AN EXTENSION OF BREMNER AND MACLEOD'S THEOREM

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ABSTRACT. Bremner and Macleod [An unusual cubic representation problem, *Ann. Math. Inform.* **43** (2014), 29-41] showed that for all odd positive integers n, the equation

$$n = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

has no solutions in the positive integers. We extend this theorem to the equation

$$2na^2b^2c^2 - a^2 - b^2 - c^2 = \frac{a^2x}{y+z} + \frac{b^2y}{z+x} + \frac{c^2z}{x+y}, \tag{1}$$

where $a, b, c \in \mathbb{Z} - \{0\}$ and $n, x, y, z \in \mathbb{Z}^+$. Furthermore, we show that the insolubility (1) (under some conditions on a, b, c, n) can be explained by a Brauer-Manin obstruction for weak approximation for an elliptic curve model of the defining equation.

1. Introduction

The following remarkable theorem was proved by Bremner and Macleod in [1].

Theorem 1.1. *Let n be an positive odd integer. Then the equation*

$$n = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \tag{2}$$

has no solutions in the positive integers.

The size of positive integer solutions to (2) for small even values of n could be large, see [1, Table 2]. The goal of this paper is to extend Theorem 1.1.

Theorem 1.2. *Let a, b, c be nonzero integers such that $-(a+b+c)$ and abc are square numbers with $2 \nmid a+b+c$ and $\gcd(abc, a+b+c) = 1$. Then for all positive integers n corpime to $a+b+c$, the equation*

$$2na^2b^2c^2 - a^2 - b^2 - c^2 = \frac{a^2x}{y+z} + \frac{b^2y}{z+x} + \frac{c^2z}{x+y} \tag{3}$$

has no solutions in the positive integers. Furthermore, the insolubility of (3) is explained by a Brauer-Manin obstruction to weak approximation for a certain elliptic curve.

Theorem 1.1 is a special case of Theorem 1.2 when $|a| = |b| = |c| = 1$.

2. Preliminaries

2.1. **The Brauer-Manin obstruction.** This section follows Colliot-Thélène and Skorobogatov [4, Chapter 13], see also Poonen [7, Chapter 8]. Let k be a number field, let Ω be the set of all places of k, and let $\mathbb{A}_k$ be the adèle ring of k. Let X be a proper, smooth, geometrically irreducible variety over k. Let $\mathrm{Br}(X)$ be the Brauer group of X, that is the group of equivalence classes of Azumaya algebras over X. In 1970, Manin [6] introduced the Brauer-Manin paring

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

sending $(P_v) \in X(\mathbb{A}_k)$ and $\mathcal{A} \in \mathrm{Br}(X)$ to

$$\mathrm{ev}_{\mathcal{A}}((P_v)) = \sum_{v\in\Omega} \mathrm{inv}_v(\mathcal{A}(P_v)) \in \mathbb{Q}/\mathbb{Z},$$

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23 Jan 2023 00:00:18 PST
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where for each valuation v and each Azumaya algebra $\mathcal{A}$ in $\mathrm{Br}(X)$, $\mathrm{inv}_v\colon \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ is the local invariant map from class field theory and $\mathcal{A}(P_v)$ is defined as follows. A point $P_v \in X(k_v)$ gives a map $\mathrm{Spec}(k_v) \to X$, and hence induces a pullback map $\mathrm{Br}(X) \to \mathrm{Br}(k_v)$. We write $\mathcal{A}(P_v)$ for the image of $\mathcal{A}$ under this map. The Brauer set of $X(\mathbb{A}_k)^{\mathrm{Br}}$ is given

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \{(P_v) \in X(\mathbb{A}_k) \text{ such that } \mathrm{ev}_{\mathcal{A}}((P_v)) = 0 \text{ for all } \mathcal{A} \in \mathrm{Br}(X)\}\,.$$

The following theorem is due to Manin, see [6].

