,z_2)=L(s,t)+z_1, \delta_2(s,t)=(s,t), m=n=1$, then Eq. (3) is obtained from Eq. (7).

If $F(s,t,z_1,z_2)=1$, $G(s,t,z_1,z_2)=L(s,t)+z_1.h(s,t,z)$, and $\delta_2(s,t)=(s,t), m=n=1$, then Eq. (7) can be converted to Eq. (4).

Again, if we put

$$F(s,t,z_1,z_2)=L(s,t)+z_1, \delta_1(s,t)=(s,t), \varphi_1(s)=\phi(t)=1, m=n=1,$$

then Eq. (7) can be converted to Eq. (5).

2 Preliminaries

In entire article, we use the following notations.

- $\mathbf{E}$: Real Banach space;
- B_ε: Open ball having origen as a center with radius ε;
- ∂B_ε : Sphere in $\mathbf{E}$ around 0 with radius ε;
- $co\bar{F}$: Closed convex hull of a set F.

Definition 2.1. [18] Let $F\subset\mathbf{E}$ and

$$\bar{\alpha}(F)=\inf\left\{\sigma>0: F=\bigcup_{i=1}^{m}F_i \text{ with diam}F_i\leq\sigma,\ i=1,2,...,n\right\}.$$

Hence, $0 \leq \bar{\alpha}(\digamma) < \infty$. $\bar{\alpha}(\digamma)$ is called the Kuratowski MNC.

Definition 2.2. [3] The Hausdorff MNC is defined as

$$\mu(\digamma) = \inf\{\sigma > 0 : \text{there exists a finite } \sigma\text{-net for } \digamma \text{ in } \mathbf{E}\}, \tag{8}$$

where a finite σ-net for $\digamma$ in $\mathbf{E}$ means that a set $\{z_1, z_2, ..., z_n\} \subset \mathbf{E}$ such that $B_\sigma(\mathbf{E}, z_1), B_\sigma(\mathbf{E}, z_2)$, ..., $B_\sigma(\mathbf{E}, z_n)$ over $\digamma$. These MNC are connected in the following way

$$\mu(\digamma) \leq \bar{\alpha}(\digamma) \leq 2\mu(\digamma),$$

for any bounded set $\digamma \subset \mathbf{E}$.

Theorem 2.1. *[25] Let $\digamma, \tilde{\digamma} \subset \mathbf{E}$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(\digamma) = 0$ *if and only if* $\digamma$ *is pre-compact;*

(ii) $\digamma \subseteq \tilde{\digamma} \implies \mu(\digamma) \leq \mu(\tilde{\digamma})$;

(iii) $\mu(\bar{co}\digamma) = \mu(\digamma)$;

(iv) $\mu(\digamma \cup \tilde{\digamma}) = \max\{\mu(\digamma), \mu(\tilde{\digamma})\}$;

(v) $\mu(\lambda\digamma) = |\lambda|\mu(\digamma)$;

(vi) $\mu(\digamma + \tilde{\digamma}) \leq \mu(\digamma) + \mu(\tilde{\digamma})$.

Here, we think the Banach space $C(D, \mathbb{R})$ with the usual norm

$$\|z\| = \sup\{|z(s,t)| : s \in [0,a], t \in [0,b]\}.$$

The modulus of continuity of $z \in D$ is described as

$$\omega(z, \sigma) = \sup\{|z(s,t) - z(\hat{s}, \hat{t})| : s, \hat{s} \in [0,a], t, \hat{t} \in [0,b], |s - \hat{s}|, |t - \hat{t}| \leq \sigma\}.$$

Further,

$$\omega(\digamma, \sigma) = \sup\{\omega(z, \sigma) : z \in \digamma\},$$

$$\omega_0(\digamma) = \lim_{\sigma \to 0} \omega(\digamma, \sigma).$$

In [3] it can be found that $\omega_0(\digamma)$ is a regular MNC in $C(D)$.

Definition 2.3. [23] Let $T : \mathbf{E} \to \mathbf{E}$ be a continuous mapping of $\mathbf{E}$. T is called a k- set contraction if for all $\digamma \subset \mathbf{E}$ with $\digamma$ bounded, $T(\digamma)$ is bounded and $\tilde{\alpha}(T\digamma) \leq k\tilde{\alpha}(\digamma), k \in (0,1)$. If

$$\tilde{\alpha}(T\digamma) < \tilde{\alpha}(\digamma), \text{for all } \tilde{\alpha}(\digamma) > 0,$$

then T is called condensing or densifying mapping.

Now, we present some lemmas and Petryshyn's fixed point theorems from [25] which are applied in the major results.

Theorem 2.2. *[25] Assume that $T : \bar{B}_\varepsilon \to \mathbf{E}$ is a condensing mapping which satisfy the boundary condition,*

$$T(z) = kz, \text{ for some } z \in \partial B_\varepsilon \text{ then } k \leq 1.$$

Then the set of fixed points of T in $\bar{B}_\varepsilon$ is non-empty.

Lemma 2.3. *[23, 25] Assume that $\mathbf{E}$ is a real Banach space. If the operators $\mathbf{F}$ and $\mathbf{G}$ satisfies the Petryshyn's condition on a bounded set $\digamma$ of $\mathbf{E}$ with constant $h < 1$ and $\tilde{h} < 1$, respectively, then the operator $T = \mathbf{F}.\mathbf{G}$ satisfies Petryshyn's condition (condensing map) on $\digamma$ with the constant $k = h.\tilde{h} < 1$; where $h < 1, \tilde{h} < 1$.*

3 Main results

In this part, we study the existence of the FIE (7) with the following assumption

(1) $F \in C(D \times \mathbb{R}^{m+1}, \mathbb{R})$, $G \in C(D \times \mathbb{R}^{n+1}, \mathbb{R})$ and $f, g \in C(\tilde{D} \times \mathbb{R}, \mathbb{R})$. Also, the functions $p_i, q_j, \delta_r : D \to D$ and $\varphi_r, \phi_r : D \to \mathbb{R}_+$ are continuous such that $\varphi_r(s) \leq C_1, \phi_r(t) \leq C_2$ for every $s \in [0, a]$, $t \in [0, b]$, where $\tilde{D} = D \times [0, C_1] \times [0, C_2]$, $1 \leq i \leq m$, $1 \leq j \leq n$, and r = 1, 2.

