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RATIONAL QUADRILATERALS

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ABSTRACT. A quadrilateral is said to be rational if its four sides, the two diagonals and the area are all expressible by rational numbers. The problem of constructing rational quadrilaterals dates back to the seventh century when Brahmagupta gave an elegant solution of the problem. In 1848 Kummer gave a method of generating all rational quadrilaterals. In this paper we present an alternative method of generating all rational quadrilaterals. For rational cyclic quadrilaterals, we obtain a complete parametrization and for noncyclic rational quadrilaterals, we give several parametrizations in terms of quadratic and quartic polynomials. The parametrizations obtained in this paper are simpler than the known parametrizations of rational quadrilaterals. We also describe how further parametrizations of rational quadrilaterals may be obtained.

1. Introduction

Since ancient times there has been considerable interest in geometric objects such as triangles, quadrilaterals and other polygons such that the lengths of their sides and diagonals, as well as their areas are expressible by rational numbers. This paper is concerned with rational quadrilaterals — a rational quadrilateral being defined as one whose four sides, the two diagonals and the area are given by rational numbers.

In the seventh century Brahmagupta gave an elegant method of constructing rational quadrilaterals. Brahmagupta's method was elaborated further by Bhaskara in the 12^{th} century and by Chasles in the 19^{th} century (as quoted by Dickson [7, pp. 216–217]). In 1848 Kummer [13] gave a method of generating all rational quadrilaterals. In 1921 Dickson [6] presented Kummer's construction in a somewhat simplified manner.

We will now briefly describe Kummer's construction. Kummer first proved that both the diagonals of a rational quadrilateral mutually divide each other into two rational segments, say α, γ, and β, δ, respectively. He then showed that, without loss of generality, we may take $\beta = 1$, and there must exist rational numbers ξ, η and c, where $|c| < 1$, such that

$$\left(\frac{(\xi+c)^2-1}{2\xi}\right)\left(\frac{(x-c)^2-1}{2x}\right) = \left(\frac{(\eta-c)^2-1}{2\eta}\right)\left(\frac{(y+c)^2-1}{2y}\right), \tag{1.1}$$

and all quadrilaterals with rational sides and diagonals may be obtained by solving the diophantine equation (1.1). An attempt to solve Eq. (1.1) yields a parametrized quartic function that must be made a perfect square. While the complete solution of this final condition is not known, Kummer was able to find parametric solutions of Eq. (1.1).

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As an example, Kummer [13, p. 270] found the following parametrization of a quadrilateral $ABCD$ in terms of arbitrary rational parameters ξ, η and c:

$$
\begin{aligned}
&AB = (\xi^2 + k^2)/(2\xi), \quad BC = (\eta^2 + k^2)/(2\eta),\\
&CD = (\eta^2(\xi\eta - 2c\xi + k^2)^2 + k^2(\xi\eta + 2c\eta + k^2)^2)/(2\eta(\xi\eta + (1+c)^2)(\xi\eta + (1-c)^2)),\\
&DA = (\xi^2(\xi\eta + 2c\eta + k^2)^2 + k^2(\xi\eta - 2c\xi + k^2)^2)/(2\xi(\xi\eta + (1+c)^2)(\xi\eta + (1-c)^2)),\\
&\alpha = ((\xi + c)^2 - 1)/(2\xi), \quad \beta = 1, \quad \gamma = ((\eta - c)^2 - 1)/(2\eta),\\
&\delta = ((\xi\eta - 2c\xi + k^2)(\xi\eta + 2c\eta + k^2))/((\xi\eta + (1+c)^2)(\xi\eta + (1-c)^2))
\end{aligned}
\tag{1.2}
$$

where $k^2 = 1 - c^2$, and AB, BC, CD, DA are the sides of the quadrilateral while, as already mentioned, α, γ, and β, δ are the segments of the two diagonals. Kummer did not give explicitly the formulae for the diagonals and area of the quadrilateral (1.2) but it is clear that the two diagonals have rational lengths $\alpha + \gamma$, and $\beta + \delta$, respectively. The area of the quadrilateral is $(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha)k/2$ and it will also be rational if $c = (\lambda^2 - 1)/(\lambda^2 + 1)$ where λ is an arbitrary rational number.

Since the publication of Kummer's classic paper, various related problems concerning rational quadrilaterals have been studied by several mathematicians ([1], [2], [3], [7, pp. 216–221], [10], [11], [14], [17]). Some authors have adopted a relaxed definition of rational quadrilaterals that requires only the four sides and the two diagonals of the quadrilateral to be rational without the additional condition of the area being rational while some others have studied only cyclic quadrilaterals. For instance, Sastry [17, pp. 170–171] has given a parametrization of cyclic quadrilaterals whose sides, diagonals and the area are all expressible by rational numbers. However, regarding the original problem of constructing quadrilaterals with rational sides, diagonals and area, there has been no progress in over a hundred years since Dickson's simplification in 1921 of the method described by Kummer.

In this paper we describe a method, different from that of Kummer, of generating all rational convex quadrilaterals. We obtain several parametrizations of rational quadrilaterals whose sides a, b, c, d, the diagonals e, f and the area A are given by multivariate polynomials. Specifically we show that the sides, diagonals and the area of an arbitrary rational cyclic quadrilateral, may be written, after appropriate scaling, in terms of independent rational parameters $p_i, q_i, r_i, i = 1, 2$, as follows:

$$
\begin{aligned}
&a = (p_1^2 + p_2^2)(r_1^2 + r_2^2)q_1q_2, \quad b = (q_1^2 + q_2^2)(r_1^2 + r_2^2)p_1p_2, \quad c = (q_1^2 + q_2^2)(p_1^2 + p_2^2)r_1r_2,\\
&d = (p_1q_1r_2 + p_1q_2r_1 + p_2q_1r_1 - p_2q_2r_2)(p_1q_1r_1 - p_1q_2r_2 - p_2q_1r_2 - p_2q_2r_1),\\
&e = (r_1^2 + r_2^2)(p_1q_2 + p_2q_1)(p_1q_1 - p_2q_2), \quad f = (q_1^2 + q_2^2)(p_1r_2 + p_2r_1)(p_1r_1 - p_2r_2),\\
&A = (p_1q_1 - p_2q_2)(p_1q_2 + p_2q_1)(p_1r_1 - p_2r_2)(p_1r_2 + p_2r_1)(q_1r_1 - q_2r_2)(q_1r_2 + q_2r_1).
\end{aligned}
\tag{1.3}
$$

