Some inequalities on ranks of cubic partitions

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Abstract. A partition is a cubic partition if its even parts come in two colors (blue and red). Reti defined the rank of a cubic partition as the difference between the number of even parts in blue color and the number of even parts in red color. Motivated by the works on inequalities of rank and crank for certain partitions proved by Andrews and Lewis, and Chern, Fu, Tang and Wang, we prove some inequalities for $N^{'}(r,m,n)$, which count the number of cubic partitions of n whose rank is congruent to r modulo m. More precisely, we establish the generating functions for $N^{'}(r,m,n)$ and determine the signs of the differences $N^{'}(r,m,n)-N^{'}(s,m,n)$ with $m\in\{2,3,4,6\}$ and $0\leq r<s\leq m-1$ by utilizing q-series technique in this paper.

Keywords: cubic partitions, rank, inequalities, q-series technique.

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1 Introduction

A partition π of a positive integer n is a sequence of positive integers $\pi_1\geq\pi_2\geq\cdots\geq\pi_k>0$ such that $\pi_1+\pi_2+\cdots+\pi_k=n$. Let $p(n)$ be the partition function, namely it counts the number of partitions of n. The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty}p(n)q^n=\frac{1}{J_1},\tag{1.1}$$

where here and throughout the rest of the paper, we use the following notation

$$\begin{aligned}(a;q)_\infty:=&\prod_{k=0}^{\infty}(1-aq^k),\\ (a_1,a_2,\ldots,a_k;q)_\infty:=&(a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty,\\ J_m:=&(q^m;q^m)_\infty.\end{aligned}$$

In 1919, Ramanujan [23] found the following famous congruences for ordinary partition function $p(n)$:

$$p(5n+4)\equiv 0\ (\text{mod }5),$$

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$$
\begin{aligned}
p(7n+5) \equiv& 0 \pmod{7},\\
p(11n+6) \equiv& 0 \pmod{11}.
\end{aligned}
$$

To provide combinatorial interpretations of Ramanujan's congruences for $p(n)$, two important partition statistics of partitions, rank and crank, were defined by Dyson [9], and Andrews and Garvan [1], respectively. In recent years, some equalities and inequalities on $N(r,m,n)$ and $M(r,m,n)$ for some small m have been established by mathematicians [2, 10, 19, 20, 28], where $N(r,m,n)$ and $M(r,m,n)$ denote the number of partitions of n with rank congruent to r modulo m and the number of partitions of n with crank congruent to r modulo m, respectively. For example, Andrews and Lewis [2] proved that for $n \geq 0$,

$$
\begin{aligned}
M(0,2,2n) \geq& M(1,2,2n),\\
M(0,2,2n+1) \leq& M(1,2,2n+1).
\end{aligned}
$$

In recent years, Chern, Tang and Wang [8], and Fu and Tang [11, 12] proved some inequalities for Garvan's bicrank function of 2-colored partitions and a generalized crank for k-colored partitions, respectively.

In this paper, we are going to focus on ranks for cubic partitions. Recall that the partitions in which even parts come in two colors blue (denoted by b) and red (denoted by r) are known as cubic partitions. For instance, the nine cubic partitions of 4 are:

$$4_b, \quad 4_r, \quad 3+1, \quad 2_b+2_b, \quad 2_b+2_r, \quad 2_r+2_r, \quad 2_r+1+1, \quad 2_b+1+1, \quad 1+1+1+1.$$

Let $a(n)$ denote the number of cubic partitions of n. The generating function for $a(n)$ is

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{J_1 J_2}.$$

In a series of papers, Chan [4, 5, 6] studied congruence properties for $a(n)$ and proved some congruences modulo powers of 3 for $a(n)$. In particular, Chan [4] proved an analog of Ramanujan's "most beautiful identity"

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{J_3^3 J_6^3}{J_1^4 J_2^4},$$

which implies that for $n \geq 0$,

$$a(3n+2) \equiv 0 \pmod{3}. \tag{1.2}$$

Hirschhorn [14], Xiong [26] and Yao [27] deduced some congruences modulo powers of 5 and 7 for $a(n)$. Chern and Dastidar [7] discovered two congruences modulo 11 for $a(n)$. For more details, see [22].

In his thesis, Reti [24] defined the rank of a cubic partition as the difference between the number of even parts in blue color and the number of even parts in red color. Let $N^{'}(m,n)$ denote the number of cubic partitions of n whose rank is m and let $N^{'}(r,m,n)$ denote the number of cubic partitions of n whose rank is congruent to r modulo m, namely,

$$N^{'}(m,n) := \sum_{\substack{\lambda\in\mathscr{C}(n),\\ \operatorname{rank}(\lambda)=m}} 1, \qquad \text{and} \qquad N^{'}(r,m,n) := \sum_{\substack{\lambda\in\mathscr{C}(n),\\ \operatorname{rank}(\lambda)\equiv r \pmod{m}}} 1, \tag{1.3}$$

where $\mathscr{C}(n)$ denote the set of cubic partitions of n. The generating function for $N^{'}(m,n)$, due to Reti [24, Theorem 1, eqn. (15)], is

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N^{'}(m,n)x^m q^n = \sum_{n=0}^{\infty}\sum_{\lambda\in\mathscr{C}(n)} x^{\mathrm{rank}(\lambda)}q^n = \frac{1}{(q;q^2)_\infty(xq^2;q^2)_\infty(q^2/x;q^2)_\infty}. \tag{1.4}$$

Reti [24] proved that

$$N^{'}(0,3,3n+2) = N^{'}(1,3,3n+2) = N^{'}(2,3,3n+2),$$

which provided a combinatorial interpretation of (1.2).