(2) There exist non-negative constants $h < 1, \tilde{h} < 1$ such that

$$|F(s, t, z_1, z_2, ..., z_{m+1}) - F(s, t, \tilde{z_1}, \tilde{z_2}, ..., \tilde{z_{m+1}})| \leq h \sum_{i=1}^{m+1} |z_i - \tilde{z_i}|$$

and

$$|G(s, t, z_1, z_2, ..., z_{n+1}) - G(s, t, \tilde{z_1}, \tilde{z_2}, ..., \tilde{z_{n+1}})| \leq \tilde{h} \sum_{j=1}^{n+1} |z_j - \tilde{z_j}|$$

hold for all $s, t \in D$ and $z_i, \tilde{z_i}, z_j, \tilde{z_j} \in \mathbb{R}$.

(3) There exists an $\varepsilon > 0$ such that the following bounded condition is satisfied

$$\sup\{H_1 \times H_2\} \leq \varepsilon,$$

where,

$$H_1 = \sup\{|F(s, t, z_1, z_2, ..., z_{m+1}))| : \text{for all} \quad s, t \in D \quad \text{and} \quad z_2, ..., z_{m+1} \in [-\varepsilon, \varepsilon], |z_1| \leq \frac{M_1 C_1^{\alpha_1} C_2^{\alpha_2}}{\Gamma(1 + \alpha_1)\Gamma(1 + \alpha_2)}\}.$$

$$M_1 = \sup\{|f(s, t, u, \tau, z)| : \text{for all} \quad (s, t, u, \tau) \in \tilde{D} \quad \text{and} \quad z \in [-\varepsilon, \varepsilon]\}.$$

$$H_2 = \sup\{|G(s, t, z_1, z_2, ..., z_{n+1})| : \text{for all} \quad s, t \in D \quad \text{and} \quad z_2, ..., z_{n+1} \in [-\varepsilon, \varepsilon], |z_1| \leq \frac{M_2 C_1^{\beta_1} C_2^{\beta_2}}{\Gamma(1 + \beta_1)\Gamma(1 + \beta_2)}\}.$$

$$M_2 = \sup\{|g(s, t, u, \tau, z)| : \text{for all} \quad (s, t, u, \tau) \in \tilde{D} \quad \text{and} \quad z \in [-\varepsilon, \varepsilon]\}.$$

Theorem 3.1. *Under the assumptions* (1) – (3), *Eq.* (7) *has at least one solution in* $\mathbf{E} = C(D, \mathbb{R})$.

Proof. Let the operators $\mathbf{F}, \mathbf{G} : \bar{B}_\varepsilon \to \mathbf{E}$ be in the following form

$$
\begin{aligned}
&(\mathbf{F}z)(s,t)\\
&= F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, z(p_1(s,t)),...,z(p_m(s,t))\right),
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbf{G}z)(s,t)\\
&= G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g(s,t,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s)-u)^{1-\beta_1}(\phi_2(t)-\tau)^{1-\beta_2}}d\tau du, z(q_1(s,t)),...,z(q_n(s,t))\right),
\end{aligned}
$$

for $(s,t) \in D$. Now, we define the operator T by setting

$$Tz = (\mathbf{F}z).(\mathbf{G}z).$$

Let $\varepsilon > 0$ and select arbitrary $z, x \in \bar{B}_\varepsilon$ such that $\|z - x\| \le \sigma$. Then for all $(s,t) \in D$, we have

$$
\begin{aligned}
&|(\mathbf{F}z)(s,t) - (\mathbf{F}x)(s,t)|\\
&= F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, z(p_1(s,t)),...,z(p_m(s,t))\right)\\
&- F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,x(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, x(p_1(s,t)),...,x(p_m(s,t))\right)\\
&\le F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, z(p_1(s,t)),...,z(p_m(s,t))\right)\\
&- F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, x(p_1(s,t)),...,x(p_m(s,t))\right)\\
&+ F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, x(p_1(s,t)),...,x(p_m(s,t))\right)\\
&- F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,x(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du, x(p_1(s,t)),...,x(p_m(s,t))\right)\\
&\le h\sum_{i=1}^{m}|z(p_i(s,t)) - x(p_i(s,t))|\\
&+ \frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{|f(s,t,u,\tau,z(\delta_1(u,\tau))) - f(s,t,u,\tau,x(\delta_1(u,\tau)))|}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du\\
&\le hm\|z - x\| + \frac{hC_1^{\alpha_1}C_2^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}\omega(f,\sigma)
\end{aligned}
$$

