Borodin-Kostochka's Conjecture on $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs

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Abstract

Let P_n denote the induced path on n vertices. A *gem* is the graph that consists of a P_4 plus a vertex which is adjacent to all the vertices of that path and a *banner* is the graph that consists of an induced cycle on four vertices and a single vertex with precisely one neighbour on the cycle. For two graphs H_1 and H_2, we use $H_1 \cup H_2$ to denote the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Let $\Delta(G)$, $\chi(G)$ and $\omega(G)$ denote the maximum degree, chromatic number and clique number of G, respectively. The Borodin-Kostochka Conjecture states that for a graph G, if $\Delta(G) \geq 9$, then $\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}$. We prove the Borodin-Kostochka Conjecture for $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs.

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1 Introduction

All our graphs are finite and have no loops or multiple edges. For classical graph theory we use the standard notation, we mainly follow standard terminology of the books [1, 13]. A *clique* in a graph is a set of pairwise adjacent vertices. Let $G = (V, E)$ be a graph. For any integer k, a *k-colouring* of G is a mapping $\varphi : V \to \{1, 2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent in G. A graph is *k-colourable* if it admits a k-colouring. The *chromatic number* of G is the minimum number k for which G is k-colourable. A vertex subset $K \subseteq V$ is a *clique cutset* if $G - K$ has more components than G and K induces a clique. We use $\chi(G)$, $\omega(G)$, $\Delta(G)$ and

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$\delta(G)$ to denote the chromatic number, clique number, maximum degree and minimum degree of G, respectively. By greedily colouring the vertices of G in any order, it is easily verified that $\chi(G) \leq \Delta(G)+1$. In 1941, Brooks [3] strengthened this bound.

Theorem 1.1 (Brooks' Theorem [3]). *Let G be a graph with $\Delta(G) \geq 3$. Then $\chi(G) \leq \max\{\Delta(G), \omega(G)\}$.*

In 1977, Borodin and Kostochka [2] conjectured that a similar result holds for $\Delta(G)-1$ colourings.

Conjecture 1.2 (Borodin-Kostochka Conjecture [2]). *Let G be a graph with $\Delta(G) \geq 9$. Then $\chi(G) \leq \max\{\Delta(G)-1, \omega(G)\}$.*

Note that if $\omega(G) \geq \Delta(G)$, then by Theorem 1.1, G satisfies the Borodin-Kostochka Conjecture. Therefore, to prove Conjecture 1.2, it suffices to prove that for a graph G, if $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G)-1$, then $\chi(G) \leq \Delta(G)-1$. Cranston, Lafayette and Rabern [6] proved that Conjecture 1.2 cannot be strengthened by making $\Delta(G) \geq 8$ or $\omega(G) \leq \Delta(G)-2$. By Theorem 1.1, each graph G with $\chi(G) > \Delta(G) \geq 9$ contains $K_{\Delta(G)+1}$. So, Conjecture 1.2 is equivalent to the statement that each graph G with $\chi(G) = \Delta(G) \geq 9$ contains $K_{\Delta(G)}$. In 1999, Reed [12] presented the strongest partial result towards Conjecture 1.2 by showing that Conjecture 1.2 is true for all graphs having maximum degree at least 10^{14}.

We say that a graph G contains a graph F if F is isomorphic to an induced subgraph of G. A graph G is *F-free* if it does not contain F. For a family $\mathcal{F}$ of graphs, G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. A *hole* of G is an induced subgraph of G which is a cycle of length at least four, and a hole is said to be an odd hole if it has odd length. An *anti-hole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$.

Let P_n and C_n denote the induced path and cycle on n vertices, respectively. For two graphs H_1 and H_2, we use $H_1 \cup H_2$ to denote the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Conjecture 1.2 has been proved for many interesting classes of graphs, particularly those defined by forbidden induced subgraphs. For example, Cranston and Rabern [7] proved it for claw-free graphs. Gupta and Pradhan [9] proved it for $\{P_5, C_4\}$-free graphs. It was also recently proved for $\{P_5, \text{gem}\}$-free graphs by Cranston, Lafayette and Rabern [6]. Lan, Liu and Zhou [11] proved Conjecture 1.2 for $\{P_2 \cup P_3, C_4\}$-free graphs.