As an example of a rational noncyclic quadrilateral, we obtain the following parametrization:

$$
\begin{aligned}
&a=(p_1^2+p_2^2)^2(q_2^2-q_1^2)q_1q_2, \quad b=p_1p_2(p_1^2+p_2^2)(q_2^4-q_1^4),\\
&c=2p_1p_2(p_1q_2+p_2q_1)(p_2q_2-p_1q_1)(q_1^2+q_2^2),\\
&d=(p_1q_2+p_2q_1)(p_2q_2-p_1q_1)(p_1q_1-p_1q_2-p_2q_1-p_2q_2)(p_1q_1+p_1q_2+p_2q_1-p_2q_2),\\
&e=(p_1^2+p_2^2)(p_2q_2-p_1q_1)(p_1q_2+p_2q_1)(q_2^2-q_1^2), \quad f=p_1p_2(p_2^2-p_1^2)(q_1^2+q_2^2)^2,\\
&A=p_1p_2(p_2^2-p_1^2)(q_2^2-q_1^2)(p_1q_2+p_2q_1)(p_2q_2-p_1q_1)(p_1q_1^2-p_1q_2^2-2p_2q_1q_2)\\
&\qquad \times(2p_1q_1q_2+p_2q_1^2-p_2q_2^2),
\end{aligned}
\tag{1.4}
$$

where $p_i,q_i,i=1,2$, are arbitrary nonzero rational parameters.

The parametrizations of rational quadrilaterals obtained in this paper are more general and simpler as compared to Kummer's formulae as well as Sastry's formulae for cyclic quadrilaterals.

2. A theorem on rational quadrilaterals

We now prove a theorem that describes a necessary and sufficient condition for the existence of a rational convex quadrilateral.

Theorem 1. *A necessary and sufficient condition for the existence of a rational convex quadrilateral is that there exist rational numbers a,b,c,d,e,f,x_1,x_2,y_1 and y_2 satisfying the simultaneous diophantine equations,*

$$x_1^2+y_1^2=a^2, \tag{2.1}$$

$$(e-x_1)^2+y_1^2=b^2, \tag{2.2}$$

$$(e-x_2)^2+y_2^2=c^2, \tag{2.3}$$

$$x_2^2+y_2^2=d^2, \tag{2.4}$$

$$(x_1-x_2)^2+(y_1-y_2)^2=f^2. \tag{2.5}$$

and such that the following rational numbers are all positive: $a,b,c,d,e,f,\ -y_1y_2, -(y_1-y_2)(x_1y_2-x_2y_1)$ and $(y_1-y_2)(ey_1-ey_2+x_1y_2-x_2y_1)$.

Proof. Without loss of generality, let $OABC$ be a rational convex quadrilateral with one vertex O at the origin of the x-y plane and with its diagonal OB along the x-axis as shown in Figure 1. Let the lengths of the sides OA,AB,BC and OC of the quadrilateral be a,b,c and d respectively, and let the lengths of the diagonals OB and AC be e and f respectively. Let D and E be the feet of the perpendiculars to the diagonal OB drawn from the vertices A and C respectively, and let lines through the vertices C and A parallel to the x- and y- axes, respectively, intersect at F. Let the coordinates of the vertices A and C be (x_1,y_1) and (x_2,y_2) respectively. It readily follows from the theorem of Pythagoras that the relations (2.1)–(2.5) must be satisfied.

Since $OABC$ is assumed to be a rational quadrilateral, a,b,c,d,e and f are rational numbers and the area A of the quadrilateral is rational. We will now prove that the four numbers $x_i,y_i,i=1,2$, are all rational. Since all the three sides of the two triangles OAB and OCB are rational, the cosines of the

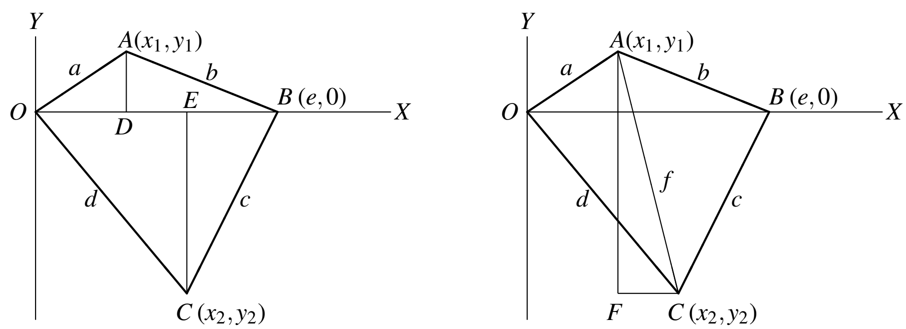

FIGURE 1. Rational quadrilateral

angles AOB and COB are rational, and hence the lengths $OA\cos AOB$ and $OC\cos COB$, that is, x_1 and x_2 are rational.

Further, it follows from Heron's formula for the area of a triangle that the areas of the two triangles OAB and OCB may be written as $\sqrt{r_1}$ and $\sqrt{r_2}$ where r_1 and r_2 are rational numbers. Since, by definition, the area A of the quadrilateral $OABC$ is rational, it follows that $\sqrt{r_1}+\sqrt{r_2}=A$, or, $\sqrt{r_1}=A-\sqrt{r_2}$, or, $r_1=A^2+r_2-2A\sqrt{r_2}$, and hence $\sqrt{r_2}$ is a rational number. Similarly, $\sqrt{r_1}$ is also a rational number. Thus, the areas of the two triangles OAB and OCB are rational. Since we may write these areas as $OB\times AD/2$ and $OB\times CE/2$, respectively, and OB is of rational length, it follows that the lengths of the line segments AD and CE are rational, that is, both y_1 and y_2 are rational numbers.

We will now prove that all the rational numbers mentioned towards the end of the theorem are positive. We first note that a,b,c,d,e,f, being the lengths of the sides and diagonals of the quadrilateral, are necessarily positive rational numbers. Further, since $OABC$ is a convex quadrilateral, the diagonal OB lies wholly within the quadrilateral, and hence the vertices A and C must be on opposite sides of the diagonal OB, and hence the ordinates y_1 and y_2 must be of opposite signs, and thus $-y_1y_2$ has to be positive. Finally, since $OABC$ is a convex quadrilateral, the point of intersection M of the two diagonals must strictly lie somewhere between the two vertices O and B, that is, the abscissa of the point M must be between 0 and e. The coordinates of the point M are readily worked out to be $(-(x_1y_2-x_2y_1)/(y_1-y_2),0)$, hence we must have $0<-(x_1y_2-x_2y_1)/(y_1-y_2)<e$. It now readily follows that the numbers $-(y_1-y_2)(x_1y_2-x_2y_1)$ and $(y_1-y_2)(ey_1-ey_2+x_1y_2-x_2y_1)$ are both positive.