Motivated by the works on inequalities for ranks and cranks for certain partitions, such as [2, 8, 11], we prove some inequalities on $N^{'}(r,m,n)$ in this paper. More precisely, we establish the generating functions for $N^{'}(r,m,n)$ and determine the signs of the differences $N^{'}(r,m,n) - N^{'}(s,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \le r < s \le m-1$ by utilizing q-series technique.

From (1.3) and (1.4), we obtain

$$N^{'}(r,m) = N^{'}(-r,m), \qquad N^{'}(r,m,n) = N^{'}(m-r,m,n).$$

Therefore, we only list the inequalities on $N^{'}(r,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \le r \le \frac{m}{2}$ in the following theorems which parallels the results proved by Andrews and Lewis [2].

Theorem 1.1 *For $n \ge 0$,*

$$N^{'}(0,2,4n) \ge N^{'}(1,2,4n), \tag{1.5}$$
$$N^{'}(0,2,4n+1) \ge N^{'}(1,2,4n+1), \tag{1.6}$$
$$N^{'}(0,2,4n+2) \le N^{'}(1,2,4n+2), \tag{1.7}$$
$$N^{'}(0,2,4n+3) \le N^{'}(1,2,4n+3). \tag{1.8}$$

Theorem 1.2 *For $n \ge 0$,*

$$N^{'}(0,3,3n) \ge N^{'}(1,3,3n), \tag{1.9}$$
$$N^{'}(0,3,3n+1) \ge N^{'}(1,3,3n+1), \tag{1.10}$$
$$N^{'}(0,3,3n+2) = N^{'}(1,3,3n+2). \tag{1.11}$$

Remark. Identity (1.11) was first proved by Reti [24, Theorem 1, p. 9] by using a different method; see also [12].

Theorem 1.3 *For $n \ge 0$,*

$$N^{'}(0,4,n) \ge N^{'}(1,4,n), \tag{1.12}$$
$$N^{'}(1,4,n) \ge N^{'}(2,4,n). \tag{1.13}$$

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Theorem 1.4 *For $n \geq 0$,*

$$N'(j,6,n) \geq N'(j+1,6,n), \tag{1.14}$$

where $j \in \{0,1,2\}$.

The rest of the paper is organized as follows. In Section 2, we establish the generating functions for $N'(r,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \leq r \leq m-1$ which will be used to prove the main results of this paper. Sections 3–5 are devoted to the proofs of Theorems 1.1–1.4. We conclude in the last section with some remarks.

2 Generating functions for $N'(r,m,n)$

The aim of this section is to establish the generating functions for $N'(r,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \leq r \leq m-1$.

Theorem 2.1 *We have*

$$\sum_{n=0}^{\infty} N'(0,4,n)q^n = \frac{1}{4J_1J_2} + \frac{J_2J_4}{2J_1J_8} + \frac{J_2^3}{4J_1J_4^2}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} N'(1,4,n)q^n = \sum_{n=0}^{\infty} N'(3,4,n)q^n = \frac{1}{4J_1J_2} - \frac{J_2^3}{4J_1J_4^2}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} N'(2,4,n)q^n = \frac{1}{4J_1J_2} - \frac{J_2J_4}{2J_1J_8} + \frac{J_2^3}{4J_1J_4^2}. \tag{2.3}$$

By Theorem 2.1 and the fact that

$$N'(r,m,n) = N'(r,2m,n) + N'(r+m,2m,n), \tag{2.4}$$

we can deduce the generating functions for $N'(r,2,n)$.

Corollary 2.2 *We have*

$$\sum_{n=0}^{\infty} N'(0,2,n)q^n = \frac{1}{2J_1J_2} + \frac{J_2^3}{2J_1J_4^2}, \tag{2.5}$$

$$\sum_{n=0}^{\infty} N'(1,2,n)q^n = \frac{1}{2J_1J_2} - \frac{J_2^3}{2J_1J_4^2}. \tag{2.6}$$