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Similarly, we get

$$
\begin{aligned}
&|(\mathbf{G}z)(s,t)-(\mathbf{G}x)(s,t)| \\
&=\left|G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,z\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, z\left(q_1\left(s,t\right)\right),...,z\left(q_n\left(s,t\right)\right)\right)\right. \\
&\left.-G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,x\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, x\left(q_1\left(s,t\right)\right),...,x\left(q_n\left(s,t\right)\right)\right)\right| \\
&\leq\left|G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,z\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, z\left(q_1\left(s,t\right)\right),...,z\left(q_n\left(s,t\right)\right)\right)\right. \\
&\left.-G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,z\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, x\left(q_1\left(s,t\right)\right),...,x\left(q_n\left(s,t\right)\right)\right)\right| \\
&+\left|G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,z\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, x\left(q_1\left(s,t\right)\right),...,x\left(q_n\left(s,t\right)\right)\right)\right. \\
&\left.-G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g\left(s,t,u,\tau,x\left(\delta_2\left(u,\tau\right)\right)\right)}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du, x\left(q_1\left(s,t\right)\right),...,x\left(q_n\left(s,t\right)\right)\right)\right| \\
&\leq\tilde{h}\sum_{i=1}^{n}|z(q_j(s,t))-x(q_j(s,t))| \\
&+\frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{\left|g\left(s,t,u,\tau,z\left(\delta_2\left(u,\tau\right)\right)\right)-g\left(s,t,u,\tau,x\left(\delta_2\left(u,\tau\right)\right)\right)\right|}{\left(\varphi_2\left(s\right)-u\right)^{1-\beta_1}\left(\phi_2\left(t\right)-\tau\right)^{1-\beta_2}}d\tau du \\
&\leq\tilde{h}n\|z-x\|+\frac{\tilde{h}C_1^{\beta_1}C_2^{\beta_2}}{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}\omega(g,\sigma),
\end{aligned}
$$

where

$$\omega(f,\sigma)=\sup\{|f(s,t,u,\tau,z)-f(s,t,u,\tau,x)|:(s,t,u,\tau)\in\tilde{D},z,x\in[-\varepsilon,\varepsilon],\|z-x\|\leq\sigma\},$$

and

$$\omega(g,\sigma)=\sup\{|g(s,t,u,\tau,z)-g(s,t,u,\tau,x)|:(s,t,u,\tau)\in\tilde{D},z,x\in[-\varepsilon,\varepsilon],\|z-x\|\leq\sigma\}.$$

By uniform continuity of the functions f and g on the subset $\tilde{D}\times[-\varepsilon,\varepsilon]$, we infer that $\omega(f,\sigma)$, and $\omega(g,\sigma)\to 0$ as $\sigma\to 0$. From the above estimate we get the operators $\mathbf{F}$ and $\mathbf{G}$ are continuous on the ball $\bar{B}_\varepsilon$. Hence the operator T is continuous on the ball $\bar{B}_\varepsilon$. Next, we prove that the operators $\mathbf{F}$ and $\mathbf{G}$ satisfy the densifying condition with respect to measure μ. For this take a nonempty a subset $\digamma$ of $\bar{B}_\varepsilon$. Further, choose an arbitrary number $\sigma>0$ and $(s_1,t_1),(s_2,t_2)\in D$ such that $|s_2-s_1|\leq\sigma,|t_2-t_1|\leq\sigma$.

Thus, we get

$$
\begin{aligned}
&|(\mathbf{F}z)(s_2,t_2)-(\mathbf{F}z)(s_1,t_1)| \\
&= F\Bigg(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_2)}\int_0^{\phi_1(t_2)}\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_2,t_2)),...,z(p_m(s_2,t_2))\Bigg) \\
&- F\Bigg(s_1,t_1,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_1,t_1)),...,z(p_m(s_1,t_1))\Bigg) \\
&\le F\Bigg(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_2)}\int_0^{\phi_1(t_2)}\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_2,t_2)),...,z(p_m(s_2,t_2))\Bigg) \\
&- F\Bigg(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_2,t_2)),...,z(p_m(s_2,t_2))\Bigg) \\
&+ F\Bigg(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_2,t_2)),...,z(p_m(s_2,t_2))\Bigg) \\
&- F\Bigg(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du, \\
&\qquad z(p_1(s_1,t_1)),...,z(p_m(s_1,t_1))\Bigg)
\end{aligned}
$$