We have thus proved that if there exists a rational convex quadrilateral, then there exist rational numbers a,b,c,d,e,f,x_1,x_2,y_1 and y_2 satisfying the relations (2.1)–(2.5) and such that the numbers $a,b,c,d,e,f,-y_1y_2,-(y_1-y_2)(x_1y_2-x_2y_1)$ and $(y_1-y_2)(ey_1-ey_2+x_1y_2-x_2y_1)$ are all positive. We have thus shown that the condition stated in the theorem is necessary.

Next let there exist rational numbers a,b,c,d,e,f,x_1,x_2,y_1 and y_2 satisfying the conditions of the theorem. We will construct a quadrilateral $OABC$ with vertices O,A,B and C at the points $(0,0),(x_1,y_1),(e,0)$ and (x_2,y_2) respectively as shown in Figure 1. In view of the relations (2.1)–(2.5), it is clear that the lengths of the four sides of the quadrilateral are $a,b,c,$ and d, respectively, and the two diagonals OB and AC have lengths e and f respectively, and hence these lengths are all rational.

Further, the line segments OB, AD and CE have rational lengths, hence both the triangles OAB and OCB have rational areas, and thus the area of the quadrilateral $OABC$ is rational. Finally it is readily seen that the conditions that the numbers $-y_1y_2, -(y_1-y_2)(x_1y_2-x_2y_1)$ and $(y_1-y_2)(ey_1-ey_2+x_1y_2-x_2y_1)$ are all positive, ensure that the quadrilateral is convex. Thus, the condition stated in the theorem is sufficient for the existence of a rational convex quadrilateral. This proves the theorem. □

It is readily seen that if $(a,b,c,d,e,f,x_1,x_2,y_1,y_2) = (a',b',c',d',e',f',x'_1,x'_2,y'_1,y'_2)$ is a solution of the simultaneous Eqs. (2.1)–(2.5), then $(\varepsilon_1a', \varepsilon_2b', \varepsilon_3c', \varepsilon_4d', \varepsilon_5e', \varepsilon_6f', \varepsilon_5x'_1, \varepsilon_5x'_2, \varepsilon_7y'_1, \varepsilon_7y'_2)$ is also a solution where each $\varepsilon_i, i = 1,\ldots,7$, is taken either as $+1$ or -1. If a solution of the diophantine Eqs. (2.1)–(2.5) does not satisfy all the conditions of a convex quadrilateral, sometimes we may be able to choose appropriate values of ε_i and obtain a solution that satisfies all the conditions of a convex quadrilateral. This is not always possible as is illustrated by the solution $(a,b,c,d,e,f,x_1,x_2,y_1,y_2) = (210,280,325,375,350,165,126,225,168,300)$ which does not satisfy the condition $-y_1y_2 > 0$ even with the permissible choices for ε_i.

3. Parametrizations of rational quadrilaterals

We will now solve the simultaneous diophantine Eqs. (2.1)–(2.5). It is not difficult to obtain the complete solution of any four of the five Eqs. (2.1)–(2.5), and in each case the final equation reduces to a parameterized quartic function to be made a perfect square. While rational solutions of this final condition can be obtained, these are generally quite cumbersome. This situation is similar to the one that arises on following Kummer's method.

In the next section we will solve the four Eqs. (2.1)–(2.4) in a manner such that the final condition arising from Eq. (2.5) has simple solutions.

3.1. ***Quadrilaterals with four sides, one diagonal and the area expressible by rational numbers.*** We will now obtain the complete solution of the four simultaneous diophantine Eqs. (2.1)–(2.4). We will repeatedly make use of the fact that the complete solution of the diophantine equation $X_1X_2 = Y_1Y_2$ is given by $X_1 = gm, X_2 = hn, Y_1 = gn, Y_2 = hm$, where g,h,m and n are arbitrary parameters (see [16, p. 69]). The complete solution of Eqs. (2.1)–(2.4) will also yield a parametrization of all quadrilaterals $OABC$ in which the four sides, the diagonal OB and the area of the quadrilateral $OABC$ are rational.

Taking the difference of the two equations, (2.1) and (2.2), we get $-e(e-2x_1) = (a+b)(a-b)$, hence we may write $-e = 2u_1v_1, e-2x_1 = 2u_2v_2, a+b = 2u_1v_2, a-b = 2u_2v_1$. We have introduced the factor 2 on the right-hand side in each case so as to get the nice solution,

$$a = u_1v_2 + u_2v_1, \quad b = u_1v_2 - u_2v_1, \quad e = -2u_1v_1, \quad x_1 = -u_1v_1 - u_2v_2, \tag{3.1}$$

where u_1, u_2, v_1, v_2 are arbitrary parameters.

Similarly, the complete solution of the difference of the two equations (2.3) and (2.4) is given by

$$d = u_3v_4 + u_4v_3, \quad c = u_3v_4 - u_4v_3, \quad e = -2u_3v_3, \quad x_2 = -u_3v_3 - u_4v_4, \tag{3.2}$$

where u_3, u_4, v_3, v_4 are arbitrary parameters.

The two solutions (3.1) and (3.2) will be consistent if the values of e given by the two solutions are the same, that is, $u_1v_1 = u_3v_3$, and hence we take

$$u_1 = m_1n_1, \quad v_1 = m_2n_2, \quad u_3 = m_1n_2, \quad v_3 = m_2n_1, \tag{3.3}$$

where m_1, m_2, n_1 and n_2 are arbitrary nonzero rational parameters. On substituting the values of $u_i, v_i, i = 1, 2$ given by (3.3) in (3.1) and (3.2), we get the following relations,

$$\begin{aligned} &a = m_1n_1v_2 + m_2n_2u_2, \quad b = m_1n_1v_2 - m_2n_2u_2,\\ &c = m_1n_2v_4 - m_2n_1u_4, \quad d = m_1n_2v_4 + m_2n_1u_4,\\ &e = -2m_1m_2n_1n_2, \quad x_1 = -m_1m_2n_1n_2 - u_2v_2, \quad x_2 = -m_1m_2n_1n_2 - u_4v_4, \end{aligned} \tag{3.4}$$

where $m_1, m_2, n_1, n_2, u_2, u_4, v_2$ and v_4 are arbitrary parameters.