Proof of Theorem 2.1. Here and throughout this paper, we always set $\zeta_m = e^{2\pi i/m}$. In view of (1.3), (1.4) and the fact that

$$\sum_{j=0}^{m-1} \zeta_m^{kj} = \begin{cases} m, & \text{if } k \equiv 0 \pmod{m}, \\ 0, & \text{if } k \not\equiv 0 \pmod{m}, \end{cases} \tag{2.7}$$

we deduce that for any integer r and any positive integer m,

$$\begin{aligned}
\sum_{n\geq 0} N'(r,m,n)q^n &= \sum_{n=0}^{\infty}\left(\sum_{\substack{\lambda\in\mathscr{C}(n),\\ \operatorname{rank}(\lambda)\equiv r \pmod m}} 1\right) q^n \\
&= \sum_{n=0}^{\infty}\sum_{\lambda\in\mathscr{C}(n)}\left(\frac{1}{m}\sum_{j=0}^{m-1}\zeta_m^{(\operatorname{rank}(\lambda)-r)j}\right) q^n \\
&= \frac{1}{m}\sum_{j=0}^{m-1}\zeta_m^{-rj}\sum_{n=0}^{\infty}\sum_{\lambda\in\mathscr{C}(n)}\zeta_m^{\operatorname{rank}(\lambda)j}q^n \\
&= \frac{1}{m}\sum_{j=0}^{m-1}\zeta_m^{-rj}\frac{1}{(q;q^2)_\infty(\zeta_m^j q^2;q^2)_\infty(q^2/\zeta_m^j;q^2)_\infty}. \qquad \text{(by (1.4))} \qquad (2.8)
\end{aligned}$$

In particular, setting $m=4$ and $r\in\{0,1,2,3\}$ in (2.8) yields

$$\sum_{n=0}^{\infty} N'(r,4,n)q^n = \frac{1}{4}\sum_{j=0}^{3}\zeta_4^{-rj}\frac{1}{(q;q^2)_\infty(\zeta_4^j q^2;q^2)_\infty(q^2/\zeta_4^j;q^2)_\infty}. \tag{2.9}$$

From the following identity,

$$(1-\zeta_4^j q^k)(1-q^k/\zeta_4^j) = 1-(\zeta_4^j+\zeta_4^{-j})q^k+q^{2k},$$

we arrive at

$$\frac{1}{(q;q^2)_\infty(\zeta_4^j q^2;q^2)_\infty(q^2/\zeta_4^j;q^2)_\infty} = \begin{cases} \dfrac{1}{J_1J_2}, & \text{if } j=0,\\ \dfrac{J_2J_4}{J_1J_8}, & \text{if } j\in\{1,3\},\\ \dfrac{J_2^3}{J_1J_4^2}, & \text{if } j=2. \end{cases} \tag{2.10}$$

By (2.9) and (2.10) and the fact that $\zeta_4 = \mathrm{i}$, we arrive at (2.1)–(2.3). The proof of Theorem 2.1 is complete. ∎

Theorem 2.3 *We have*

$$\sum_{n=0}^{\infty} N'(0,6,n)q^n = \frac{1}{6J_1J_2}+\frac{J_4J_6}{3J_1J_{12}}+\frac{J_2^2}{3J_1J_6}+\frac{J_2^3}{6J_1J_4^2}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} N'(1,6,n)q^n = \sum_{n=0}^{\infty} N'(5,6,n)q^n = \frac{1}{6J_1J_2}+\frac{J_4J_6}{6J_1J_{12}}-\frac{J_2^2}{6J_1J_6}-\frac{J_2^3}{6J_1J_4^2}, \tag{2.12}$$

$$\sum_{n=0}^{\infty} N'(2,6,n)q^n = \sum_{n=0}^{\infty} N'(4,6,n)q^n = \frac{1}{6J_1J_2}-\frac{J_4J_6}{6J_1J_{12}}-\frac{J_2^2}{6J_1J_6}+\frac{J_2^3}{6J_1J_4^2}, \tag{2.13}$$

$$\sum_{n=0}^{\infty} N'(3,6,n)q^n = \frac{1}{6J_1J_2}-\frac{J_4J_6}{3J_1J_{12}}+\frac{J_2^2}{3J_1J_6}-\frac{J_2^3}{6J_1J_4^2}. \tag{2.14}$$

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In view of (2.4) and Theorem 2.3, we obtain the following corollary.

Corollary 2.4 *We have*

$$\sum_{n=0}^{\infty} N^{'}(0,3,n)q^n = \frac{1}{3J_1J_2} + \frac{2J_2^2}{3J_1J_6}, \tag{2.15}$$

$$\sum_{n=0}^{\infty} N^{'}(1,3,n)q^n = \sum_{n=0}^{\infty} N^{'}(2,3,n)q^n = \frac{1}{3J_1J_2} - \frac{J_2^2}{3J_1J_6}. \tag{2.16}$$

Proof of Theorem 2.3. Setting $m=6$ and $r \in \{0,1,2,3,4,5\}$ in (2.8), we deduce that

$$\sum_{n=0}^{\infty} N^{'}(r,6,n)q^n = \frac{1}{6}\sum_{j=0}^{5} \zeta_6^{-rj} \frac{1}{(q;q^2)_\infty(\zeta_6^j q^2;q^2)_\infty(q^2/\zeta_6^j;q^2)_\infty}. \tag{2.17}$$