A *gem* is the graph that consists of a P_4 plus a vertex which is adjacent to all the vertices of that path. A *banner* is the graph that consists of a hole on four vertices and a single vertex with precisely one neighbour on the hole (see Figure 1 for a depiction.) Note that the banner-free graphs generalize the well-studied class of claw-free graphs. In this article, we study $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs. More precisely, we prove Conjecture 1.2 for $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs. We do this by reducing the problem to imperfect $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs via the Strong Perfect Graph Theorem [5]. Our result is stated in the following theorem.

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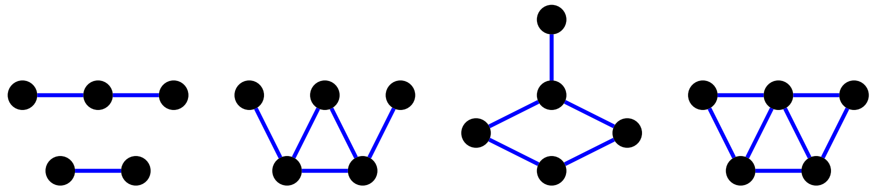

Figure 1: Illustration of some forbidden configurations.

Theorem 1.3. *Let G be a $\{P_2 \cup P_3, gem, banner\}$-free graph with $\Delta(G) \geq 9$. Then $\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}$.*

This paper is organized as follows. In the remainder of this section, we describe notation and terminology that we will be used in our proof. We present some preliminaries in Section 2 and recall some lammas. In Section 3, by considering a smallest counterexample (smallest in terms of the number on vertices), we prove Theorem 1.3.

If X is the set of vertices in G, denote by $G[X]$ the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. For any $x \in V(G)$, let $N(x)$ denote the set of all neighbors of x in G; and let $d_G(x) := |N(x)|$, and we often write $d(x)$ if the context is clear. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \backslash X : u$ is adjacent to a vertex of $X\}$. Let X and Y be any two subsets of $V(G)$. We write $[X, Y]$ to denote the set of edges that has one end in X and other end in Y. We say that X is *complete* to Y or $[X, Y]$ is *complete* if every vertex in X is adjacent to every vertex in Y; and X is *anti-complete* to Y if $[X, Y] = \emptyset$. If X is singleton, say $\{u\}$, we simply write u is complete (anti-complete) to Y instead of writing $\{u\}$ is complete (anti-complete) to Y. For a given positive integer k, we use the standard notation $[k]$ to denote the set $\{1, 2, \ldots, k\}$. In the rest of this paper, every subscript is understood to be modulo 5.

2 Preliminaries

In this section, we present some lemmas that will be use to prove Theorem 1.3. For a given positive integer k, a graph G is said to be *k-vertex-critical* if $\chi(G) = k$ and $\chi(G - v) \leq k - 1$ for each vertex v of G. The following lemma, given by Dirac, states a useful property of k-vertex-critical graphs.

Lemma 2.1 ([8])**.** *If a graph G is k-vertex-critical, then $\delta(G) \geq k - 1$.*

A *buoy* is a graph G, whose vertex set can be partitioned into five nonempty sets, X_1, X_2, X_3, X_4, and X_5 such that $[X_i, X_{i+1}]$ is complete and $[X_i, X_{i+2}] = [X_i, X_{i+3}] = \emptyset$ for all $i \in [5]$, A buoy is said to be a *complete buoy* if X_i is a clique for each $i \in [5]$. In [6], it is proved that if

G is a complete bouy, then $\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}$, that is, a complete buoy satisfies the Borodin-Kostochka Conjecture. We restate this in the following lemma to use later.

Lemma 2.2 ([6]). *Conjecture 1.2 holds for any complete buoy.*

A class $\mathcal{F}$ of graphs is *hereditary* if for each $G \in \mathcal{F}$ we have $H \in \mathcal{F}$ for each induced subgraph H of G. Every class of graphs characterized by a list of forbidden induced subgraphs is a hereditary class. In particular, the class of $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs is hereditary. Kostochka and Catlin independently presented a useful result, by which one can choose a counterexample for the Borodin-Kostochka Conjecture in a hereditary graph class $\mathcal{G}$, if exists, to have maximum degree 9.