On substituting the values of a and x_1 given by (3.4) in (2.1), we get $y_1^2 = -(m_2n_2 - v_2)(m_2n_2 + v_2)(m_1n_1 - u_2)(m_1n_1 + u_2)$. Thus the right-hand side of this last equation must be a nonzero perfect square, and this is possible if and only if there exists a nonzero rational number h_1 such that $h_1^2(m_1n_1 - u_2)(m_1n_1 + u_2) = -(m_2n_2 - v_2)(m_2n_2 + v_2)$, hence there exist nonzero rational parameters $p_i, q_i, i = 1, 2$, such that

$$h_1(m_1n_1 - u_2) = 2p_1q_2, \; h_1(m_1n_1 + u_2) = 2p_2q_1, \; -(m_2n_2 - v_2) = 2p_1q_1, \; m_2n_2 + v_2 = 2p_2q_2. \tag{3.5}$$

On solving Eqs. (3.5) for m_1, m_2, u_2 and v_2, we get,

$$\begin{aligned} &m_1 = (p_1q_2 + p_2q_1)/(h_1n_1), \quad m_2 = -(p_1q_1 - p_2q_2)/n_2,\\ &u_2 = -(p_1q_2 - p_2q_1)/h_1, \quad v_2 = p_1q_1 + p_2q_2. \end{aligned} \tag{3.6}$$

Similarly, on substituting the values of d and x_2 given by (3.4) in (2.4), we get $y_2^2 = -(m_2n_1 - v_4)(m_2n_1 + v_4)(m_1n_2 - u_4)(m_1n_2 + u_4)$, and proceeding as in the previous paragraph, we get the solution,

$$\begin{aligned} &m_1 = (r_1s_2 + r_2s_1)/(h_2n_2), \quad m_2 = -(r_1s_1 - r_2s_2)/n_1,\\ &u_4 = -(r_1s_2 - r_2s_1)/h_2, \quad v_4 = r_1s_1 + r_2s_2, \end{aligned} \tag{3.7}$$

where h_2, r_1, r_2, s_1, s_2 are arbitrary nonzero rational parameters.

For the solutions (3.6) and (3.7) to be consistent, we must impose the conditions that the respective values of m_1 and m_2 given by these two solutions coincide. These conditions are satisfied if and only if we choose h_1, h_2, n_1 and n_2 as follows:

$$\begin{aligned} &h_1 = (p_1q_2 + p_2q_1)(p_1q_1 - p_2q_2), \quad h_2 = (r_1s_2 + r_2s_1)(r_1s_1 - r_2s_2),\\ &n_1 = r_1s_1 - r_2s_2, \quad n_2 = p_1q_1 - p_2q_2. \end{aligned} \tag{3.8}$$

Using the values of $h_i, n_i, i = 1, 2$, given by (3.8) we can now find the values of m_1, m_2, u_2, v_2, u_4 and v_4 from the relations (3.6) and (3.7), and we can solve Eqs. (2.1) and (2.4) to obtain rational values of y_1, y_2, and further, using the relations (3.4), we get, on appropriate scaling, the following solution of the simultaneous diophantine Eqs. (2.1)–(2.4):

$$
\begin{aligned}
a &= q_1q_2(r_1s_2+r_2s_1)(r_1s_1-r_2s_2)(p_1^2+p_2^2),\\
b &= p_1p_2(r_1s_2+r_2s_1)(r_1s_1-r_2s_2)(q_1^2+q_2^2),\\
c &= r_1r_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(s_1^2+s_2^2),\\
d &= s_1s_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(r_1^2+r_2^2),\\
e &= (p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(r_1s_2+r_2s_1)(r_1s_1-r_2s_2),\\
x_1 &= q_1q_2(r_1s_2+r_2s_1)(r_1s_1-r_2s_2)(p_1-p_2)(p_1+p_2),\\
x_2 &= s_1s_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(r_1-r_2)(r_1+r_2),\\
y_1 &= 2p_1p_2q_1q_2(r_1s_2+r_2s_1)(r_1s_1-r_2s_2),\\
y_2 &= -2r_1r_2s_1s_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2).
\end{aligned}
\tag{3.9}
$$

where $p_i, q_i, r_i, s_i, i = 1, 2$, are arbitrary rational parameters.

While solving the simultaneous diophantine equations (2.1)–(2.4), we had to solve several intermediate equations and for each of these equations, we obtained the complete solution. Hence, the formulae (3.9) give the complete solution of the simultaneous equations (2.1)–(2.4).

It follows that the formulae (3.9) generate all convex quadrilatetrals whose four sides a, b, c, d, one diagonal and the area are given by rational numbers. We must, of course, choose the parameters such that the rational numbers mentioned towards the end of Theorem 1 are all positive. The area of the quadrilateral defined by (3.9) is given by

$$
\begin{aligned}
A = {}&(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(r_1s_2+r_2s_1)(r_1s_1-r_2s_2)[p_1^2q_1q_2r_1r_2s_1s_2\\
&+\{q_1^2r_1r_2s_1s_2+(r_1s_2+r_2s_1)(r_1s_1-r_2s_2)q_1q_2-q_2^2r_1r_2s_1s_2\}p_1p_2-p_2^2q_1q_2r_1r_2s_1s_2].
\end{aligned}
\tag{3.10}
$$

3.2. ***Rational quadrilaterals.*** We will now obtain parametrizations of rational quadrilaterals by obtaining parametric solutions of the simultaneous diophantine Eqs. (2.1)–(2.5).