Moreover, it is easy to verify that

$$\frac{1}{(q;q^2)_\infty(\zeta_6^j q^2;q^2)_\infty(q^2/\zeta_6^j;q^2)_\infty} = \begin{cases} \dfrac{1}{J_1J_2}, & \text{if } j=0, \\ \dfrac{J_4J_6}{J_1J_{12}}, & \text{if } j\in\{1,5\}, \\ \dfrac{J_2^2}{J_1J_6}, & \text{if } j\in\{2,4\}, \\ \dfrac{J_2^3}{J_1J_4^2}, & \text{if } j=3. \end{cases} \tag{2.18}$$

In light of (2.17), (2.18) and the fact that $\zeta_6 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, we get (2.11)–(2.14). This completes the proof of Theorem 2.3. ∎

3 Proof of Theorem 1.1

Throughout this paper, for two power series $p_1(q) := \sum_{n=-\infty}^{\infty} b_1(n)q^n$ and $p_2(q) := \sum_{n=-\infty}^{\infty} b_2(n)q^n$, we say that $p_1(q) \succeq p_2(q)$ if $b_1(n) \geq b_2(n)$ holds for any integer n.

It follows from (2.5) and (2.6) that

$$\sum_{n=0}^{\infty}(N^{'}(0,2,n) - N^{'}(1,2,n))q^n = \frac{J_2^3}{J_1J_4^2}. \tag{3.1}$$

In [25], Xia and Yao proved that

$$\frac{J_2^3}{J_1J_4} = \frac{J_4}{(-q^2;q^{16})_\infty(q^8;q^{16})_\infty(-q^{14};q^{16})_\infty} + q\frac{J_4}{(-q^6;q^{16})_\infty(q^8;q^{16})_\infty(-q^{10};q^{16})_\infty}. \tag{3.2}$$

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Substituting (3.2) into (3.1) and then extracting the terms of the form q^{2n} and q^{2n+1}, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}(N^{'}(0,2,2n)-N^{'}(1,2,2n))q^n =&\frac{1}{(-q;q^8)_\infty(q^4;q^8)_\infty(-q^7;q^8)_\infty}\\ =&\frac{f(-q,-q^7)}{(q^2,q^{14};q^{16})_\infty(q^4;q^4)_\infty}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(N^{'}(0,2,2n+1)-N^{'}(1,2,2n+1))q^n =&\frac{1}{(-q^3;q^8)_\infty(q^4;q^8)_\infty(-q^5;q^8)_\infty}\\ =&\frac{f(-q^3,-q^5)}{(q^6,q^{10};q^{16})_\infty(q^4;q^4)_\infty},\end{aligned} \tag{3.4}$$

where

$$f(a,b)=(-a,-b,ab;ab)_\infty. \tag{3.5}$$

The following identity follows from [3, Entry 30, (ii) and (iii), p. 46]:

$$f(a,b)=f(a^3b,ab^3)+af(b/a,a^5b^3). \tag{3.6}$$

Setting $a=-q$ and $b=-q^7$ in (3.6) yields

$$f(-q,-q^7)=f(q^{10},q^{22})-qf(q^6,q^{26}). \tag{3.7}$$

Substituting (3.7) into (3.3) and then extracting the terms of the form q^{2n} and q^{2n+1}, we arrive at

$$\sum_{n=0}^{\infty}(N^{'}(0,2,4n)-N^{'}(1,2,4n))q^n =\frac{(-q^5,-q^{11},q^{16};q^{16})_\infty}{(q,q^7;q^8)_\infty(q^2;q^2)_\infty} \tag{3.8}$$

and

$$\sum_{n=0}^{\infty}(N^{'}(0,2,4n+2)-N^{'}(1,2,4n+2))q^n =-\frac{(-q^3,-q^{13},q^{16};q^{16})_\infty}{(q,q^7;q^8)_\infty(q^2;q^2)_\infty}. \tag{3.9}$$

Inequalities (1.5) and (1.7) follow from (3.8) and (3.9), respectively.

Setting $a=-q^3$ and $b=-q^5$ in (3.6) yields

$$f(-q^3,-q^5)=f(q^{14},q^{18})-q^3f(q^2,q^{30}). \tag{3.10}$$

If we substitute (3.10) into (3.4) and then extract the terms of the form q^{2n} and q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty}(N^{'}(0,2,4n+1)-N^{'}(1,2,4n+1))q^n =\frac{(-q^7,-q^9,q^{16};q^{16})_\infty}{(q^3,q^5;q^8)_\infty(q^2;q^2)_\infty} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty}(N^{'}(0,2,4n+3)-N^{'}(1,2,4n+3))q^n =-\frac{q(-q,-q^{15},q^{16};q^{16})_\infty}{(q^3,q^5;q^8)_\infty(q^2;q^2)_\infty}. \tag{3.12}$$

which imply (1.6) and (1.8). This completes the proof of Theorem 1.1. ∎

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4 Proofs of Theorems 1.2 and 1.3

We first present a proof of Theorem 1.2.