Theorem 2.1. *Let k be a number field and let X be a variety over k. The Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$ contains the closure of the image of the diagonal map $X(k) \to X(\mathbb{A}_k)$.*

Assume that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$, then one says there is a Brauer-Manin obstruction to weak approximation for X. Our model problem is that for a given variety X over k, we would like to show $X(k)_{\mathcal{P}} = \emptyset$, where $X(k)_{\mathcal{P}}$ is the set of all points in $X(k)$ having property $\mathcal{P}$. The guiding principle is to construct an Azumaya algebra $\mathcal{A} \in \mathrm{Br}(X)$ such that

$$\mathrm{ev}_{\mathcal{A}}((P_v)) \neq 0 \text{ for all } P \in X(k)_{\mathcal{P}},$$

where $(P_v) \in X(\mathbb{A}_k) = \prod_{v\in\Omega} X(k_v)$ is the image of $P \in X(k)$ under the diagonal map.

2.2. **The local Hilbert symbol.** This section follows Cohen [3, Section 5.2]. Let p be a prime number. For a p-adic number $a \neq 0$, let $v_p(a)$ denote the p-adic valuation of a; that is, the exponent of the highest power of the prime number p dividing a. Let $k = \mathbb{Q}_p$ or $k = \mathbb{R}$. For a and b in k^*, the local Hilbert symbol $(a,b)_p$ is defined by

$$(a,b)_p = \begin{cases} 1 \text{ if } ax^2 + by^2 = z^2 \text{ has a point in } \mathbb{P}^2(k), \\ -1 \text{ otherwise.} \end{cases}$$

Then

- For $a,\, b,\, c \in \mathbb{Q}_p^*$,

$$(a,\, b^2)_p = 1,$$
$$(a, bc)_p = (a,b)_p(a,c)_p.$$

- For $a = p^\alpha u$, $b = p^\beta v$, where $\alpha = v_p(a)$ and $\beta = v_p(b)$,

$$(a,b)_p = (-1)^{\alpha\beta(p-1)/2}\left(\frac{u}{p}\right)^{\beta}\left(\frac{v}{p}\right)^{\alpha} \text{ if } p \neq 2,$$
$$(a,b)_p = (-1)^{(u-1)(v-1)/4+\alpha(v^2-1)/8+\beta(u^2-1)/8} \text{ if } p = 2,$$

  where $\left(\frac{u}{p}\right)$ denotes the Legendre symbol.

Let $\mathbb{Z}^2 = \{x^2 \colon x \in \mathbb{Z}\}$, $\mathbb{Z}_p^2 = \{x^2 \colon x \in \mathbb{Z}_p\}$, $\mathbb{Q}_p^2 = \{x^2 \colon x \in \mathbb{Q}_p\}$, $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \colon v_p(x) = 0\}$.

3. Proof of Theorem 1.2

Assume that there exist positive integers x_0, y_0, z_0 satisfying (3). Then $[x_0 : y_0 : z_0]$ is a point on the projective cubic curve $\mathcal{F}$ defined by

$$(2na^2b^2c^2-a^2-b^2-c^2)(x+y)(y+z)(z+x)-a^2x(x+y)(x+z)-b^2y(y+z)(y+x)-c^2z(z+x)(z+y)=0.$$

A Weierstrass form is

$$\mathcal{E}\colon y^2 = x(x^2 + Ax + B), \tag{4}$$

where

$$A = 16n^2a^4b^4c^4 - 8na^2b^2c^2(a^2+b^2+c^2) + a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2,$$
$$B = 64na^4b^4c^4.$$

A map ϕ from $\mathcal{F}$ to $\mathcal{E}$ is given by

$$\phi(x:y:z) = (u:v:1),$$

where

$$\begin{cases} u = -\dfrac{(x+y)(64n^2a^6b^2c^4 + 2na^6c^2 - 48na^4b^2c^2 + 8a^2b^2 - a^2c^2) + (y+z)c^4(2na^2c^2-1)}{(x+y)(4na^2c^2-1)}, \\ v = \dfrac{8c^4(2a^4 + 2na^2c^2 - 1)^2(a^2(x+y)(2na^2c^2-1) - 2na^2c^4(x-y) + c^2x)}{(x+y)(4na^2c^2-1)^3}. \end{cases} \tag{5}$$

