New q-supercongruences from a quadratic transformation of Rahman

Na Tang

School of Mathematics and Statistics, Huaiyin Normal University, Huai'an 223300, Jiangsu, People's Republic of China

hytn999@126.com

Abstract. We give three families of q-supercongruences from a quadratic transformation of Rahman. As a conclusion, we obtain the following supercongruence: for $0 < r < d \leqslant 2r$ and any prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(rp+r-d)/d} (3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k 4^k} \equiv 0 \pmod{p^3},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

Keywords: q-supercongruences; supercongruences; Rahman's transformation; creative microscoping

AMS Subject Classifications: 33D15, 11A07, 11B65

1. Introduction

For any non-negative integer n and complex number a, let $(a)_n = a(a+1)\cdots(a+n-1)$ be the Pochhammer symbol. For any odd prime p and p-adic integer x, let $\Gamma_p(x)$ denote the p-adic Gamma function [11]. Motivated by Van Hamme's work on supercongruenes [13], He [7] gave the following supercongruence:

$$\sum_{k=0}^{(p-1)/2} (6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4} p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4}. \end{cases} \tag{1.1}$$

Liu [8] further proved that the above supercongruence is true modulo p^3.

Recently, using the method of 'creative microscoping' introduced by Guo and Zudilin [5], together with Rahman's quadratic transformation (see (1.8)), Liu and Wang [10] established the following q-analogue of Liu's refinement of (1.1): for any positive odd

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integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4,\end{cases} \tag{1.2}$$

where $M=n-1$ or $(n-1)/2$. Here and in what follows, the *q-shifted factorial* is defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for integers $n\geqslant 1$ or $n=\infty$. For convenience, we also adopt the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for integers $n\geqslant 0$ or $n=\infty$. The *q-integer* is defined as $[n]=[n]_q=(1-q^n)/(1-q)$. Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. We say that two rational functions $A(q)$ and $B(q)$ in q are congruent modulo a polynomial $P(q)$, denoted by $A(q)\equiv B(q) \pmod{P(q)}$, if the numerator of the reduced form of $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$. For some other recent work on q-supercongruences, we refer the reader to [1,4,6,9,10,12,14].

Very recently, Guo [3] gave some generalizations of (1.2) modulo $\Phi_n(q)^3$. Motivated by Guo's work, in this paper we shall give two new generalizations of the $n\equiv 3 \pmod 4$ case of (1.2) modulo $\Phi_n(q)^3$.

Theorem 1.1. *Let d and r be positive integers with $r<d\leqslant 2r$. Let n be a positive integer with $n\equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(rn+r-d)/d}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.3}$$

Theorem 1.2. *Let d and r be positive integers with $d\geqslant 2r$ and $d\equiv r+1\equiv 0 \pmod 2$. Let n be a positive integer with $n\equiv d+1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(dn+rn-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.4}$$

It is obvious that both (1.3) and (1.4) for $(d,r)=(2,1)$ reduce to the second case of (1.2) modulo $\Phi_n(q)^3$. Moreover, letting $n=p$ be a prime and $q\to 1$ in (1.3) and (1.4), we obtain the following two supercongruences: for $0<r<d\leqslant 2r$ and any prime $p\equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(rp+r-d)/d}(3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k4^k}\equiv 0 \pmod{p^3}, \tag{1.5}$$

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and for $d \geqslant 2r$ and any prime $p \equiv d+1 \pmod{2d}$,

$$\sum_{k=0}^{(dp+rp-r)/(2d)} (3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k 4^k} \equiv 0 \pmod{p^3}. \tag{1.6}$$

It is clear that both (1.5) and (1.6) are generalizations of (1.1) for $p \equiv 3 \pmod 4$.

We shall also give a new generalization of the $n \equiv 1 \pmod 4$ case of (1.2) modulo $\Phi_n(q)^2$ as follows.

Theorem 1.3. *Let d and r be positive integers with $d \geqslant 2r$. Let n be a positive integer with $n \equiv 1 \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$\equiv [rn]\frac{(q^d;q^{2d})_{r(n-1)/(2d)}}{(q^{d+2r};q^{2d})_{r(n-1)/(2d)}}q^{(r-d)r(n-1)/(2d)} \pmod{\Phi_n(q)^2}. \tag{1.7}$$

Likewise, the first case of (1.2) modulo $\Phi_n(q)^2$ follows from (1.7) by taking $(d,r) = (2,1)$. Moreover, letting $n = p$ be a prime and $q \to 1$ in (1.7), we arrive at the following supercongruence: for $0 < 2r \leqslant d$ and any prime $p \equiv 1 \pmod{2d}$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k 4^k} \equiv rp\frac{(\frac{1}{2})_{r(p-1)/(2d)}}{(\frac{d+2r}{2d})_{r(p-1)/(2d)}} \pmod{p^2}.$$

Note that the q-supercongruences in Theorems 1.1–1.3 do not hold modulo $[n]$ in general. When $r = 1$ the supercongruence (1.7) seems to be true modulo $\Phi_n(q)^3$ (which is the $N = n-1$ and $e \to 0$ case of [10, Theorem 4], but the proof of the $N = n-1$ case of [10, Theorem 4] is not correct). However, this is not the case for general r.