Proof of Theorem 1.2. From [3, Corollary (ii), p. 49], we have

$$\frac{J_2^2}{J_1} = \frac{J_6 J_9^2}{J_3 J_{18}} + q\frac{J_{18}^2}{J_9}. \tag{4.1}$$

By (2.15), (2.16) and (4.1),

$$\begin{aligned}\sum_{n=0}^{\infty}(N'(0,3,n) - N'(1,3,n))q^n =&\frac{J_2^2}{J_1 J_6}\\ =&\frac{1}{J_6}\left(\frac{J_6 J_9^2}{J_3 J_{18}} + q\frac{J_{18}^2}{J_9}\right)\\ =&\frac{1}{J_9 J_{18}}\sum_{n=0}^{\infty} t_3(n)q^{3n} + \frac{q}{J_9 J_{18}}\sum_{n=0}^{\infty} t_3(n)q^{6n}, \end{aligned}\tag{4.2}$$

where $t_s(n)$ is the number of s-core partitions of n and the generating functions of $t_s(n)$ is

$$\sum_{n=0}^{\infty} t_s(n)q^n = \frac{J_s^s}{J_1}.$$

From [16, Theorem 1],

$$\sum_{n=0}^{\infty} t_3(n)q^n = \frac{1}{3}\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}$$

Thus, $t_3(n)$ is nonnegative and Theorem 1.2 follows from (4.2). This completes the proof. ∎

Now, we turn to prove Theorem 1.3.

Proof of Theorem 1.3. It follows from (2.1) and (2.2) that

$$2\sum_{n=0}^{\infty}(N'(0,4,n) - N'(1,4,n))q^n =\frac{1}{J_4}\left(\frac{J_2 J_4^2}{J_1 J_8} + \frac{J_2^3}{J_1 J_4}\right). \tag{4.3}$$

Note that

$$\frac{J_2 J_4^2}{J_1} =\sum_{n=0}^{\infty} t_2(n)q^n \sum_{n=0}^{\infty} t_2(n)q^{2n},$$

and since $t_2(n) \geq 0$ for all $n \geq 0$ [13], we have

$$\frac{J_2 J_4^2}{J_1} \succeq 1 + q + q^2 + 2q^3 + q^5 + 2q^6 + q^7 + q^8 + q^9 + q^{10} + 3q^{12},$$

and therefore,

$$\frac{J_2 J_4^2}{J_1 J_8} =\frac{1}{J_8}\frac{J_2 J_4^2}{J_1}$$

$$
\begin{aligned}
&\succeq \frac{1}{J_8}(1+q+q^2+2q^3+q^5+2q^6+q^7+q^8+q^9+q^{10}+3q^{12}) \\
&\succeq 1+q+q^2+2q^3+q^5+2q^6+q^7+q^8+\sum_{n\geq 9} q^n. \qquad (4.4)
\end{aligned}
$$

The following identity is the well-known Euler's pentagonal number theorem:

$$
J_1 = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}(1-q^{2n+1}). \qquad (4.5)
$$

Replacing q by $-q$ in (4.5) and using the fact that

$$
(-q;-q)_\infty = \prod_{n=1}^{\infty}(1-(-q)^n) = \left(\prod_{n=1}^{\infty}(1+q^{2n-1})\right)\left(\prod_{n=1}^{\infty}(1-q^{2n})\right) = \frac{J_2^3}{J_1J_4}
$$

yields

$$
\begin{aligned}
\frac{J_2^3}{J_1J_4} &= \sum_{n=0}^{\infty}(-1)^n(-q)^{n(3n+1)/2}(1+q^{2n+1}) \\
&= 1+q-q^2-q^5-q^7-q^{12}+\sum_{n=3}^{\infty}(-1)^n(-q)^{n(3n+1)/2}(1+q^{2n+1}). \qquad (4.6)
\end{aligned}
$$

Combining (4.4) and (4.6), we arrive at

$$
\begin{aligned}
\frac{J_2J_4^2}{J_1J_8} + \frac{J_2^3}{J_1J_4} &\succeq 1+q+q^2+2q^3+q^5+2q^6+q^7+q^8+\sum_{n\geq 9} q^n \\
&\quad + \left(1+q-q^2-q^5-q^7-q^{12}+\sum_{n=3}^{\infty}(-1)^n(-q)^{n(3n+1)/2}(1+q^{2n+1})\right) \\
&\succeq 0,
\end{aligned}
$$

which yields (1.12) after combining (4.3).

In view of (2.2), (2.3), (4.4) and (4.6),

$$
\begin{aligned}
&2\sum_{n=0}^{\infty}(N'(1,4,n)-N'(2,4,n))q^n = \frac{1}{J_4}\left(\frac{J_2J_4^2}{J_1J_8} - \frac{J_2^3}{J_1J_4}\right) \\
&= \frac{1}{J_4}\Bigg(1+q+q^2+2q^3+q^5+2q^6+q^7+q^8+\sum_{n\geq 9} q^n \\
&\quad - \left(1+q-q^2-q^5-q^7-q^{12}+\sum_{n=3}^{\infty}(-1)^n(-q)^{n(3n+1)/2}(1+q^{2n+1})\right)\Bigg) \\
&\succeq 0,
\end{aligned}
$$

which yields (1.13). This completes the proof. ∎

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5 Proof of Theorem 1.4

In order to prove Theorem 1.4, we first prove two lemmas.