Lemma 2.3 ([4, 10]). *If $\mathcal{G}$ is a hereditary graph class and if Conjecture 1.2 is true for all graphs $G \in \mathcal{G}$ having $\Delta(G) = 9$, then Conjecture 1.2 is true for all graphs in $\mathcal{G}$.*

A graph G is said to be *vertex-critical* if $\chi(G - v) \leq \chi(G)$ for each vertex v of G. The following lemma guarantees the existence of a vertex-critical counterexample of Conjecture 1.2 if a counterexample exists for the conjecture.

Lemma 2.4 ([4, 10]). *If G is a smallest counterexample (smallest in terms of the number on vertices) for Conjecture 1.2 with $\Delta(G) = 9$, then G must be vertex-critical.*

A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G, and *imperfect* otherwise. The Strong Perfect Graph Theorem [5] says that a graph is perfect if and only if it does not contain an odd hole or an odd anti-hole as an induced subgraph. Since a $\{P_2 \cup P_3\}$-free graph contains no hole of length at least 7, and a gem-free graph contain no anti-hole of length at least 7, we have the following.

Lemma 2.5. *Every imperfect $\{P_2 \cup P_3, \text{gem}\}$-free graph contains an induced C_5.*

3 Proof of Theorem 1.3

In this section, by means of the Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5, we prove our main theorem. A general technique to prove results of the type we consider in this work is to consider a smallest counterexample (smallest in terms of the number on vertices), prove some properties that such a graph must satisfy, and finally derive a contradiction. So, we will assume that Theorem 1.3 is false and choose G to be a smallest counterexample. Ultimately, we reach a contradiction.

Proof of Theorem 1.3. Let $\mathcal{G}$ be the class of all $\{P_2 \cup P_3, \text{gem}, \text{banner}\}$-free graphs. It is sufficient to prove the theorem for connected graphs only. If possible, then let $G = (V, E)$ be a smallest counterexample in $\mathcal{G}$ (smallest in terms of the number on vertices) for Conjecture 1.2

with $\Delta(G) = 9$, (note that such G exists by Lemma 2.3). By Lemma 2.4, G is vertex-critical. In what follows, we let ω denote the clique number of a graph under consideration. If $\omega \geq 9$, then the result holds due to Theorem 1.1. So we may assume that $\chi(G) = 9$ and $\omega \leq 8$. By Lemma 2.1, each vertex in G has degree either 8 or 9. If G is perfect, then $\chi(G) = \omega \leq \max\{\Delta(G)-1, \omega\}$. This is a contradiction to the fact that G is a counterexample of Conjecture 1.2. Hence, G is imperfect. By Lemma 2.5, G must contain an induced C_5. Let $C := u_1u_2u_3u_4u_5$ be the vertex set of an induced C_5 in G with edge set $\{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$. We define the following sets.

$$\mathcal{N} := \{u \in V \backslash C : N(u) \cap C \neq \emptyset\},$$
$$\mathcal{R} := V \backslash (\mathcal{N} \cup C).$$

Clearly, $V = C \cup \mathcal{N} \cup \mathcal{R}$.

Since G is $\{\text{gem}, \text{banner}\}$-free, we may easily observe that each vertex in $\mathcal{N}$ is either adjacent to exactly one vertex in C, or exactly two consecutive vertices in C, or exactly three consecutive vertices in C. It follows that $\mathcal{N}$ can be partitioned into subsets as follows.

$$A_i := \{u \in \mathcal{N}_2 : N(u) \cap C = \{u_i\}\},$$
$$B_i := \{u \in \mathcal{N}_2 : N(u) \cap C = \{u_i, u_{i+1}\}\},$$
$$D_i := \{u \in \mathcal{N}_2 : N(u) \cap C = \{u_i, u_{i+1}, u_{i+2}\}\}.$$

See Figure 2 for an illustration of these sets.