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see [2]) is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q,\, z\right] = \sum_{k=0}^{\infty}\frac{(a_1, a_2, \dots, a_{r+1};q)_k z^k}{(q, b_1, \dots, b_r;q)_k}.$$

A quadratic transformation of Rahman [2, (3.8.13)] may be stated as follows:

$$\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a,d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(aq/d,d,q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^k$$
$$= \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix} b, c, aq/bc \\ dq, aq^2/d \end{matrix}; q^2, q^2\right], \tag{1.8}$$

provided that d and aq/d are not of the form q^{-2n} (n is a non-negative integer).

We shall prove Theorems 1.1–1.3 by employing the method of 'creative microscoping' and Rahman's transformation (1.8) again.

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2. Proof of Theorem 1.1

We first give a generalization of Theorem 1.1 with an additional parameter a.

Theorem 2.1. *Let d and r be positive integers with $r < d \leqslant 2r$. Let n be a positive integer with $n \equiv -1 \pmod{2d}$ and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^{(d-r)n})(a-q^{(d-r)n})$,*

$$\sum_{k=0}^{(rn+r-d)/d}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0. \tag{2.1}$$

Proof. Letting $d\to 0$ in (1.8), we get

$$\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^{(k^2+k)/2}=\frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}. \tag{2.2}$$

We then take $q\mapsto q^d$, $a=q^r$, $b=q^{r+(d-r)n}$, $c=q^{r-(d-r)n}$ in the above formula to obtain

$$\begin{aligned}&\sum_{k=0}^{(dn-rn-r)/d}\frac{(1-q^{3dk+r})(q^r;q^{2d})_k(q^{r+(d-r)n},q^{r-(d-r)n},q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k(q^{2d-(d-r)n},q^{2d+(d-r)n},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\\&=\frac{(q^{2d+r},q^{d+r+(d-r)n},q^{d+r-(d-r)n},q^{2d-r};q^{2d})_\infty}{(q^d,q^{2d-(d-r)n},q^{2d+(d-r)n},q^{d+2r};q^{2d})_\infty}\\&=0,\end{aligned} \tag{2.3}$$

where we have used the fact that $(q^{r-(d-r)n};q^d)_k=0$ for $k>(dn-rn-r)/d$, and $(q^{d+r-(d-r)n};q^{2d})_\infty=0$. Since $(rn+r-d)/d\geqslant(dn-rn-r)/d$, we see that the left-hand side of (2.1) is equal to 0 for $a=q^{-(d-r)n}$ or $a=q^{(d-r)n}$. Namely, the q-congruence (2.1) is true modulo $1-aq^{(d-r)n}$ and $a-q^{(d-r)n}$.

On the other hand, letting $q\mapsto q^d$, $a=q^{r-rn}$, $b=aq^r$, $c=q^r/a$ in (1.8), we get

$$\begin{aligned}&\sum_{k=0}^{(rn+r-d)/d}\frac{(1-q^{3dk+r-rn})(q^{r-rn};q^{2d})_k(aq^r,q^r/a,q^{d-r-rn};q^d)_k}{(1-q^{r-rn})(q^d;q^d)_k(q^{2d-rn}/a,aq^{2d-rn},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\\&=\frac{(q^{2d+r-rn},aq^{d+r},q^{d+r}/a,q^{2d-r-rn};q^{2d})_\infty}{(q^d,q^{2d-rn}/a,aq^{2d-rn},q^{d+2r};q^{2d})_\infty}\\&=0,\end{aligned} \tag{2.4}$$

where we have utilized $(q^{2d-r-rn};q^{2d})_\infty=0$ and $(q^{d-r-rn};q^d)_k=0$ for $k>(rn+r-d)/d$. Since $n\equiv -1 \pmod{2d}$, we have $\gcd(2d,n)=1$. Thus, the minimal positive integer k such that $(q^m;q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $(2d-m)(n+1)/(2d)$ for m in the range $0<m<2d$. This means that the polynomial $(q^{d+2r};q^{2d})_k$ is always relatively prime to