Lemma 5.1 *We have*

$$\frac{J_4J_6}{J_1J_{12}} - 6\frac{J_2^2}{J_1J_6} \succeq -5 - 5q + 2q^2 - 3q^3 + 4q^4 + 6q^5 - 4q^6. \tag{5.1}$$

Proof. In [15], Hirschhorn and Sellers proved that

$$\frac{J_2}{J_1^2} = \frac{J_6^4J_9^6}{J_3^8J_{18}^3} + 2q\frac{J_6^3J_9^3}{J_3^7} + 4q^2\frac{J_6^2J_{18}^3}{J_3^6}. \tag{5.2}$$

By (5.2),

$$\begin{aligned}
\frac{J_4J_6}{J_1J_{12}} - 6\frac{J_2^2}{J_1J_6} =& \frac{J_2^2}{J_1}\left(\frac{J_4J_6}{J_2^2J_{12}} - \frac{6}{J_6}\right)\\
=& \frac{J_2^2}{J_1}\left(\frac{J_6}{J_{12}}\left(\frac{J_{12}^4J_{18}^6}{J_6^8J_{36}^3} + 2q^2\frac{J_{12}^3J_{18}^3}{J_6^7} + 4q^4\frac{J_{12}^2J_{36}^3}{J_6^6}\right) - \frac{6}{J_6}\right)\\
\succeq& \frac{J_2^2}{J_1}\left(\sum_{n\geq 0} r_1(n)q^{6n} + \sum_{n\geq 0} q^{6n+2} + \sum_{n\geq 0} q^{6n+4}\right),
\end{aligned} \tag{5.3}$$

where $r_1(n)$ is defined by

$$\sum_{n\geq 0} r_1(n)q^n := \frac{J_2^3J_3^6}{J_1^7J_6^3} - \frac{6}{J_1}. \tag{5.4}$$

It is easy to check that

$$\frac{J_2^3J_3^6}{J_1^6J_6^3} \succeq 1 + 6q + 24q^2 + 73q^3 + \sum_{n\geq 4} q^n. \tag{5.5}$$

Based on (1.1) and (5.5),

$$\begin{aligned}
\sum_{n\geq 0} r_1(n)q^n =& \frac{J_2^3J_3^6}{J_1^6J_6^3}\sum_{n\geq 0} p(n)q^n - \frac{6}{J_1}\\
\succeq& \sum_{n\geq 0} p(n)q^n\left(1 + 6q + 24q^2 + 73q^3 + \sum_{n\geq 4} q^n\right) - 6\sum_{n\geq 0} p(n)q^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
r_1(n) \geq& -5p(n) + 6p(n-1) + 24p(n-2) + 73p(n-3) + \sum_{k\geq 4} p(n-k)\\
=& -5(p(n) - 2p(n-1) + p(n-3)) - 4(p(n-1) - 2p(n-2) + p(n-4))
\end{aligned}$$

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$$+16p(n-2)+78p(n-3)+4p(n-4)+\sum_{k\geq 4}p(n-k). \tag{5.6}$$

In [21], Merca proved for $n\geq 2$,

$$p(n)-2p(n-1)+p(n-3)\leq 0. \tag{5.7}$$

In view of (5.6) and (5.7), we find that for $n\geq 4$

$$r_1(n)\geq 16p(n-2)+78p(n-3)+\sum_{k\geq 4}p(n-k) \tag{5.8}$$

In addition,

$$r_1(0)=-5,\qquad r_1(1)=1,\qquad r_1(2)=20,\qquad r_1(3)=95. \tag{5.9}$$

Combining (5.8) and (5.9) yields

$$\sum_{n\geq 0}r_1(n)q^n\succeq -5+q+20q^2+95q^3+\sum_{n\geq 4}\sum_{k=4}^{n}p(n-k)q^n\succeq -5+\sum_{n\geq 1}q^n. \tag{5.10}$$

In light of (5.3) and (5.10),

$$\begin{aligned}\frac{J_4J_6}{J_1J_{12}}-6\frac{J_2^2}{J_1J_6}\succeq&\frac{J_2^2}{J_1}\left(-5+\sum_{n\geq 1}q^{6n}+q^2\sum_{n\geq 0}q^{6n}+q^4\sum_{n\geq 0}q^{6n}\right)\\ =&\sum_{k\geq 0}q^{k(k+1)/2}\left(-5+\sum_{n\geq 1}q^{2n}\right),\end{aligned} \tag{5.11}$$

where here we have used the following identity, due to Gauss [3, Entry 22, p. 36]:

$$\sum_{k\geq 0}q^{k(k+1)/2}=\frac{J_2^2}{J_1}. \tag{5.12}$$

Define

$$\sum_{n\geq 0}w(n)q^n:=\frac{J_4J_6}{J_1J_{12}}-6\frac{J_2^2}{J_1J_6}. \tag{5.13}$$