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$$
\begin{aligned}
&+F\left(s_2,t_2,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du,\right.\\
&\qquad\left. z(p_1(s_1,t_1)),...,z(p_m(s_1,t_1))\right)\\
&-F\left(s_1,t_1,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du,\right.\\
&\qquad\left. z(p_1(s_1,t_1)),...,z(p_m(s_1,t_1))\right)\\
&\leq\frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left|\int_0^{\varphi_1(s_2)}\int_0^{\phi_1(t_2)}\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}d\tau du\right.\\
&\qquad\left.-\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\frac{f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du\right|\\
&+h\sum_{i=1}^{m}|z(p_i(s_2,t_2))-z(p_i(s_1,t_1))|+\omega_F(\sigma,\varepsilon)\\
&\leq\frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\left|\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))-f(s_1,t_1,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}\right|d\tau du\\
&+\frac{hM_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_0^{\phi_1(t_1)}\left|\frac{1}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}\right.\\
&\qquad\left.-\frac{1}{(\varphi_1(s_1)-u)^{1-\alpha_1}(\phi_1(t_1)-\tau)^{1-\alpha_2}}d\tau du\right|\\
&+\frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{\varphi_1(s_1)}^{\varphi_1(s_2)}\int_0^{\phi_1(t_1)}\left|\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}\right|d\tau du\\
&+\frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s_1)}\int_{\phi_1(t_1)}^{\phi_1(t_2)}\left|\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}\right|d\tau du\\
&+\frac{h}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{\varphi_1(s_1)}^{\varphi_1(s_2)}\int_{\phi_1(t_1)}^{\phi_1(t_2)}\left|\frac{f(s_2,t_2,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s_2)-u)^{1-\alpha_1}(\phi_1(t_2)-\tau)^{1-\alpha_2}}\right|d\tau du\\
&+h\sum_{i=1}^{m}|z(p_i(s_2,t_2))-z(p_i(s_1,t_1))|+\omega_F(\sigma,\varepsilon)\\
&\leq\frac{h\omega_f(\sigma,\varepsilon)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left[\frac{(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}}{\alpha_1}-\frac{\varphi_1(s_2)^{\alpha_1}}{\alpha_1}\right]\left[\frac{(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}}{\alpha_2}-\frac{\phi_1(t_2)^{\alpha_2}}{\alpha_2}\right]\\
&+\frac{hM_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left[\frac{\varphi_1(s_1)^{\alpha_1}\phi_1(t_1)^{\alpha_2}}{\alpha_1\alpha_2}\right.\\
&\qquad\left.-\left(\frac{(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}-\varphi_1(s_2)^{\alpha_1}}{\alpha_1}\right)\left(\frac{(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}-\phi_1(t_2)^{\alpha_2}}{\alpha_2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{hM_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\Big[\frac{(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}}{\alpha_1\alpha_2}\\
&\qquad+\frac{(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}\left[\varphi_1(s_2)^{\alpha_1}-(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}\right]}{\alpha_1\alpha_2}\\
&\qquad+\frac{(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}\left[\phi_1(t_2)^{\alpha_2}-(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}\right]}{\alpha_1\alpha_2}\Big]\\
&+h\omega(z,\omega(p,\sigma))+\omega_F(\sigma,\varepsilon)\\
&\le\frac{h\omega_f(\sigma,\varepsilon)\varphi_1(s_1)^{\alpha_1}\phi_1(t_1)^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big[\varphi_1(s_1)^{\alpha_1}\phi_1(t_2)^{\alpha_2}+(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}\\
&\times\{\phi_1(t_2)^{\alpha_2}-(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}\}+\varphi_1(s_2)^{\alpha_1}(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}-\varphi_1(s_2)^{\alpha_1}\phi_1(t_2)^{\alpha_2}\Big]\\
&+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Bigg[\varphi_1(s_2)^{\alpha_1}(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}\\
&\qquad+(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}\{\phi_1(t_2)^{\alpha_2}-(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}\}\Bigg]\\
&+h\omega(z,\omega(p,\sigma))+\omega_F(\sigma,\varepsilon)\\
&\le\frac{h\omega_f(\sigma,\varepsilon)C_1^{\alpha_1}C_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Bigg[(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}C_2^{\alpha_2}\\
&\quad+(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}C_2^{\alpha_1}+(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}C_1^{\alpha_1}\Bigg]\\
&+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big[(\varphi_1(s_2)-\varphi_1(s_1))^{\alpha_1}C_2^{\alpha_2}+(\phi_1(t_2)-\phi_1(t_1))^{\alpha_2}C_1^{\alpha_1}\Big]\\
&+h\omega(z,\omega(p,\sigma))+\omega_F(\sigma,\varepsilon).
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&|(\mathbf{G}z)(s_2,t_2)-(\mathbf{G}z)(s_1,t_1)|\\
&=G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_2)}\int_0^{\phi_2(t_2)}\frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}}d\tau du,\\
&\qquad z(q_1(s_2,t_2)),...,z(q_n(s_2,t_2))\Bigg)\\
&-G\Bigg(s_1,t_1,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,\\
&\qquad z(q_1(s_1,t_1)),...,z(q_n(s_1,t_1))\Bigg)
\end{aligned}
$$

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$$\leq G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_2)}\int_0^{\phi_2(t_2)}\frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_2,t_2)),...,z(q_n(s_2,t_2))\Bigg)$$

$$-G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_2,t_2)),...,z(q_n(s_2,t_2))\Bigg)$$

$$+G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_2,t_2)),...,z(q_n(s_2,t_2))\Bigg)$$

$$-G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_1,t_1)),...,z(q_n(s_1,t_1))\Bigg)$$

$$+G\Bigg(s_2,t_2,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_1,t_1)),...,z(q_n(s_1,t_1))\Bigg)$$

$$-G\Bigg(s_1,t_1,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du,$$

$$z(q_1(s_1,t_1)),...,z(q_n(s_1,t_1))\Bigg)$$

$$\leq \frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)}\Bigg|\int_0^{\varphi_2(s_2)}\int_0^{\phi_2(t_2)}\frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}}d\tau du$$

$$-\int_0^{\varphi_2(s_1)}\int_0^{\phi_2(t_1)}\frac{g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}}d\tau du\Bigg|$$

$$+\tilde{h}\sum_{j=1}^{n}|z(q_j(s_2,t_2))-z(q_j(s_1,t_1))|+\omega_G(\sigma,\varepsilon)$$