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$\Phi_n(q)$ for $0 \leqslant k \leqslant (rn+r-d)/d$ (since $0 \leqslant (d+2r)-2d < 2d-r$ according to the condition in the theorem). In view of $q^n \equiv 1 \pmod{\Phi_n(q)}$, we conclude from (2.4) that

$$\sum_{k=0}^{(rn+r-d)/d} [3dk+r] \frac{(q^r;q^{2d})_k (aq^r, q^r/a, q^{d-r};q^d)_k}{(q^d;q^d)_k (aq^{2d}, q^{2d}/a, q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2} \equiv 0 \pmod{\Phi_n(q)}.$$

Since $1-aq^n$, $a-q^n$ and $\Phi_n(q)$ are pairwise relatively prime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.1. Note that the polynomial $(q^{2d};q^{2d})_k$ is relatively prime to $\Phi_n(q)$ for any $0 \leqslant k \leqslant n-1$. Moreover, the polynomial $(1-q^n)^2$ has the factor $\Phi_n(q)^2$. The proof of (1.3) then follows from (2.1) by specializing $a=1$. □

3. Proof of Theorem 1.2

Similarly as before, we first establish the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let d and r be positive integers with $d \geqslant 2r$ and $d \equiv r+1 \equiv 0 \pmod 2$. Let n be a positive integer with $n \equiv d+1 \pmod{2d}$ and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^{rn})(a-q^{rn})$,*

$$\sum_{k=0}^{(dn+rn-r)/(2d)} [3dk+r] \frac{(q^r;q^{2d})_k (aq^r, q^r/a, q^{d-r};q^d)_k}{(q^d;q^d)_k (aq^{2d}, q^{2d}/a, q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2} \equiv 0. \tag{3.1}$$

Proof. The proof is similar to that of Theorem 2.1. This time we take $q \mapsto q^d$, $a = q^r$, $b = q^{r+rn}$, $c = q^{r-rn}$ in (2.2) to obtain

$$\begin{aligned} &\sum_{k=0}^{r(n-1)/d} \frac{(1-q^{3dk+r})(q^r;q^{2d})_k (q^{r+rn}, q^{r-rn}, q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k (q^{2d-rn}, q^{2d+rn}, q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2} \\ &= \frac{(q^{2d+r}, q^{d+r+rn}, q^{d+r-rn}, q^{2d-r};q^{2d})_\infty}{(q^d, q^{2d-rn}, q^{2d+rn}, q^{d+2r};q^{2d})_\infty} \\ &= 0, \end{aligned} \tag{3.2}$$

where we have used the fact that $(q^{r-rn};q^d)_k = 0$ for $k > r(n-1)/d$, and $(q^{d+r-rn};q^{2d})_\infty = 0$. This proves that the left-hand side of (3.1) is equal to 0 for $a = q^{-rn}$ or $a = q^{rn}$ (since $r(n-1)/d < (dn+rn-r)/(2d)$). Namely, the q-congruence (3.1) is true modulo $1-aq^{rn}$ and $a-q^{rn}$.

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On the other hand, letting $q\mapsto q^d, a=q^{r-(d+r)n}, b=aq^r$, $c=q^r/a$ in (1.8), we get

$$\sum_{k=0}^{(dn+rn-r)/(2d)}\frac{(1-q^{3dk+r-dn-rn})(q^{r-dn-rn};q^{2d})_k(aq^r,q^r/a,q^{d-r-dn-rn};q^d)_k}{(1-q^{r-dn-rn})(q^d;q^d)_k(q^{2d-dn-rn}/a,aq^{2d-dn-rn},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$=\frac{(q^{2d+r-dn-rn},aq^{d+r},q^{d+r}/a,q^{2d-r-dn-rn};q^{2d})_\infty}{(q^d,q^{2d-dn-rn}/a,aq^{2d-dn-rn},q^{d+2r};q^{2d})_\infty}$$
$$=0, \tag{3.3}$$

where we have utilized $(q^{2d+r-dn-rn};q^{2d})_\infty=0$ and $(q^{r-dn-rn};q^{2d})_k=0$ for $k>(dn+rn-r)/(2d)$. It is easy to see that $\gcd(2d,n)=1$, and the minimal positive integer k such that $(q^{d+2r};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $(d+2r)(n-1)/(2d)$, which is greater than $(dn+rn-r)/(2d)$. Thus, the polynomial $(q^{d+2r};q^{2d})_k$ is always relatively prime to $\Phi_n(q)$ for $0\leqslant k\leqslant (dn+rn-r)/(2d)$. In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, we conclude from (3.3) that

$$\sum_{k=0}^{(dn+rn-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)}.$$

This proves (3.1). □

Proof of Theorem 1.3. Since $\gcd(2d,n)=1$ and $q^n\equiv 1 \pmod{\Phi_n(q)}$, the proof of (1.4) immediately follows from the $a=1$ case of (3.1). □

4. Proof of Theorem 1.3

Likewise, we have a parametric generalization of Theorem 1.3 as follows.