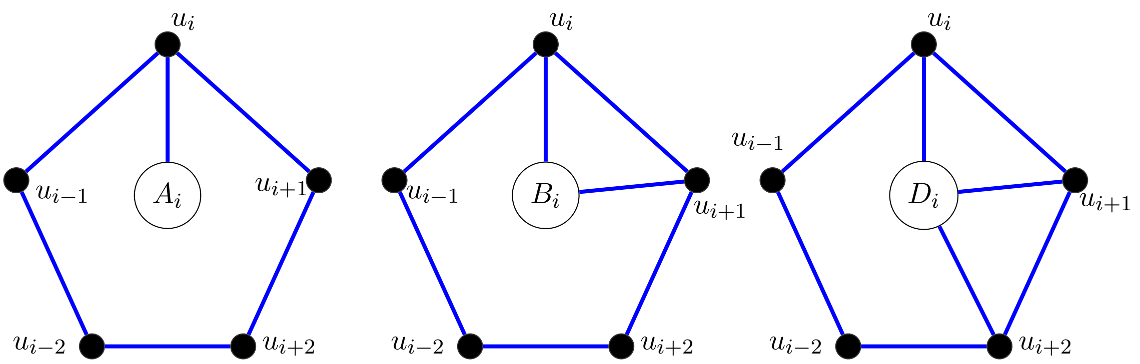

Figure 2: Illustration of the sets A_i, B_i and D_i.

Then, $V = (\cup_{i=1}^5 A_i) \bigcup (\cup_{i=1}^5 B_i) \bigcup (\cup_{i=1}^5 D_i) \bigcup C \bigcup \mathcal{R}$. Below we deduce some structural properties of G. Each of the next properties is followed by a short proof.

(3.1) $|A_i| \leq 1$ for all $i \in [5]$.

To the contrary, assume that A_i contains two vertices x and y. Then either $\{x, y\} \cup \{u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a $P_2 \cup P_3$ or $\{u_{i+2}, u_{i+3}\} \cup \{x, u_i, y\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction.

(3.2) $[A_i, A_{i+1}] = \emptyset$ for all $i \in [5]$.

To the contrary, assume that there are two vertices $x \in A_i$ and $y \in A_{i+1}$ such that $xy \in E$. Then $\{x, y\} \cup \{u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a $P_2 \cup P_3$. This is a contradiction.

(3.3) $[A_i, A_{i+2}]$ is complete for all $i \in [5]$.

To the contrary, assume that there are two vertices $x \in A_i$ and $y \in A_{i+2}$ such that $xy \notin E$. Then $\{x, u_i\} \cup \{y, u_{i+2}, u_{i+3}\}$ induces a $P_2 \cup P_3$. This is a contradiction.

(3.4) $|B_i| \leq 1$ for all $i \in [5]$.

To the contrary, assume that B_i contains two vertices x and y. Then either $\{x, y\} \cup \{u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a $P_2 \cup P_3$ or $\{u_{i+2}, u_{i+3}\} \cup \{x, u_i, y\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction.

(3.5) Either $A_i = \emptyset$ or $B_i = B_{i-1} = \emptyset$ for all $i \in [5]$.

To the contrary, assume that $x \in A_i$ and $y \in B_i$. Then either $\{x, y\} \cup \{u_{i+2}, u_{i+3}, u_{i+4}\}$ or $\{u_{i+2}, u_{i+3}\} \cup \{x, u_i, y\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction. The case that $y \in B_{i-1}$ is symmetric.

(3.6) Either $A_i = \emptyset$ or $B_{i+1} = B_{i-2} = \emptyset$ for all $i \in [5]$.

To the contrary, assume that $x \in A_i$ and $y \in B_{i+1}$. Then either $\{u_{i-1}, u_i, x, y, u_{i+1}\}$ induces a banner or $\{x, u_i\} \cup \{y, u_{i+2}, u_{i+3}\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction. The case that $y \in B_{i-2}$ is symmetric.

(3.7) Each D_i is a clique for all $i \in [5]$.