$$
\begin{aligned}
&\leq \frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^{\varphi_2(s_1)} \int_0^{\phi_2(t_1)} \left| \frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau))) - g(s_1,t_1,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}} \right| d\tau du \\
&+ \frac{\tilde{h}M_2}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^{\varphi_2(s_1)} \int_0^{\phi_2(t_1)} \left| \frac{1}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\alpha_2}} \right. \\
&\qquad\qquad \left. - \frac{1}{(\varphi_2(s_1)-u)^{1-\beta_1}(\phi_2(t_1)-\tau)^{1-\beta_2}} \right| d\tau du \\
&+ \frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{\varphi_2(s_1)}^{\varphi_2(s_2)} \int_0^{\phi_2(t_1)} \left| \frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}} \right| d\tau du \\
&+ \frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^{\varphi_2(s_1)} \int_{\phi_2(t_1)}^{\phi_2(t_2)} \left| \frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}} \right| d\tau du \\
&+ \frac{\tilde{h}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{\varphi_2(s_1)}^{\varphi_2(s_2)} \int_{\phi_2(t_1)}^{\phi_2(t_2)} \left| \frac{g(s_2,t_2,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s_2)-u)^{1-\beta_1}(\phi_2(t_2)-\tau)^{1-\beta_2}} \right| d\tau du \\
&+ \tilde{h} \sum_{j=1}^{n} |z(q_j(s_2,t_2)) - z(q_j(s_1,t_1))| + \omega_G(\sigma,\varepsilon) \\
&\leq \frac{\tilde{h}\omega_g(\sigma,\varepsilon)}{\Gamma(\beta_1)\Gamma(\beta_2)} \left[\frac{(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}}{\beta_1} - \frac{\varphi_2(s_2)^{\beta_1}}{\beta_1} \right] \left[\frac{(\phi_2(t_2)-\phi_2(t_1))^{\beta_2}}{\beta_2} - \frac{\phi_2(t_2)^{\beta_2}}{\beta_2} \right] \\
&+ \frac{\tilde{h}M_2}{\Gamma(\beta_1)\Gamma(\beta_2)} \left[\frac{\varphi_2(s_1)^{\beta_1}\phi_2(t_1)^{\beta_2}}{\beta_1\beta_2} \right. \\
&\qquad\qquad \left. - \left(\frac{(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1} - \varphi_2(s_2)^{\beta_1}}{\beta_1} \right) \left(\frac{(\phi_2(t_2)-\phi_2(t_1))^{\beta_2} - \phi_2(t_2)^{\beta_2}}{\beta_2} \right) \right] \\
&+ \frac{\tilde{h}M_2}{\Gamma(\beta_1)\Gamma(\beta_2)} \left[\frac{(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}(\phi_2(t_2)-\phi_2(t_1))^{\beta_2}}{\beta_1\beta_2} \right. \\
&+ \frac{(\phi_2(t_2)-\phi_2(t_1))^{\beta_2}\left[\varphi_2(s_2)^{\beta_1} - (\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}\right]}{\beta_1\beta_2} \\
&+ \left. \frac{(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}\left[\phi_2(t_2)^{\beta_2} - (\phi_2(t_2)-\phi_2(t_1))^{\beta_2}\right]}{\beta_1\beta_2} \right] + \tilde{h}\omega(z,\omega(q,\sigma)) + \omega_G(\sigma,\varepsilon) \\
&\leq \frac{\tilde{h}\omega_g(\sigma,\varepsilon)\varphi_2(s_1)^{\beta_1}\phi_2(t_1)^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} + \frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \left[\varphi_2(s_1)^{\beta_1}\phi_2(t_2)^{\beta_2} + (\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1} \right. \\
&\times \left. \{\phi_2(t_2)^{\beta_2} - (\phi_2(t_2)-\phi_2(t_1))^{\beta_2}\} + \varphi_2(s_2)^{\beta_1}(\phi_2(t_2)-\phi_2(t_1))^{\beta_2} - \varphi_2(s_2)^{\beta_1}\phi_2(t_2)^{\beta_2} \right] \\
&+ \frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \left[\varphi_2(s_2)^{\beta_1}(\phi_2(t_2)-\phi_2(t_1))^{\beta_2} \right. \\
&\qquad\qquad \left. + (\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}\{\phi_2(t_2)^{\beta_2} - (\phi_2(t_2)-\phi_2(t_1))^{\beta_2}\} \right]
\end{aligned}
$$

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$$
\begin{aligned}
&+\tilde{h}\omega(z,\omega(q,\sigma))+\omega_G(\sigma,\varepsilon)\\
&\leq \frac{\tilde{h}\omega_g(\sigma,\varepsilon)C_1^{\beta_1}C_2^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}+\frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big[(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}C_2^{\beta_2}\\
&+(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}C_2^{\beta_2}+(\phi_2(t_2)-\phi_2(t_1))^{\beta_2}C_1^{\beta_1}\Big]\\
&+\frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big[(\phi_2(t_2)-\phi_2(t_1))^{\beta_2}C_1^{\beta_1}+(\varphi_2(s_2)-\varphi_2(s_1))^{\beta_1}C_2^{\beta_2}\Big]\\
&+\tilde{h}\omega(z,\omega(q,\sigma))+\omega_G(\sigma,\varepsilon).
\end{aligned}
$$

Here

$$\omega(p_i,\sigma)=\sup\{|p_i(s,t)-p_i(\tilde{s},\tilde{t})|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b]\},$$

$$\omega(q_j,\sigma)=\sup\{|q_j(s,t)-q_j(\tilde{s},\tilde{t})|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b]\},$$

$$\omega(p,\sigma)=\max_{1\leq i\leq m}\omega(p_i,\sigma),$$

$$\omega(q,\sigma)=\max_{1\leq j\leq n}\omega(q_j,\sigma),$$

$$\omega(\varphi_k,\sigma)=\sup\{|\varphi_i(s)-\varphi_i(\tilde{s})|:|s-\tilde{s}|\leq\sigma,s,\tilde{s}\in[0,a]\},$$

$$\omega(\phi_k,\sigma)=\sup\{|\phi_i(t)-\phi_i(\tilde{t})|:|t-\tilde{t}|\leq\sigma,t,\tilde{t}\in[0,b]\},$$

$$
\begin{aligned}
\omega_f(\sigma,\varepsilon)=\sup\{&|f(s,t,u,\tau,z)-f(\tilde{s},\tilde{t},u,\tau,z)|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,\\
&s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b],u\in[0,C_1],\tau\in[0,C_2],z\in[-\varepsilon,\varepsilon]\},
\end{aligned}
$$

$$
\begin{aligned}
\omega_g(\sigma,\varepsilon)=\sup\{&|g(s,t,u,\tau,z)-g(\tilde{s},\tilde{t},u,\tau,z)|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,\\
&s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b],u\in[0,C_1],\tau\in[0,C_2],z\in[-\varepsilon,\varepsilon]\},
\end{aligned}
$$

$$
\begin{aligned}
\omega_F(\sigma,\varepsilon)=\sup\{&|F(s,t,z_1,z_2,...,z_{m+1})-F(\tilde{s},\tilde{t},z_1,z_2,...,z_{m+1})|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,\\
&s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b],|z_1|\leq\frac{M_1C_1^{\alpha_1}C_2^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)},z_2,...,z_{m+1}\in[-\varepsilon,\varepsilon]\},
\end{aligned}
$$

$$
\begin{aligned}
\omega_G(\sigma,\varepsilon)=\sup\{&|G(s,t,z_1,z_2,...,z_{n+1})-G(\tilde{s},\tilde{t},z_1,z_2,...,z_{n+1})|:|s-\tilde{s}|\leq\sigma,|t-\tilde{t}|\leq\sigma,\\
&s,\tilde{s}\in[0,a],t,\tilde{t}\in[0,b],|z_1|\leq\frac{M_2C_1^{\beta_1}C_2^{\beta_2}}{\Gamma(1+\beta_1)\Gamma(1+\beta_2)},z_2,...,z_{n+1}\in[-\varepsilon,\varepsilon]\},
\end{aligned}
$$

for $i=1,2,...,m$, $j=1,2,...,n$ and $k=1,2$.