On substituting the values of x_1, x_2, y_1 and y_2 given by (3.9) in Eq. (2.5), we get the condition,

$$
\begin{aligned}
&q_1^2q_2^2r_1^2r_2^2(p_1^2+p_2^2)^2s_1^4-2p_1p_2q_1q_2r_1r_2\{(q_1r_1+q_1r_2+q_2r_1-q_2r_2)p_1+(q_1r_1-q_1r_2\\
&\quad-q_2r_1-q_2r_2)p_2\}\{(q_1r_1-q_1r_2-q_2r_1-q_2r_2)p_1-(q_1r_1+q_1r_2+q_2r_1-q_2r_2)p_2\}s_1^3s_2\\
&\quad+\{2p_1^4q_1^2q_2^2r_1^2r_2^2+8q_1q_2r_1r_2(q_1r_2+q_2r_1)(q_1r_1-q_2r_2)p_1^3p_2+(q_1^4r_1^4+2q_1^4r_1^2r_2^2+q_1^4r_2^4\\
&\qquad+8q_1^3q_2r_1^3r_2-8q_1^3q_2r_1r_2^3+2q_1^2q_2^2r_1^4-24q_1^2q_2^2r_1^2r_2^2+2q_1^2q_2^2r_2^4-8q_1q_2^3r_1^3r_2\\
&\quad+8q_1q_2^3r_1r_2^3+q_2^4r_1^4+2q_2^4r_1^2r_2^2+q_2^4r_2^4)p_1^2p_2^2-8q_1q_2r_1r_2(q_1r_2+q_2r_1)(q_1r_1-q_2r_2)p_1p_2^3\\
&\quad+2p_2^4q_1^2q_2^2r_1^2r_2^2\}s_1^2s_2^2+2p_1p_2q_1q_2r_1r_2\{(q_1r_1+q_1r_2+q_2r_1-q_2r_2)p_1+(q_1r_1-q_1r_2\\
&\quad-q_2r_1-q_2r_2)p_2\}\{(q_1r_1-q_1r_2-q_2r_1-q_2r_2)p_1-(q_1r_1+q_1r_2+q_2r_1-q_2r_2)p_2\}s_1s_2^3\\
&\quad+q_1^2q_2^2r_1^2r_2^2(p_1^2+p_2^2)^2s_2^4=f^2.
\end{aligned}
\tag{3.11}
$$

We need to solve Eq. (3.11) such that the rational numbers mentioned towards the end of Theorem 1 are positive. Thus the parameters $p_i, q_i, r_i, s_i, i = 1, 2$, must all be nonzero and simple solutions of

Eq. (3.11) such as $p_1 = p_2q_2/q_1$ or $p_1 = -p_2q_1/q_2$ or $r_1 = r_2s_2/s_1$ or $r_1 = -r_2s_1/s_2$ are also ruled out. We will give four solutions of (3.11) that satisfy the conditions of Theorem 1, and thus obtain four parametrizations of rational quadrilaterals. We will also indicate how infinitely many parametrizations of rational quadrilaterals may be obtained.

We will obtain two solutions by first writing $s_1 = ts_2, f = (p_1^2+p_2^2)q_1q_2r_1r_2s_2^2Y$, when Eq. (3.11) may be written as

$$Y^2 = t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 \tag{3.12}$$

where the values of $a_i, i = 1,2,3,4$, are readily obtained. Three simple solutions of Eq. (3.12) are given by $(-q_1/q_2, \pm Y_1), (-r_1/r_2, \pm Y_2)$ and $(r_2/r_1, \pm Y_3)$, where

$$\begin{aligned} Y_1 &= (q_1^2+q_2^2)(p_1r_2+p_2r_1)(p_1r_1-p_2r_2)/(q_2^2r_1r_2(p_1^2+p_2^2)),\\ Y_2 &= (r_1^2+r_2^2)(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)/(q_1q_2r_2^2(p_1^2+p_2^2)),\\ Y_3 &= (r_1^2+r_2^2)(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)/(q_1q_2r_1^2(p_1^2+p_2^2)). \end{aligned} \tag{3.13}$$

According to a theorem proved by Choudhry [5, Theorem 4.1, pp. 789–790], if $(t_i, Y_i), i = 1,2$ are two rational solutions of Eq. (3.12), a new rational solution is given by (t_{12}, Y_{12}) where

$$\begin{aligned} t_{12} = \{&-2Y_1Y_2 + 2(t_1-t_2)(t_2Y_1-t_1Y_2) + a_1(t_1+t_2)t_1t_2 + 2a_2t_1t_2 + a_3(t_1+t_2) + 2a_4\\ &+2(t_1^2-t_1t_2+t_2^2)t_1t_2\}/\{(t_1-t_2)(2Y_1-2Y_2+a_1(t_1-t_2)+2t_1^2-2t_2^2)\}, \end{aligned} \tag{3.14}$$

provided the denominator on the right-hand side of (3.14) $\neq 0$. We omit the value of Y_{12} given in [5, Theorem 4.1, pp. 789–790] since it is cumbersome to write and is not needed for our computations.

Applying Choudhry's theorem to the two known solutions $(-q_1/q_2, -Y_1)$ and $(-r_1/r_2, -Y_2)$ yields a solution of Eq. (3.12) in which

$$t = (p_1q_1r_2 + p_1q_2r_1 + p_2q_1r_1 - p_2q_2r_2)/(p_1q_1r_1 - p_1q_2r_2 - p_2q_1r_2 - p_2q_2r_1). \tag{3.15}$$

Since $s_1 = s_2t$, this immediately yields the following values of s_1 and s_2 which make the left-hand side of Eq. (3.11) a perfect square:

$$s_1 = p_1(q_1r_2+q_2r_1) + p_2(q_1r_1-q_2r_2), \quad s_2 = p_1(q_1r_1-q_2r_2) - p_2(q_1r_2+q_2r_1). \tag{3.16}$$

Similarly by applying Choudhry's aforesaid theorem to the two known solutions $(-r_1/r_2, Y_2)$ and $(r_2/r_1, Y_3)$ of Eq. (3.12), we obtain a new solution which yields the following values s_1 and s_2 which make the left-hand side of Eq. (3.11) a perfect square:

$$s_1 = r_1r_2\{2p_1q_1q_2 + (q_1^2-q_2^2)p_2\}, \quad s_2 = q_1q_2\{(r_1^2-r_2^2)p_1 - 2p_2r_1r_2\}. \tag{3.17}$$

We can obtain more solutions of Eq. (3.11) by using the aforementioned theorem of Choudhry. Further, Eq. (3.12) may be considered as representing the quartic model of an elliptic curve over the function field $\mathbb{Q}(p_1.p_2, q_1, q_2, r_1, r_2)$ and we found a point on this curve P whose abscissa is given by (3.15). We can, by following a well-known procedure (see, for instance, [15, p. 77] or [4, pp. 35–36]), reduce the elliptic curve (3.12) by a birational transformation to the usual cubic model, and find a point P_1 on the cubic model corresponding to the point P on the quartic model. It is easily established that the point P_1 is not of finite order. Thus, by repeated application of the group law, we can find infinitely many rational points on the cubic model of the curve, and hence also on the curve (3.12).

The infinitely many rational solutions of Eq. (3.12) would yield imfinitely many parametrizations of rational quadrilaterals.