To the contrary, assume that there exist two vertices x and y in D_i such that $xy \notin E$. Then $\{u_i, x, y, u_{i+2}, u_{i+3}\}$ induces a banner. This is a contradiction.

(3.8) $[D_i, D_{i+2} \cup D_{i-2}] = \emptyset$ for all $i \in [5]$.

To the contrary, assume that there exist two vertices $x \in D_i$ and $y \in D_{i+2}$ such that $xy \in E$. Then $\{x, u_i, u_{i+1}, u_{i+2}, y\}$ induces a gem. This is a contradiction. The case that $y \in D_{i-2}$ is symmetric.

(3.9) If $D_i \neq \emptyset$, then $[D_{i+2}, D_{i-2}]$ is complete for all $i \in [5]$.

To the contrary, assume that $x \in D_i$, $y \in D_{i+2}$ and $z \in D_{i-2}$ such that $yz \notin E$. Then, by (3.8), we have $xy \notin E$ and $xz \notin E$. This implies that $\{x, u_{i+1}\} \cup \{y, u_{i+3}, z\}$ induces a $P_2 \cup P_3$. This is a contradiction.

(3.10) Either $A_i = \emptyset$ or $D_{i+1} = D_{i+2} = \emptyset$ for all $i \in [5]$.

To the contrary, assume that there exist two vertices $x \in A_i$ and $y \in D_{i+1}$. Then either $\{x, u_i, u_{i+1}, y, u_{i+3}\}$ induces a banner or $\{y, u_{i+2}\} \cup \{x, u_i, u_{i-1}\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction. The case that $y \in D_{i+2}$ is symmetric.

(3.11) $[B_i, D_i \cup D_{i-1}]$ is complete for all $i \in [5]$.

To the contrary, assume that there exist two vertices $x \in B_i$ and $y \in D_i$ such that $xy \notin E$. Then $\{u_{i+3}, u_{i+4}\} \cup \{x, u_{i+1}, y\}$ induces a $P_2 \cup P_3$. This is a contradiction. The case that $y \in D_{i-1}$ is similar.

(3.12) Either $B_i = \emptyset$ or $D_{i+1} = D_{i-2} = \emptyset$ for all $i \in [5]$.

To the contrary, assume that both B_i and D_{i+1} are not empty and let $x \in B_i$ and $y \in D_{i+1}$. Then either $\{y, x, u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a gem or $\{y, u_{i+2}\} \cup \{x, u_i, u_{i-1}\}$ induces a $P_2 \cup P_3$, depending on whether x and y are adjacent. This is a contradiction. The case that $D_{i-2} \neq \emptyset$ is symmetric.

(3.13) $[B_i, D_{i+2}] = \emptyset$ for all $i \in [5]$.

To the contrary, assume that there exist two vertices $x \in B_i$ and $y \in D_{i+2}$ such that $xy \in E$. Then $\{x, u_{i+1}, u_{i+2}, y, u_{i+4}\}$ induces a banner. This is a contradiction.

(3.14) $\mathcal{R} = \emptyset$.

To the contrary, assume that $w \in \mathcal{R}$. Then there is some path connecting w to C since G is connected. That path must pass through $\mathcal{N}$. Let xy be an edge of that path such that $x \in \mathcal{R}$ and $y \in \mathcal{N}$. If $y \in A_i$, then $\{x, y\} \cup \{u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a $P_2 \cup P_3$; if $y \in B_i$, then $\{x, y\} \cup \{u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a $P_2 \cup P_3$; if $y \in D_i$, then $\{u_{i+3}, u_{i+4}\} \cup \{x, y, u_{i+1}\}$ induces a $P_2 \cup P_3$. This is a contradiction.

Now we continue our proof by showing a contradiction to the fact that G is a smallest counterexample to Conjecture 1.2. By (3.14), $V = (\cup_{i=1}^5 A_i) \bigcup (\cup_{i=1}^5 B_i) \bigcup (\cup_{i=1}^5 D_i) \bigcup C$.