Then, we get the estimate

$$\omega(\mathbf{F}z,\sigma)\le\frac{h\omega_f(\sigma,\varepsilon)C_1^{\alpha_1}C_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big[\omega(\varphi_1,\sigma)^{\alpha_1}C_2^{\alpha_2}$$
$$+\omega(\varphi_1,\sigma)^{\alpha_1}C_2^{\alpha_1}+\omega(\phi_1,\sigma)^{\alpha_2}C_1^{\alpha_1}\Big]$$
$$+\frac{hM_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big[\omega(\varphi_1,\sigma)^{\alpha_1}C_2^{\alpha_2}+\omega(\phi_1,\sigma)^{\alpha_2}C_1^{\alpha_1}\Big]$$
$$+h\omega(z,\omega(p,\sigma))+\omega_F(\sigma,\varepsilon)$$

and

$$\omega(\mathbf{G}z,\sigma)\le\frac{\tilde{h}\omega_g(\sigma,\varepsilon)C_1^{\beta_1}C_2^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}+\frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big[\omega(\varphi_2,\sigma)^{\beta_1}C_2^{\beta_2}$$
$$+\omega(\varphi_2,\sigma)^{\beta_1}C_2^{\beta_1}+\omega(\phi_2,\sigma)^{\beta_2}C_1^{\beta_1}\Big]$$
$$+\frac{\tilde{h}M_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big[\omega(\varphi_2,\sigma)^{\beta_1}C_2^{\beta_2}+\omega(\phi_2,\sigma)^{\beta_2}C_1^{\beta_1}\Big]$$
$$+\tilde{h}\omega(z,\omega(q,\sigma))+\omega_G(\sigma,\varepsilon).$$

Taking limit $\sigma\to 0$, we have

$$\mu(\mathbf{F}\digamma)\le h\mu(\digamma) \tag{9}$$

and

$$\mu(\mathbf{G}\digamma)\le\tilde{h}\mu(\digamma). \tag{10}$$

By Eq. (9) and (10), we obtain T is a densifying map. Let $z\in\partial B_\varepsilon$ and if $Tz=kz$, then $k\varepsilon=k\|\varepsilon\|=\|Tz\|$ and by assumption (3)

$$\|Tz(s,t)\|$$
$$=F\left(s,t,\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\varphi_1(s)}\int_0^{\phi_1(t)}\frac{f(s,t,u,\tau,z(\delta_1(u,\tau)))}{(\varphi_1(s)-u)^{1-\alpha_1}(\phi_1(t)-\tau)^{1-\alpha_2}}d\tau du,z(p_1(s,t)),...,z(p_m(s,t))\right)$$
$$\times G\left(s,t,\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^{\varphi_2(s)}\int_0^{\phi_2(t)}\frac{g(s,t,u,\tau,z(\delta_2(u,\tau)))}{(\varphi_2(s)-u)^{1-\beta_1}(\phi_2(t)-\tau)^{1-\beta_2}}d\tau du,z(q_1(s,t)),...,z(q_n(s,t))\right),$$

for all $(s,t)\in D$, hence $\|Tz\|\le\varepsilon$. This shows $k\le 1$ and completes the proof. □

Now, we offer some illustrations of 2D fractional integral equations to show the merits of our results.

Example 3.1. *Let we consider the following 2D fractional integral equation in* $C\left([0,1]\times\left[0,\frac{\sqrt{2}}{2}\right],\mathbb{R}\right)$

$$\begin{aligned} z(s,t) &= \left(\frac{1}{1+t^2+s^2}+\frac{1}{3\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^s\int_0^t \frac{e^{-\left(t^2+s^2\right)}\left|z\left(u,\cos\tau\right)\right|}{\sqrt{(s-u)(t-\tau)}}d\tau du+\frac{2z(s,t)}{5}+\frac{z\left(\frac{s^2}{2},\frac{t^3}{3}\right)}{3}\right) \\ &\times\frac{1}{\pi^3}\left(e^{-t-s}+\frac{1}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}\int_0^s\int_0^{t^2}\frac{\left(1+u^2\right)\left|z(u,\tau)\right|}{\sqrt[4]{\left((s-u)\left(t^2-\tau\right)\right)^3}}d\tau du+z\left(\sqrt{s},\sqrt{t}\right)\right) \end{aligned} \tag{11}$$

where $m=2,\ n=1,\alpha_1=\alpha_2=1/2,\beta_1=\beta_2=1/4,$

$$F(s,t,z_1,z_2,z_3)=\frac{1}{1+t^2+s^2}+\frac{2z_2}{5}+\frac{z_1+z_3}{3}$$

and

$$G(s,t,z_1,z_2)=\frac{e^{-t-s}+z_1+z_2}{\pi^3}.$$

Also

$$\begin{gathered} \delta_1(u,\tau)=(u,\cos\tau),\quad \delta_2(u,\tau)=(u,\tau), \\ f(s,t,u,\tau,z)=e^{-\left(t^2+s^2\right)}|z|,\quad g(s,t,u,\tau,z)=\left(1+u^2\right)|z|, \\ p_1(s,t)=(s,t),\quad p_2(s,t)=\left(\frac{s^2}{2},\frac{t^3}{3}\right),\quad q_1(s,t)=\left(\sqrt{s},\sqrt{t}\right), \\ \varphi_1(s)=\varphi_2(s)=s,\quad \phi_1(t)=t,\quad \phi_2(t)=t^2,\quad C_1=C_2=1. \end{gathered}$$