We will now describe another method of obtaining simple solutions of Eq. (3.11). We first rewrite the left-hand side of Eq. (3.11) as a quartic function of r_1 and r_2, that is, we write it as $c_0r_1^4+c_1r_1^3r_2+c_2r_1^2r_2^2+c_3r_1r_2^2+c_4r_2^4$. We now note that c_1 and c_3 vanish simultaneously if we take $s_1=(p_1-p_2)q_1-(p_1+p_2)q_2$ and $s_2=-(p_1+p_2)q_1-(p_1-p_2)q_2$. With these values of s_1,s_2, the left-hand side of Eq. (3.11) has a squared factor $(q_1^2+q_2^2)^2$ which may be removed to obtain a quartic function of q_1,q_2 whose discriminant with respect to q_1 vanishes if we take $r_1=-p_2$ and $r_2=p_1$, and now the left-hand side of Eq. (3.11) becomes a perfect square. We thus get a solution of Eq. (3.11) by taking

$$r_1=-p_2,\ r_2=p_1,\ s_1=(p_1-p_2)q_1-(p_1+p_2)q_2,\ s_2=-(p_1+p_2)q_1-(p_1-p_2)q_2. \tag{3.18}$$

Similarly, we may write the left-hand side as a quartic function of q_1,q_2 when we observe that the coefficients of q_1^3 and q_1 both vanish if we take $s_1=(p_1-p_2)r_1-(p_1+p_2)r_2$, $s_2=-(p_1+p_2)r_1-(p_1-p_2)r_2$. With these values of s_1,s_2, the left-hand side of Eq. (3.11) has a squared factor $(r_1^2+r_2^2)^2$ on removing which we get a quartic function of r_1,r_2 whose discriminant with respect to r_1 vanishes if we take $q_1=-p_2,q_2=p_1$, and now the left-hand side of Eq. (3.11) becomes a perfect square. We thus get a solution of Eq. (3.11) by taking

$$q_1=-p_2,\ q_2=p_1,\ s_1=(p_1-p_2)r_1-(p_1+p_2)r_2,\ s_2=-(p_1+p_2)r_1-(p_1-p_2)r_2. \tag{3.19}$$

Further solutions of Eq. (3.11) may be obtained by writing the left-hand side as a quartic function of p_1,p_2 or of s_1,s_2 and proceeding in a similar manner.

While the methods described above could be used to obtain additional solutions of Eq. (3.11), we restrict ourselves to the four explicit solutions given above since with these solutions all the conditions mentioned towards the end of Theorem 1 are satisfied and we actually obtain rational convex quadrilaterals. In the next four subsections we consider the quadrilaterals generated by the four solutions of Eq. (3.11) obtained above.

3.2.1. *Rational cyclic quadrilaterals.* We first give a parametrization of all rational cyclic quadrilaterals. We have already seen that the left-hand side of Eq. (3.11) becomes a perfect square if we assign to s_1,s_2 the values given by (3.16). This yields a rational quadrilateral whose sides a,b,c,d, the diagonals e,f and the area A may be written, after appropriate scaling, as given by (1.3).

Further, the related values of x_1,x_2,y_1,y_2, obtained from (3.9) are as follows:

$$\begin{aligned}
x_1&=(p_1+p_2)(p_1-p_2)(r_1^2+r_2^2)q_1q_2,\\
x_2&=(r_1-r_2)(r_1+r_2)(p_1q_1r_2+p_1q_2r_1+p_2q_1r_1-p_2q_2r_2)\\
&\qquad\times(p_1q_1r_1-p_1q_2r_2-p_2q_1r_2-p_2q_2r_1)/(r_1^2+r_2^2),\\
y_1&=2(r_1^2+r_2^2)p_1p_2q_1q_2,\\
y_2&=-2r_1r_2(p_1q_1r_2+p_1q_2r_1+p_2q_1r_1-p_2q_2r_2)\\
&\qquad\times(p_1q_1r_1-p_1q_2r_2-p_2q_1r_2-p_2q_2r_1)/(r_1^2+r_2^2),
\end{aligned} \tag{3.20}$$

If we assign positive rational values to all the parameters $p_i, q_i, r_i, i = 1, 2$, such that $q_1 > q_2 r_2 / r_1$ and $p_1 > p_2(q_1 r_2 + q_2 r_1)/(q_1 r_1 - q_2 r_2)$, it readily follows that all conditions mentioned in Theorem 1 are satisfied so that the formulae (1.3) define a rational convex quadrilateral. As a numerical example, when we take $(p_1, p_2, q_1, q_2, r_1, r_2) = (4, 1, 3, 1, 2, 1)$, we get a rational convex quadrilateral whose sides, diagonals and the area, after appropriate scaling, are given by $(a, b, c, d, e, f, A) = (51, 40, 68, 75, 77, 84, 3234)$.

We will now show that the formulae (1.3) give the sides, diagonals and the area of a cyclic quadrilateral. In fact, it is useful to recall here that if a, b, c, d are the consecutive sides of a cyclic quadrilateral, the area K and the lengths e and f of the two diagonals of the quadrilateral are given by the following formulae proved by Brahmagupta [8, p. 187]:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{3.21}$$

and

$$\begin{aligned} e &= \sqrt{(ab+cd)(ac+bd)/(ad+bc)}, \\ f &= \sqrt{(ac+bd)(ad+bc)/(ab+cd)}, \end{aligned} \tag{3.22}$$

where s is the semi-perimeter, that is, $s = (a+b+c+d)/2$. Further, the circumradius R of the cyclic quadrilateral is given by the following formula of Paramesvara [9]:

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}. \tag{3.23}$$

We also note that the area Δ of an arbitrary convex quadrilateral having sides a, b, c, d, and the sum of any one pair of opposite angles equal to $2u$, is given by the following formula [12, p. 82]:

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2 u, \tag{3.24}$$

where s is the semiperimeter.