We first claim that $\cup_{i=1}^5 A_i = \emptyset$. Suppose not. Without loss of generality, we may assume that $A_1 \neq \emptyset$. Then, by (3.5) and (3.6), we have $B_1 = B_2 = B_4 = B_5 = \emptyset$ and by (3.10), $D_2 = D_3 = \emptyset$. On the other hand, since $8 \leq d(u_3) \leq 9$ and $8 \leq d(u_4) \leq 9$, we have $|D_1| \geq 4$ and $|D_4| \geq 4$ by (3.1)-(3.4). It follows that $d(u_1) \geq 4 + 4 + 2 + 1 = 11$, which is a contradiction to the fact that $\Delta(G) = 9$. Therefore, $V = (\cup_{i=1}^5 B_i) \bigcup (\cup_{i=1}^5 D_i) \bigcup C$.

We now claim that $\cup_{i=1}^5 B_i = \emptyset$. Suppose not. Without loss of generality, we may assume that $B_1 = \{b_1\}$ by (3.4). Then, by (3.12), $D_2 = D_4 = \emptyset$. If $B_2 \neq \emptyset$, then $D_3 = D_5 = \emptyset$ by (3.12). It follows that $d(u_5) \leq 4$ by (3.4), which is a contradiction to the fact that $\delta(G) = 8$. So, $B_2 = \emptyset$. Similarly, $B_5 = \emptyset$. Furthermore, if $B_3 \neq \emptyset$, then $D_1 = \emptyset$ by (3.12). Since $8 \leq d(u_2) \leq 9$ and $8 \leq d(u_3) \leq 9$, we have $|D_3| \geq 5$ and $|D_5| \geq 5$ by (3.4). It follows that $d(u_5) \geq 5 + 5 + 2 = 12$, which is a contradiction to the fact that $\Delta(G) = 9$. So, $B_3 = \emptyset$. Similarly, $B_4 = \emptyset$. That is,

$V = B_1 \cup C \cup D_1 \cup D_3 \cup D_5$. On the other hand, by (3.7), (3.8) and (3.13), we have $6 \leq |D_3| \leq 7$. Since $8 \leq d(u_3) \leq 9$ and $8 \leq d(u_5) \leq 9$, we have $6 \leq |D_3| + |D_1| \leq 7$ and $6 \leq |D_3| + |D_5| \leq 7$. It follows that $0 \leq |D_1| \leq 1$ and $0 \leq |D_5| \leq 1$. But then $d(b_1) \leq 4$ by (3.11) and (3.13), which is a contradiction to the fact that $\delta(G) = 8$. Therefore, $V = (\cup_{i=1}^5 D_i) \bigcup C$.

Suppose first that $D_i \neq \emptyset$ for all $i \in [5]$. Let $X_i = D_i \cup \{u_{i+1}\}$. Then, $V = \cup_{i=1}^5 X_i$ and by (3.7) and (3.8), each X_i is a clique and $[X_i, X_{i+2}] = [X_i, X_{i+3}] = \emptyset$. Moreover, for all $i \in [5]$, $[X_i, X_{i+1}]$ is complete by (3.9). Therefore, G is a complete buoy. By Lemma 2.2, G satisfies Conjecture 1.2, which is a contradiction to the fact that G is a counterexample.

Suppose now that $D_i = \emptyset$ for some $i \in [5]$. Without loss of generality, we may assume that $D_1 = \emptyset$. Since $8 \leq d(u_1) \leq 9$ and $8 \leq d(u_5) \leq 9$, we have $6 \leq |D_4| + |D_5| \leq 7$ and $6 \leq |D_3| + |D_4| + |D_5| \leq 7$. This implies that $|D_3| \leq 1$. On the other hand, since $8 \leq d(u_3) \leq 9$, $|D_2| \geq 5$. Furthermore, since $8 \leq d(u_2) \leq 9$, $|D_5| \leq 2$. Hence, $|D_4| \geq 4$. It follows that $d(u_4) \geq 4 + 5 + 2 = 11$, which is a contradiction to the fact that $\Delta(G) = 9$.

This completes the proof of Theorem 1.3. □

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Declaration

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data sharing not applicable to this paper as no datasets were generated or analysed during the current study.