Note that

$$A^2 - 4B = DEFG,$$

where

$$D = (a+b+c)^2 - 4na^2b^2c^2,\ E = (a-b+c)^2 - 4na^2b^2c^2,$$
$$F = (a+b-c)^2 - 4na^2b^2c^2,\ G = (-a+b+c)^2 - 4na^2b^2c^2.$$

The Magma code verifying the map ϕ and the factorization of $A^2 - 4B$ is available at `https://www.overleaf.com/read/wwmkcknjfkbv`.

Lemma 3.1. *$D < 0$, $E < 0$, $F < 0$, and $G < 0$.*

Proof. We show that $D < 0$. The cases $E < 0$, $F < 0$, and $G < 0$ are treated similarly. Without loss of generality, we assume $|a| = \max\{|a|, |b|, |c|\}$.

Case 1: $|bc| > 1$. Then

$$(a+b+c)^2 \le (|a|+|b|+|c|)^2 \le 9|a|^2 < 4na^2b^2c^2.$$

Hence $D < 0$.

Case 2: $|bc| = 1$. Then $(b,c) = (1,1), (1,-1), (-1,1), (-1,-1)$. But $(b,c) = (1,1)$ is impossible due to the condition that $-(a+b+c)$ and abc are both perfect squares.

If $(b,c) = (1,-1), (-1,1)$, then

$$D = a^2 - 4na^2 < 0.$$

If $(b,c) = (-1,-1)$, since $a+b+c < 0$ and $abc > 0$, $0 < a < 2$. Hence $a = 1$. Therefore

$$D = 1 - 4n < 0.$$

□

By Lemma 3.1,

$$A^2 - 4B = DEFG > 0.$$

Therefore $\mathcal{E}$ is an elliptic curve and the set $\mathcal{E}(\mathbb{R})$ has two components: the bounded component with $x < 0$ and the unbounded component with $x \ge 0$. A remarkable property of the curve $\mathcal{E}$ is that it has no rational points on the bounded component $x < 0$. This surprising property also holds for many other curves, see [1, 2, 5, 8, 9, 10].

Theorem 3.2. *Let S be the set of points $(x,y) \in \mathcal{E}(\mathbb{Q})$ with $x < 0$. Then S is empty. Furthermore, the emptiness of S is explained by a Brauer-Manin obstruction to weak approximation for $\mathcal{E}$.*

We need some lemmas.

Lemma 3.3. *Let $\mathbb{Q}(\mathcal{E})$ be the function field of $\mathcal{E}$. Let $d \in \mathbb{Q}^*$. Let $\mathcal{A}$ be the class of quaternion algebras in $\mathrm{Br}(\mathbb{Q}(\mathcal{E}))$ defined by*

$$\mathcal{A} = (x, d).$$

Then $\mathcal{A}$ is an Azumaya algebra of $\mathcal{E}$; that is, $\mathcal{A}$ belongs to the subgroup $\mathrm{Br}(\mathcal{E})$ of $\mathrm{Br}(\mathbb{Q}(\mathcal{E}))$. Furthermore, the quaternion algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, where

$$\mathcal{B} = (x^2 + Ax + B, d),\ \mathcal{C} = (\frac{x^2 + Ax + B}{x^2}, d)$$

all represent the same class in $\mathrm{Br}(\mathbb{Q}(\mathcal{E}))$.