It follows from the formulae (3.21) and (3.24) that a quadrilateral with sides a, b, c, d, is cyclic if and only if its area is given by the formula (3.21). It is readily verified that the area A of the quadrilateral defined by (1.3) is exactly the area of a cyclic quadrilateral whose sides a, b, c, d, are given by (1.3). Further, the lengths of the two diagonals, given by the formulae (3.22), coincide with the lengths e and f of the two diagonals given by (1.3). Thus, the quadrilateral defined by (1.3) is a cyclic quadrilateral. We also note that the circumradius R of the quadrilateral defined by (1.3) is guven by

$$R = (p_1^2 + p_2^2)(q_1^2 + q_2^2)(r_1^2 + r_2^2)/4. \tag{3.25}$$

It is pertinent to consider at this stage whether there are any other values of the parameters $p_i, q_i, r_i, s_i, i = 1, 2$, such the quadrilateral defined by (3.9) and (3.10) becomes a cyclic quadrilateral. The condition that the area A of this quadrilateral becomes equal to the area of a cyclic quadrilateral with sides a, b, c, d, may be written as follows:

$$\begin{aligned} &p_1 p_2 q_1 q_2 r_1 r_2 s_1 s_2 (r_1 s_2 + r_2 s_1)^2 (r_1 s_1 - r_2 s_2)^2 (p_1 q_2 + p_2 q_1)^2 (p_1 q_1 - p_2 q_2)^2 \\ &\quad \times \{(p_1 q_1 r_1 - p_1 q_2 r_2 - p_2 q_1 r_2 - p_2 q_2 r_1)s_1 - (p_1 q_1 r_2 + p_1 q_2 r_1 + p_2 q_1 r_1 - p_2 q_2 r_2)s_2\}^2 = 0. \end{aligned} \tag{3.26}$$

Equating to 0 any factor on the left-hand side of (3.26), except the last, does not yield a quadrilateral since at least one of the four sides a,b,c,d, becomes 0, and the last factor, when equated to 0, yields the cyclic quarilateral given by (1.3).

Even if we consider the possibility that for a certain choice of parameters, the formulae (3.9) may yield negative values for one or more of the numbers a,b,c,d, and we obtain a rational convex quadrilateral by changing the signs of a,b,c,d, as necessary, when we impose the condition that the area of the quadrilateral becomes equal to the area of a cyclic quadrilateral, and proceed as above, we eventually get the same parametrization of a quadrilateral as is given by (1.3). It follows that the formulae (1.3) give the complete parametrization of all cyclic quadrilaterals.

We note that Euler (as quoted by Dickson [7, p. 221]) had given complicated expressions for the sides and diagonals of a cyclic quadrilateral to be rational but the area is not rational. Sastry's parametrization [17, pp. 170–171], mentioned in the Introduction, gives the sides a,b,c,d, the diagonals e,f and the area A of a rational cyclic quadrilateral in terms of arbitrary rational parameters t,t_1 and t_2 as follows:

$$\begin{aligned}
&a=(t(t_1+t_2)+(1-t_1t_2))(t_1+t_2-t(1-t_1t_2)), \quad b=(1+t_1^2)(t_2-t)(1+tt_2),\\
&c=t(1+t_1^2)(1+t_2^2), \quad d=(1+t_2^2)(t_1-t)(1+tt_1),\\
&e=t_1(1+t^2)(1+t_2^2), \quad f=t_2(1+t^2)(1+t_1^2),\\
&A=t_1t_2(2t(1-t_1t_2)-(t_1+t_2)(1-t^2))(2(t_1+t_2)t+(1-t_1t_2)(1-t^2)).
\end{aligned} \tag{3.27}$$

Our formulae (1.3) are clearly simpler and more symmetric as compared to Sastry's formulae.

3.2.2. *A parametrization of noncyclic convex quadrilaterals.* The values of s_1, s_2 given by (3.17) yield a rational quadrilateral whose sides a, b, c, d, the diagonals e, f and the area A are as follows:

$$
\begin{aligned}
a &= q_1q_2(p_1^2+p_2^2)\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_2-2q_1q_2r_1-q_2^2r_2)p_2r_2\}\\
&\qquad\times\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_1+2q_1q_2r_2-q_2^2r_1)p_2r_1\},\\
b &= p_1p_2(q_1^2+q_2^2)\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_2-2q_1q_2r_1-q_2^2r_2)p_2r_2\}\\
&\qquad\times\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_1+2q_1q_2r_2-q_2^2r_1)p_2r_1\},\\
c &= (p_1q_2+p_2q_1)(p_1q_1-p_2q_2)\{(r_1^2+r_2^2)^2p_1^2q_1^2q_2^2+4p_1p_2q_1q_2r_1r_2\\
&\qquad\times(q_1r_1+q_2r_2)(q_1r_2-q_2r_1)+(q_1^2+q_2^2)^2p_2^2r_1^2r_2^2\},\\
d &= q_1q_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)(r_1^2+r_2^2)\{2p_1q_1q_2+(q_1^2-q_2^2)p_2\}\\
&\qquad\times\{(r_1^2-r_2^2)p_1-2p_2r_1r_2\},\\
e &= (p_1q_2+p_2q_1)(p_1q_1-p_2q_2)\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_2-2q_1q_2r_1\\
&\qquad-q_2^2r_2)p_2r_2\}\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_1+2q_1q_2r_2-q_2^2r_1)p_2r_1\},\\
f &= q_1q_2\{(r_1^2+r_2^2)^2p_1^4q_1^2q_2^2+2\{(r_1^4+r_2^4)q_1^2+2r_1r_2(r_1^2-r_2^2)q_1q_2-(r_1^4+r_2^4)q_2^2\}p_1^3p_2q_1q_2\\
&\qquad+\{(r_1^4-r_1^2r_2^2+r_2^4)q_1^4-(r_1^4+8r_1^2r_2^2+r_2^4)q_1^2q_2^2+(r_1^4-r_1^2r_2^2+r_2^4)q_2^4\}p_1^2p_2^2\\
&\qquad-2\{(r_1^2-r_2^2)q_1^4+2q_1^3q_2r_1r_2-2q_1q_2^3r_1r_2+(r_1^2-r_2^2)q_2^4\}p_1p_2^3r_1r_2+(q_1^2+q_2^2)^2p_2^4r_1^2r_2^2),\\
A &= q_1q_2(p_1q_2+p_2q_1)(p_1q_1-p_2q_2)\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_2-2q_1q_2r_1\\
&\qquad-q_2^2r_2)p_2r_2\}\{(r_1^2+r_2^2)p_1q_1q_2+(q_1^2r_1+2q_1q_2r_2-q_2^2r_1)p_2r_1\}\\
&\qquad\times\{2p_1^2q_1q_2r_1r_2+(q_1r_2+q_2r_1)(q_1r_1-q_2r_2)p_1p_2-2p_2^2q_1q_2r_1r_2\}\\
&\qquad\times\{(r_1^2-r_2^2)p_1^2q_1q_2+(r_1^2-r_2^2)(q_1^2-q_2^2)p_1p_2-(q_1^2-q_2^2)p_2^2r_1r_2\},
\end{aligned}
\tag{3.28}
$$

where $p_i, q_i, r_i, i = 1, 2$, are arbitrary nonzero rational parameters.