It can be seen that

$$\begin{aligned} \left|F(s,t,z_1,z_2,z_3)-F(s,t,\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)\right| &\le \frac{|z_1-\tilde{z}_1|+|z_3-\tilde{z}_3|}{3}+\frac{2|z_2-\tilde{z}_2|}{5} \\ &\le \frac{2}{5}\left(|z_1-\tilde{z}_1|+|z_2-\tilde{z}_2|+|z_3-\tilde{z}_3|\right) \end{aligned}$$

and

$$\left|G(s,t,z_1,z_2)-G(s,t,\tilde{z}_1,\tilde{z}_2)\right|\le\frac{1}{\pi^3}\left(|z_1-\tilde{z}_1|+|z_2-\tilde{z}_2|\right).$$

So we can choose

$$h=\frac{2}{5}\ \textit{and}\ \tilde{h}=\frac{1}{\pi^3}$$

and so the conditions (1) *and* (2) *hold.*

On the other hand we can conclude that

$$\begin{aligned} M_1 &= \sup\{|f(s,t,u,\tau,z)| : \textit{for all}\quad (s,t,u,\tau)\in\tilde{D}\quad \textit{and}\quad z\in[-\varepsilon,\varepsilon]\} \\ &= \sup\{e^{-\left(t^2+s^2\right)}|z| : \textit{for all}\quad (s,t,u,\tau)\in\tilde{D}\quad \textit{and}\quad z\in[-\varepsilon,\varepsilon]\} \\ &= \varepsilon, \end{aligned}$$

$$
\begin{aligned}
H_1 &= \sup\{|F(s,t,z_1,z_2,z_3)| : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2,z_3\in[-\varepsilon,\varepsilon], |z_1| \le \frac{M_1 C_1^{\alpha_1} C_2^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}\} \\
&= \sup\{|\frac{1}{1+t^2+s^2} + \frac{2z_2}{5} + \frac{z_1+z_3}{3}| : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2,z_3\in[-\varepsilon,\varepsilon], |z_1| \le \frac{\varepsilon}{\Gamma(\frac{3}{2})^2}\} \\
&\le \frac{15\pi + (20+11\pi)\,\varepsilon}{15\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
M_2 &= \sup\{|g(s,t,u,\tau,z)| : \textit{for all} \quad (s,t,u,\tau)\in\tilde{D} \quad \textit{and} \quad z\in[-\varepsilon,\varepsilon]\} \\
&= \sup\{\left(1+u^2\right)|z| : \textit{for all} \quad (s,t,u,\tau)\in\tilde{D} \quad \textit{and} \quad z\in[-\varepsilon,\varepsilon]\} \\
&= 2\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
H_2 &= \sup\{|G(s,t,z_1,z_2)| : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2\in[-\varepsilon,\varepsilon], |z_1| \le \frac{M_2 C_1^{\beta_1} C_2^{\beta_2}}{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}\} \\
&= \sup\{\frac{\left|e^{-(t+s)} + z_1 + z_2\right|}{\pi^3} : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2\in[-\varepsilon,\varepsilon], |z_1| \le \frac{2\varepsilon}{\Gamma(\frac{5}{4})^2}\} \\
&\le \frac{\Gamma(\frac{5}{4})^2 + \left(2+\Gamma(\frac{5}{4})^2\right)\varepsilon}{\pi^3\Gamma(\frac{5}{4})^2}.
\end{aligned}
$$

It can be seen that, existence of solution of the inequality

$$
\left(\frac{15\pi + (20+11\pi)\,\varepsilon}{15\pi}\right) \times \left(\frac{\Gamma(\frac{5}{4})^2 + \left(2+\Gamma(\frac{5}{4})^2\right)\varepsilon}{\pi^3\Gamma(\frac{5}{4})^2}\right) \le \varepsilon \tag{12}
$$

guarantees that the inequality mentioned in condition (3) *has a positive solution* ε. *Also, the inequality (12) has a positive solution* ε *such that*

$$
0.0380767 \le \varepsilon \le 6.60509.
$$

Hence, the all conditions of Theorem 3.1 hold and, by Theorem 3.1, the equation 11 has at least one solution in $C\left([0,1]\times\left[0,\frac{\sqrt{2}}{2}\right],\mathbb{R}\right)$.

Example 3.2. *Let we consider the following 2D fractional equation in* $C\left(\left[0,\frac{\pi}{4}\right]\times[0,1],\mathbb{R}\right)$.

$$
\begin{aligned}
z(s,t) &= \left(\frac{\cos^2(s+t)}{30+(s+t)^2} + \frac{1}{6+\left|\frac{1}{\Gamma(\frac{1}{2})^2}\int_0^s\int_0^t \frac{|z(u,\tau)|}{\sqrt{(s-u)(t-\tau)}}d\tau du\right|} + \frac{z\left(\cos^2 s, \sin^2 t\right)}{\pi}\right) \\
&\quad \times \left(\frac{t^2+s^2}{3} + \frac{1}{\Gamma\left(\frac{1}{3}\right)^2}\int_0^s\int_0^t \frac{\left(1+u^2\right)|z(u,\tau)|}{4\sqrt[3]{(s-u)^2(t-\tau)^2}}d\tau du + \frac{z(s,t)}{3}\right)
\end{aligned} \tag{13}
$$

where $m = n = 1, \alpha_1 = \alpha_2 = 1/2, \beta_1 = \beta_2 = 1/3,$

$$F(s,t,z_1,z_2) = \frac{\cos^2(s+t)}{30+(s+t)^2} + \frac{1}{6+|z_1|} + \frac{z_2}{\pi}$$

and

$$G(s,t,z_1,z_2) = \frac{t^2+s^2}{3} + \frac{z_1}{4} + \frac{z_2}{3}.$$

Also

$$\delta_1(u,\tau) = \delta_2(u,\tau) = (u,\tau),$$

$$f(s,t,u,\tau,z) = |z|, \quad g(s,t,u,\tau,z) = \left(1+u^2\right)|z|,$$

$$p_1(s,t) = \left(\cos^2 s, \sin^2 t\right), \quad q_1(s,t) = (s,t),$$

$$\varphi_1(s) = \varphi_2(s) = s, \quad \phi_1(t) = \phi_2(t) = t, \quad C_1 = \frac{\pi}{4}, \ C_2 = 1$$