23 Jan 2023 00:00:18 PST
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Proof. Since $\mathcal{A}+\mathcal{B}=(y^2,d)$, $\mathcal{A}=\mathcal{B}$. Since $\mathcal{B}-\mathcal{C}=(x^2,d)$, $\mathcal{B}=\mathcal{C}$. Hence $\mathcal{A}=\mathcal{B}=\mathcal{C}$. Let U_1,U_2,U_3 be the maximal open subsets of $\mathcal{E}$ where x, x^2+Ax+B, and $(x^2+Ax+B)/x^2$ have neither zeroes nor poles respectively. Then $\mathcal{A}\in \mathrm{Br}(U_1)$, $\mathcal{B}\in \mathrm{Br}(U_2)$, and $\mathcal{C}\in \mathrm{Br}(U_3)$. We just need to show that

$$\mathcal{E}=U_1\cup U_2\cup U_3. \tag{6}$$

Since $U_1=\mathcal{E}-\{(0,0),\infty\}$, $U_2=\mathcal{E}-\{(\alpha_1,0),(\alpha_2,0),\infty\}$, where α_1 and α_2 are roots of $x^2+Ax+B=0$,

$$U_1\cup U_2=\mathcal{E}-\{\infty\}. \tag{7}$$

However, at ∞ then $x^{-1}(\infty)=0$. Therefore

$$\frac{x^2+Ax+B}{x^2}(\infty)=(1+\frac{A}{x}+\frac{B}{x^2})(\infty)=1\neq 0.$$

Thus

$$\infty\in U_3. \tag{8}$$

Then (6) follows from (7) and (8). □

Let $\mathcal{M}$ be the class of quaternion algebras in $\mathrm{Br}(\mathbb{Q}(\mathcal{E}))$ given by $\mathcal{M}=(x,D)$. By Lemma 3.1, $D\neq 0$. By Lemma 3.3, $\mathcal{M}$ belongs to the Brauer group $\mathrm{Br}(\mathcal{E})$.

Fix $P=(x,y)\in S$. Then $x<0$ and

$$y^2=x(x^2+Ax+B). \tag{9}$$

Lemma 3.4. *Let p be an odd prime. Then*

$$\mathrm{inv}_p(\mathcal{M}(P_p))=0.$$

Proof. It is enough to show that $(x,D)_p=1$.

Case 1: $v_p(x)<0$. Let $x=x_1/p^r$, where $r\in\mathbb{Z}^+$ and $v_p(x_1)=0$. From (9),

$$y^2=\frac{x_1(x_1^2+p^rAx_1+Bp^{2r})}{p^{3r}}. \tag{10}$$

Therefore $v_p(y^2)=-3r$. Thus $2|r$. From (10),

$$(p^{3r/2}y)^2=x_1(x_1^2+p^rAx_1+Bp^{2r}). \tag{11}$$

Reducing (11) modulo p gives $x_1\equiv$ square $\pmod p$. Therefore $x_1\in\mathbb{Z}_p^2$. Hence $x=x_1/p^r\in\mathbb{Q}_p^2$. Thus $(x,D)_p=1$.

Case 2: $v_p(x)=0$.

Case 2.1: $p\nmid D$. Since x and D are units in $\mathbb{Z}_p$, $(x,D)_p=1$

Case 2.2: $p|D$. Since $D|A^2-4B$, $p|A^2-4B$. Therefore

$$\begin{aligned} x^2+Ax+B &= (x+\frac{A}{2})^2+\frac{4B-A^2}{4} \\ &\equiv (x+\frac{A}{2})^2 \pmod p. \end{aligned} \tag{12}$$

- $p\nmid x+A/2$. From (12), $x^2+Ax+B\in\mathbb{Z}_p^2$. Therefore

$$x=\frac{y^2}{x^2+Ax+B}\in\mathbb{Q}_p^2.$$

Thus $(x,D)_p=1$.