Thanks to (5.11) and (5.13),

$$\begin{aligned}w(n)&\geq -5|\{r|r(r+1)/2=n,\ r\geq 0\}|+|S(n)|\\ &\geq |S(n)|-5,\end{aligned} \tag{5.14}$$

where

$$S(n):=\{(r,s)|r(r+1)/2+2s=n,\ r\geq 0,s\geq 1\}$$

Note that if n is odd with $n\geq 68$, then

$$\{(1,(n-1)/2),(2,(n-3)/2),(5,(n-15)/2),\ (6,(n-21)/2),(9,(n-45)/2)\}\subseteq S(n)$$

and if n is even with $n \geq 68$, then

$$\{(3,(n-6)/2),(4,(n-10)/2),(7,(n-28)/2),\ (8,(n-36)/2),(11,(n-66)/2)\} \subseteq S(n).$$

Therefore, for $n \geq 68$, $|S(n)| \geq 5$ and

$$w(n) \geq 0. \tag{5.15}$$

With Maple, we find that

$$w(0)=-5,\ w(1)=-5,\ w(2)=2,\ w(3)=-3, w(4)=4, w(5)=6,\ w(6)=-4 \tag{5.16}$$

and for $7 \leq n \leq 67$,

$$w(n) \geq 0. \tag{5.17}$$

Lemma 5.1 follows from (5.15)–(5.17). This completes the proof of this lemma. ∎

Lemma 5.2 *We have*

$$\frac{J_4J_6}{J_1J_{12}}+4\frac{J_2^3}{J_1J_4^2} \succeq 5+5q-2q^2+3q^3+8q^4+6q^5+4q^6+7q^7, \tag{5.18}$$

$$\frac{J_4J_6}{J_1J_{12}}+2\frac{J_2^3}{J_1J_4^2} \succeq 3+3q+3q^3+6q^4+6q^5+6q^6+9q^7. \tag{5.19}$$

Proof. Here we only prove (5.18). Inequality (5.19) can be shown analogously, so we omit the details. It is easy to check that

$$\frac{J_4^2J_6}{J_1} = \frac{J_6}{J_2}\sum_{n=0}^{\infty} t_2(n)q^n \sum_{n=0}^{\infty} t_2(n)q^{2n} \succeq F(q), \tag{5.20}$$

where

$$\begin{aligned} F(q) =& 1+q+2q^2+3q^3+3q^4+5q^5+6q^6+8q^7+9q^8+12q^9+15q^{10}+18q^{11} \\ &+22q^{12}+26q^{13}+31q^{14}+37q^{15}+45q^{16}+52q^{17}. \end{aligned} \tag{5.21}$$

By (5.20) and (4.6),

$$\begin{aligned} \frac{J_4^2J_6}{J_1J_{12}}+4\frac{J_2^3}{J_1J_4} =& \frac{1}{J_{12}}\frac{J_4^2J_6}{J_1}+4\sum_{n=-\infty}^{\infty}(-1)^n(-q)^{n(3n-1)/2} \\ \succeq& \frac{F(q)}{J_{12}}+4\left(1+q-q^2-q^5-\sum_{n\geq 7}q^n\right) \\ \succeq& \left(1+q+2q^2+3q^3+3q^4+5q^5+6q^6+4\sum_{n\geq 7}q^n\right) \end{aligned}$$

$$
\begin{aligned}
&+4\left(1+q-q^2-q^5-\sum_{n\geq 7}q^n\right)\\
=&5+5q-2q^2+3q^3+3q^4+q^5+6q^6. \qquad (5.22)
\end{aligned}
$$

Define

$$\sum_{n\geq 0} r_2(n)q^n := \frac{J_4J_6}{J_1J_{12}}+4\frac{J_2^3}{J_1J_4^2}. \qquad (5.23)$$

In light of (1.1), (5.22) and (5.23),

$$
\begin{aligned}
\sum_{n\geq 0} r_2(n)q^n \succeq & \frac{1}{J_4}(5+5q-2q^2+3q^3+3q^4+q^5+6q^6)\\
=&\sum_{n\geq 0} p(n)q^{4n}(5+5q-2q^2+3q^3+3q^4+q^5+6q^6),
\end{aligned}
$$

which implies that for $n \geq 0$,

$$
\begin{aligned}
r_2(4n) \geq & 5p(n)+3p(n-1), & (5.24)\\
r_2(4n+1) \geq & 5p(n)+p(n-1), & (5.25)\\
r_2(4n+2) \geq & -2p(n)+6p(n-1), & (5.26)\\
r_2(4n+3) \geq & 3p(n). & (5.27)
\end{aligned}
$$