It can be seen that

$$\begin{aligned}
|F(s,t,z_1,z_2) - F(s,t,\tilde{z}_1,\tilde{z}_2)| &= \left|\frac{1}{6+|z_1|} + \frac{z_2}{\pi} - \frac{1}{6+|\tilde{z}_1|} - \frac{\tilde{z}_2}{\pi}\right| \\
&= \left|\frac{|\tilde{z}_1| - |z_1|}{(6+|z_1|)(6+|\tilde{z}_1|)} + \frac{z_2 - \tilde{z}_2}{\pi}\right| \\
&\le \frac{|\tilde{z}_1 - z_1|}{36} + \frac{|z_2 - \tilde{z}_2|}{\pi} \\
&\le \frac{1}{\pi}\left(|z_1 - \tilde{z}_1| + |z_2 - \tilde{z}_2|\right)
\end{aligned}$$

and

$$|G(s,t,z_1,z_2) - G(s,t,\tilde{z}_1,\tilde{z}_2)| \le \frac{1}{3}\left(|z_1 - \tilde{z}_1| + |z_2 - \tilde{z}_2|\right).$$

So we can choose

$$h = \frac{1}{\pi} \text{ and } \tilde{h} = \frac{1}{3}$$

and so the conditions (1) *and* (2) *hold.*

On the other hand we can conlude that

$$\begin{aligned}
M_1 &= \sup\{|f(s,t,u,\tau,z)| : \text{for all} \quad (s,t,u,\tau) \in \tilde{D} \quad \text{and} \quad z \in [-\varepsilon,\varepsilon]\} \\
&= \sup\{|z| : \text{for all} \quad (s,t,u,\tau) \in \tilde{D} \quad \text{and} \quad z \in [-\varepsilon,\varepsilon]\} \\
&= \varepsilon,
\end{aligned}$$

$$\begin{aligned}
H_1 &= \sup\{|F(s,t,z_1,z_2)| : \text{for all} \quad s,t \in D \quad \text{and} \quad z_2 \in [-\varepsilon,\varepsilon], |z_1| \le \frac{M_1 C_1^{\alpha_1} C_2^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}\} \\
&= \sup\{|\frac{\cos^2(s+t)}{30+(s+t)^2} + \frac{1}{6+|z_1|} + \frac{z_2}{\pi}| : \text{for all} \quad s,t \in D \quad \text{and} \quad z_2 \in [-\varepsilon,\varepsilon], |z_1| \le \frac{\varepsilon\sqrt{\frac{\pi}{4}}}{\Gamma(\frac{3}{2})^2}\} \\
&\le \frac{\pi + 5\varepsilon}{5\pi}
\end{aligned}$$

and

$$\begin{aligned} M_2 &= \sup\{|g(s,t,u,\tau,z)| : \textit{for all} \quad (s,t,u,\tau)\in \tilde{D} \quad \textit{and} \quad z\in[-\varepsilon,\varepsilon]\} \\ &= \sup\{\left(1+u^2\right)|z| : \textit{for all} \quad (s,t,u,\tau)\in \tilde{D} \quad \textit{and} \quad z\in[-\varepsilon,\varepsilon]\} \\ &= \left(\frac{16+\pi^2}{16}\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} H_2 &= \sup\{|G(s,t,z_1,z_2)| : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2\in[-\varepsilon,\varepsilon], |z_1|\le \frac{M_2 C_1^{\beta_1} C_2^{\beta_2}}{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}\} \\ &= \sup\{\left|\frac{t^2+s^2}{3}+\frac{z_1}{4}+\frac{z_2}{3}\right| : \textit{for all} \quad s,t\in D \quad \textit{and} \quad z_2\in[-\varepsilon,\varepsilon], |z_1|\le \frac{\left(16+\pi^2\right)\varepsilon\sqrt[3]{\frac{\pi}{4}}}{16\Gamma(\frac{4}{3})^2}\} \\ &\le \frac{192\Gamma(\frac{4}{3})^2+64\Gamma(\frac{4}{3})^2\varepsilon+\left(48+3\pi^2\right)\varepsilon}{192\Gamma(\frac{4}{3})^2}. \end{aligned}$$

It can be seen that, existence of solution of the inequality

$$\left(\frac{\pi+5\varepsilon}{5\pi}\right)\times\left(\frac{192\Gamma(\frac{4}{3})^2+64\Gamma(\frac{4}{3})^2\varepsilon+\left(48+3\pi^2\right)\varepsilon}{192\Gamma(\frac{4}{3})^2}\right)\le\varepsilon \tag{14}$$

guarantees that the inequality mentioned in condition (3) *has a positive solution* ε*. Also, the inequality (14) has a positive solution* ε *such that*

$$0.0542782 \le \varepsilon \le 1.37769.$$

Hence, the all conditions of Theorem 3.1 hold and, by Theorem 3.1, the equation 13 has at least one solution in $C\left(\left[0,\frac{\pi}{4}\right]\times[0,1],\mathbb{R}\right)$.

4 Conclusion

The interested researchers may analyze the result of the Eq. (7) in different Banach function spaces, e.g. Sobolev space, Orlicz space, Hölder space, etc. The significance of protecting the existence result in the study of FIEs is one of the benefits for authors. So distant, many methods have been developed for this view. This study is established on an additional known form of the FIE, which concerns some other appropriate results as nicely. In the proposed approach, Petryshyn's fixed point theorem and the concept of MNC with additional restricted conditions were involved.

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