Theorem 4.1. *Let d and r be positive integers with $d>r$. Let n be a positive integer with $n\equiv 1 \pmod{2d}$ and a an indeterminate. Then, modulo $(1-aq^{rn})(a-q^{rn})$,*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$\equiv [rn]\frac{(q^d;q^{2d})_{r(n-1)/(2d)}}{(q^{d+2r};q^{2d})_{r(n-1)/(2d)}}q^{(r-d)r(n-1)/(2d)}. \tag{4.1}$$

Proof. We again take $q\mapsto q^d$, $a=q^r$, $b=q^{r+rn}$, $c=q^{r-rn}$ in (2.2) as in the proof of

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Theorem 3.1. But this time

$$\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(1-q^{3dk+r})(q^r;q^{2d})_k(q^{r+rn},q^{r-rn},q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k(q^{2d-rn},q^{2d+rn},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\\
&=\frac{(q^{2d+r},q^{d+r+rn},q^{d+r-rn},q^{2d-r};q^{2d})_\infty}{(q^d,q^{2d-rn},q^{2d+rn},q^{d+2r};q^{2d})_\infty}\\
&=\frac{(q^{2d+r},q^{d+r-rn};q^{2d})_{r(n-1)/(2d)}}{(q^{d+2r},q^{2d-rn};q^{2d})_{r(n-1)/(2d)}}\\
&=\frac{(1-q^{rn})(q^d;q^{2d})_{r(n-1)/(2d)}}{(1-q^r)(q^{d+2r};q^{2d})_{r(n-1)/(2d)}}q^{(r-d)r(n-1)/(2d)},
\end{aligned}$$

where we have used the fact that $(q^{r-rn};q^d)_k=0$ for $k>r(n-1)/d$. This proves that both sides of (4.1) are equal for $a=q^{-rn}$ or $a=q^{rn}$. Namely, the q-congruence (4.1) is true modulo $1-aq^{rn}$ and $a-q^{rn}$. □

Proof of Theorem 1.3. In view of $n\equiv 1 \pmod{2d}$, we have $\gcd(2d,n)=1$. Thus, the minimal positive integer k such that $(q^m;q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $m(n-1)/(2d)+1$ for m in the range $0<m\leqslant 2d$. This indicates that the denominator of the reduced form of $(q^r;q^{2d})_k/(q^{d+2r};q^{2d})_k$ is always relatively prime to $\Phi_n(q)$ for $0\leqslant k\leqslant n-1$ (since $0<r<d+2r\leqslant 2d$). The proof of (1.7) then follows from the $a=1$ case of (4.1). □

5. Some open problems

Numerical calculation implies that the following stronger versions of Theorems 1.1–1.3 should be true.

Conjecture 5.1. *Let d and r be positive integers with $d>r$. Let n be a positive integer with $n\equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{5.1}$$

Conjecture 5.2. *Let d and r be positive integers with $d>r$ and $d\equiv r+1\equiv 0 \pmod 2$. Let n be a positive integer with $n\equiv d+1 \pmod{2d}$. Then (5.1) holds.*

Conjecture 5.3. *Let d be a positive odd integer, $r=(d+1)/2$, and n a positive integer satisfying $n\equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(2n+2-d)(d-1)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^4}. \tag{5.2}$$

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Conjecture 5.4. *Let d and r be positive integers with $d > r$. Let n be a positive integer with $n \equiv 1 \pmod{2d}$. Then (1.7) holds.*

Conjecture 5.5. *Let $d > 1$ be an odd integer, $r = (d-1)/2$, and n a positive integer satisfying $n \equiv 1 \pmod{2d}$. Then (1.7) holds modulo $\Phi_n(q)^3$.*

Note that Conjecture 5.3 was provided by one of the referees, and it is a generalization of [3, Conjecture 5.2]. This referee also asked us to find a similar conjecture related to Theorem 1.3. After a simple try, we found the above Conjecture 5.5. However, we are unable to confirm these two conjectures, since it is difficult to find the corresponding parametric versions of the q-supercongruences (5.2) and the modulus $\Phi_n(q)^3$ case of (1.7). Finally, we point out that Conjectures 5.3 and 5.5 are not yet true modulo $[n]$ in general, For example, the q-congruence (5.2) does not hold modulo $[n]$ for $(d, r, n) = (3, 2, 35)$, and the q-congruence (1.7) does not hold modulo $[n]$ for $(d, r, n) = (5, 2, 51)$ either.

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