- $p|x+A/2$. Then

$$x\equiv -\frac{A}{2} \pmod p. \tag{13}$$

Since $p|D=(a+b+c)^2-4na^2b^2c^2$,

$$4na^2b^2c^2\equiv (a+b+c)^2 \pmod p.$$

Therefore

$$
\begin{aligned}
A &= 16n^2a^4b^4c^4 - 8na^2b^2c^2(a^2+b^2+c^2) + a^4+b^4+c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\
&= (4na^2b^2c^2 - a^2 - b^2 - c^2)^2 - 4(a^2b^2+b^2c^2+c^2a^2) \\
&\equiv ((a+b+c)^2 - a^2 - b^2 - c^2)^2 - 4(a^2b^2+b^2c^2+c^2a^2) \pmod p \qquad (14)\\
&\equiv (2(ab+bc+ca))^2 - 4(a^2b^2+b^2c^2+c^2a^2) \pmod p \\
&\equiv 8abc(a+b+c) \pmod p.
\end{aligned}
$$

From (13) and (14),

$$x \equiv -4abc(a+b+c) \pmod p.$$

Since $-abc(a+b+c) \in \mathbb{Z}^2$ and $p \nmid x$, $x \in \mathbb{Z}_p^2$. Therefore $(x, D)_p = 1$.

Case 3: $v_p(x) > 0$. Let $x = p^r x_1$, where $r \in \mathbb{Z}^+$ and $v_p(x_1) = 0$. From (9),

$$y^2 = p^r x_1(p^{2r}x_1^2 + p^r A x_1 + B). \qquad (15)$$

Case 3.1: $p|B$. Then $p|nabc$. Therefore

$$D = (a+b+c)^2 - 4na^2b^2c^2 \equiv (a+b+c)^2 \pmod p. \qquad (16)$$

Since $\gcd(nabc, a+b+c) = 1$ and $p|nacb$, $p \nmid a+b+c$. Then (16) shows that $D \in \mathbb{Z}_p^2$. Thus $(x, D)_p = 1$.

Case 3.2: $p \nmid B$. From (15), $v_2(y^2) = r$. Thus $2|r$.

• $p \nmid D$. Then

$$
\begin{aligned}
(x, D)_p &= (p^r x_1, D)_p \\
&= (x_1, D)_p \quad (\text{since } 2|r) \\
&= 1 \quad (\text{since } x_1, D \in \mathbb{Z}_p^\times).
\end{aligned}
$$

• $p|D$. Then $4na^2b^2c^2 \equiv (a+b+c)^2 \pmod p$. Since $\gcd(nabc, a+b+c) = 1$ and $2 \nmid a+b+c$, $p \nmid abc(a+b+c)$. Therefore

$$
\begin{aligned}
x^2 + Ax + B &\equiv B \pmod p \\
&\equiv 64na^4b^4c^4 \pmod p \\
&\equiv 16a^2b^2c^2(a+b+c)^2 \pmod p \\
&\not\equiv 0 \pmod p.
\end{aligned}
$$

Hence $x^2 + Ax + B \in \mathbb{Z}_p^2$. Thus

$$x = \frac{y^2}{x^2 + Ax + B} \in \mathbb{Q}_p^2.$$

Therefore $(x, D)_p = 1$.

□

Lemma 3.5.

$$\mathrm{inv}_2(\mathcal{M}(P_2)) = 0.$$

Proof. If $2|nabc$ then

$$D \equiv (a+b+c)^2 - 4na^2b^2c^2 \equiv 1 \pmod 8.$$

Hence $D \in \mathbb{Z}_2^2$. Therefore $(x, D)_2 = 1$.

We consider the case $2 \nmid nabc$. Then $2 \nmid n$ and $2 \nmid abc$.

Case 1: $2|v_2(x)$. Let $x = 2^r x_1$, where $2|r$ and $v_2(x_1) = 0$. Then

$$
\begin{aligned}
(x, D)_2 &= (2^r x_1, D)_2 \\
&= (x_1, D)_2 \quad (\text{since } 2|r) \\
&= (-1)^{(x_1-1)(D-1)/4} \\
&= 1 \quad (\text{since } 4|D-1).
\end{aligned}
$$