By (5.7) and (5.24)–(5.27), we see that for $n \geq 2$ and $0 \leq i \leq 3$,

$$r_2(4n+i) \geq 0. \qquad (5.28)$$

In addition,

$$r_2(0)=5,\ r_2(1)=5,\ r_2(2)=-2,\ r_2(3)=3,\ r_2(4)=8,\ r_2(5)=6,\ r_2(6)=4,\ r_2(7)=7,$$

from which with (5.28), (5.18) follows. The proof of Lemma 5.2 is complete. ∎

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By (2.11), (2.12), (5.12) and (5.19), it follows that

$$\sum_{n\geq 0}(N'(0,6,n)-N'(1,6,n))q^n = \frac{J_4J_6}{6J_1J_{12}}+\frac{J_2^2}{2J_1J_6}+\frac{J_2^3}{3J_1J_4^2} \succeq 0,$$

which implies that for $n \geq 0$,

$$N'(0,6,n)-N'(1,6,n) \geq 0. \qquad (5.29)$$

By (2.12), (2.13), (4.6) and (5.20),

$$3\sum_{n\geq 0}(N'(1,6,n)-N'(2,6,n))q^n = \frac{1}{J_4}\left(\frac{J_4^2J_6}{J_1J_{12}}-\frac{J_2^3}{J_1J_4}\right)$$

$$
\begin{aligned}
&=\frac{1}{J_4}\left(\frac{1}{J_{12}}\frac{J_4^2J_6}{J_1}-\sum_{n=-\infty}^{\infty}(-1)^n(-q)^{n(3n-1)/2}\right)\\
&\succeq\frac{1}{J_4}\left(\frac{F(q)}{J_{12}}-\left(1+q-q^2-q^5+\sum_{n\geq 7}q^n\right)\right)\\
&\succeq\frac{1}{J_4}\left(\left(1+q+2\sum_{n\geq 2}q^n\right)-\left(1+q-q^2-q^5+\sum_{n\geq 7}q^n\right)\right)\\
&\succeq 0,
\end{aligned}
$$

where $F(q)$ is defined by (5.21). The above inequality implies that for $n\geq 0$,

$$
N^{'}(1,6,n)-N^{'}(2,6,n)\geq 0. \tag{5.30}
$$

Thanks to (2.13), (2.14), (5.1) and (5.18),

$$
\sum_{n\geq 0}(N^{'}(2,6,n)-N^{'}(3,6,n))q^n=\frac{1}{12}\left(\frac{J_4J_6}{J_1J_{12}}-6\frac{J_2^2}{J_1J_6}\right)+\frac{1}{12}\left(\frac{J_4J_6}{J_1J_{12}}+4\frac{J_2^3}{J_1J_4^2}\right)\succeq 0,
$$

which yields that for $n\geq 0$,

$$
N^{'}(2,6,n)-N^{'}(3,6,n)\geq 0. \tag{5.31}
$$

Theorem 1.4 follows from (5.29)–(5.31). The proof is complete. ∎

6 Conclusions

As seen in Introduction, equalities and inequalities on statistics of certain partition functions have received a lot of attention in the past decade. In this paper, we establish the generating functions for $N^{'}(r,m,n)$ and then prove some inequalities between $N^{'}(r,m,n)$ and $N^{'}(s,m,n)$ with $m\in\{2,3,4,6\}$ and $0\leq r<s\leq m-1$ by utilizing q-series technique. In [19], Lewis presented combinatorial proofs of the following two inequalities:

$$
\begin{aligned}
N(0,2,2n)&<N(1,2,2n), \qquad n\geq 2,\\
N(0,2,2n+1)&>N(1,2,2n+1), \qquad n\geq 1.
\end{aligned}
$$

Therefore, it would be interesting to find combinatorial proofs of Theorems 1.1–1.4.

In 2011, Kim [17] introduced a cubic partition crank which explains infinitely many congruences for powers of 3. As a precise definition of crank for the cubic partition function is quite complicated and not necessary for the rest of the paper, we do not give it here. One may find a precise definition in [17]. Define

$$
M^{'}(m,n):=\sum_{\substack{\lambda\in\mathscr{C}(n),\\ \mathrm{crank}(\lambda)=m}}1, \qquad \text{and} \qquad M^{'}(r,m,n):=\sum_{\substack{\lambda\in\mathscr{C}(n)\\ \mathrm{crank}(\lambda)\equiv r \pmod{m}}}1, \tag{6.1}
$$

where $\mathscr{C}(n)$ is the set of cubic partitions of n. In 2016, Kim, Kim and Namc [18] deduced asymptotic formulas for $M^{'}(m,n)$ and $N^{'}(m,n)$ and proved that for a fixed integer m,

$$N^{'}(m,n) > M^{'}(m,n)$$

holds for large enough integers n. Using the same method given in this paper, we can deduce the generating functions for $M^{'}(r,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \leq r \leq m-1$. Unfortunately, we can not determine the signs of the differences $M^{'}(r,m,n) - M^{'}(s,m,n)$ with $m \in \{2,3,4,6\}$ and $0 \leq r < s \leq m-1$ by utilizing q-series technique. One may use asymptotic formulas to determine the signs.

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