As a numerical example, if we take $(p_1, p_2, q_1, q_2, r_1, r_2) = (3, 1, 2, 1, 3, 1)$, we get a rational convex quadrilateral whose sides, diagonals and the area, after appropriate scaling, are given by $(a, b, c, d, e, f, A) = (748, 561, 615, 1000, 935, 1068, 490314)$.

We note that if, in the formulae (3.28), we take q_1, q_2, r_1, r_2 as numerical constants, the sides and diagonals of the quadrilateral are given by quartic polynomials in the parameters p_1 and p_2. Similarly, we may consider the formulae for the sides and diagonals of the above quadrilateral as quartic polynomials in the parameters r_1 and r_2.

3.2.3. *A second parametrization of noncyclic convex quadrilaterals.* If we take r_1, r_2, s_1, s_2 as in (3.18), we get a rational quadrilateral whose sides a, b, c, d, the diagonals e, f and the area A are given by (1.4).

The values of a,b,c,d,e and f given by (1.4) and the values of x_1,x_2,y_1,y_2 given by

$$\begin{aligned}
x_1 &= (p_2^4-p_1^2)(q_2^2-q_1^2)q_1q_2,\\
x_2 &= (p_1q_1-p_1q_2-p_2q_1-p_2q_2)(p_1q_1+p_1q_2+p_2q_1-p_2q_2)(p_2^2-p_1^2)\\
&\quad\times(p_1q_2+p_2q_1)(p_2q_2-p_1q_1)/(p_1^2+p_2^2),\\
y_1 &= 2(p_1^2+p_2^2)(q_2^2-q_1^2)p_1p_2q_1q_2,\\
y_2 &= -2p_1p_2(p_1q_1-p_1q_2-p_2q_1-p_2q_2)(p_1q_1+p_1q_2+p_2q_1-p_2q_2)\\
&\quad\times(p_1q_2+p_2q_1)(p_2q_2-p_1q_1)/(p_1^2+p_2^2)
\end{aligned}\tag{3.29}$$

satisfy the simultaneous Eqs. (2.1)–(2.5).

It follows from Theorem 1 that if we assign positive rational values to the parameters $p_i,q_i,i=1,2$, such that $q_2>q_1$, and $p_2>p_1(q_1+q_2)/(q_2-q_1)$, the quadrilateral defined by (1.4) is a convex quadrilateral.

As a numerical example, on taking $(p_1,p_2,q_1,q_2)=(1,2,1,5)$, we get a rational convex quadrilateral whose sides, diagonals and the area, after appropriate scaling, are given by $(a,b,c,d,e,f,A)=(125,260,273,84,315,169,26334)$.

3.2.4. *Rational quadrilaterals with two equal sides.* While two sides of paralellograms, rhombi amd kites are equal, these quadrilaterals have additional geometric properties which may be used to obtain their complete parametrization. For instance, the problem of finding a rational paralellogram is equivalent to finding a rational triangle with a rational median, and its complete solution has been given by Dickson [6, pp. 249–250]. We give below a parametrization of a more general quadrilateral in which two of the four sides are equal but the quadrilateral does not have any other specific geometric property.

If we assign to q_1,q_2,s_1,s_2 the values given by (3.19), we get a rational quadrilateral whose sides a,b,c,d, the diagonals e,f and the area A are as follows:

$$\begin{aligned}
a &= b = (p_1^2+p_2^2)(r_1^2+r_2^2), \quad c = 4(p_1^2+p_2^2)r_1r_2,\\
d &= 2(p_1r_1+p_1r_2+p_2r_1-p_2r_2)(p_1r_1-p_1r_2-p_2r_1-p_2r_2),\\
e &= 2(r_1^2+r_2^2)(p_2^2-p_1^2), \quad f = (p_1^2+p_2^2)(r_1^2+r_2^2),\\
A &= 2(p_2^2-p_1^2)(2p_1r_1r_2+p_2r_1^2-p_2r_2^2)(p_1r_1^2-p_1r_2^2-2p_2r_1r_2),
\end{aligned}\tag{3.30}$$

where $p_i,r_i,i=1,2$, are arbitrary parameters.

It follows from Theorem 1 that if we assign positive rational values to the parameters $p_i,q_i,i=1,2$, such that $r_2>r_1$ and $p_2>p_1(r_1+r_2)/(r_2-r_1)$, the formulae (3.30) will always generate rational convex quadrilaterals with two equal sides. As a numerical example, on taking $(p_1,p_2,r_1,r_2)=(1,3,1,3)$, we get a rational convex quadrilateral whose sides, diagonals and the area, after appropriate scaling, are given by $(a,b,c,d,e,f,A)=(25,25,30,14,40,25,468)$.

4. Concluding remarks

In this paper we described a method of generating all quadrilaterals whose sides, diagonals and the area are given by rational numbers. This method is different from the classic construction of rational quadrilaterals given by Kummer in 1848. We were able to obtain a complete parametrization of all rational cyclic quadrilaterals. For general rational quadrilaterals, we obtained several parametrizations in terms of quartic polynomials. When two sides of the quadrilateral are equal, we obtained a parametrization in terms of quadratic polynomials. The formulae for rational quadrilaterals obtained in this paper are simpler than the known parametrizations of such quadrilaterals.

We conclude the paper with an open problem concerning rational quadrilaterals. We first note that it readily follows from the triangle inequality that, in any convex quadrilateral, the sum of any three sides is greater than the remaining side. Further, if four rational numbers a, b, c, d are given such that $a \leq b \leq c \leq d$ and $a+b+c > d$, then $d-c < a+b$ and we can always find a rational number e such $\max(b-a, d-c) < e < a+b \leq c+d$. Both the triads of numbers a, b, e and c, d, e satisfy the triangle inequalities, hence we can construct two triangles with rational sides a, b, e and c, d, e such that we have a quadrilateral whose sides have lengths a, b, c, d and which has a diagonal of length e. However, there seems to be no simple way of ensuring that the second diagonal or the area of the quadrilateral is also rational. We thus have the following open question:

Problem: Is it possible to construct a rational quadrilateral whose sides have lengths given by four arbitrary rational numbers such that the sum of any three of them is greater than the fourth?

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