Case 2: $2 \nmid v_2(x)$.
Case 2.1: $v_2(x) < 0$. Let $x = x_1/2^r$, where $r \in \mathbb{Z}^+$ and $v_2(x_1) = 0$.
From (9),

$$y^2 = \frac{x_1(x_1^2 + 2^r As + 2^{2r}B)}{2^{3r}}.$$

Therefore $v_2(y^2) = 3r$, which impossible since $2 \nmid r$.
Case 2.2: $v_2(x) > 0$. Let $x = 2^r x_1$, where $r \in \mathbb{Z}^+$ and $v_2(x_1) = 0$. From (9),

$$y^2 = 2^r x_1(2^{2r}x_1^2 + 2^r Ax_1 + 2^6 na^4b^4c^4). \tag{17}$$

Since $2 \nmid abc$,

$$\begin{aligned} A &= 16n^2a^4b^4c^4 - 8a^2b^2c^2(a^2+b^2+c^2) + a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\ &\equiv 5 \pmod 8. \end{aligned}$$

Thus $v_2(A) = 0$.
• $r > 6$. From (17),

$$y^2 = 2^{r+6}x_1(2^{2r-6}x_1^2 + 2^{r-6}Ax_1 + na^4b^4c^4).$$

Therefore $v_2(y^2) = r + 6$, which is impossible since $2 \nmid r$.
•$r < 6$. From (17),

$$y^2 = 2^{2r}x_1(2^r x_1^2 + Ax_1 + 2^{6-r}na^4b^4c^4). \tag{18}$$

Thus $v_2(y) = r$. From (18),

$$(2^{-r}y)^2 = x_1(2^r x_1^2 + Ax_1 + 2^{6-r}na^4b^4c^4). \tag{19}$$

Note that in (19) we have $A \equiv 5 \pmod 8$, $2 \nmid r$, $0 < r < 6$, $2 \nmid nabc$.

(i) $r = 1$. Reducing (19) modulo 4 gives

$$1 \equiv x_1(2 + x_1) \equiv 2x_1 + 1 \pmod 4.$$

which is impossible since $2 \nmid x_1$.

(ii) $r = 3$. Reducing (19) modulo 8 gives

$$1 \equiv 5 \pmod 8,$$

which is impossible.

(iii) $r = 5$. Reducing (19) modulo 4 gives

$$1 \equiv x_1(x_1 + 2) \equiv 1 + 2x_1 \pmod 4,$$

which is impossible since $2 \nmid x_1$.

□

Lemma 3.6.

$$\mathrm{inv}_\infty(\mathcal{M}(P_\infty)) = \frac{1}{2}.$$

Proof. Since $D < 0$ (proved in Lemma 3.1) and $x < 0$, $(x, D)_\infty = -1$. Therefore $\mathrm{inv}_\infty(\mathcal{M}(P_\infty)) = 1/2$. □

We are now ready to prove Theorem 3.2.

Proof. Lemmas 3.4, 3.5, and 3.6 show that for all $P \in S$,

$$\mathrm{inv}_v(\mathcal{M}(P_v)) = \begin{cases} 0 & \text{if } v \neq \infty, \\ \frac{1}{2} & \text{if } v = \infty. \end{cases}$$

Therefore

$$\mathrm{ev}_{\mathcal{M}}((P_v)) = \mathrm{inv}_\infty(\mathcal{M}(P_\infty)) + \sum_{p<\infty} \mathrm{inv}_p(\mathcal{M}(P_p)) = \frac{1}{2} \quad \forall P \in S. \tag{20}$$

On the other hand, by Theorem 2.1, $\mathcal{E}(\mathbb{Q}) \subset \mathcal{E}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$. In particular, $S \subset \mathcal{E}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$. Hence

$$\mathrm{ev}_{\mathcal{M}}((P_v)) = 0 \quad \forall P \in S. \tag{21}$$

It follows from (20) and (21) that $S = \emptyset$. The proof is complete. □

Theorem 1.2 is now a consequence of Theorem 3.2. Let $(u_0 : v_0 : 1) = \phi(x_0 : y_0 : z_0)$. By (5),

$$u_0 = -\frac{(x_0 + y_0)(64n^2a^6b^2c^4 + 2na^6c^2 - 48na^4b^2c^2 + 8a^2b^2 - a^2c^2) + (y_0 + z_0)c^4(2na^2c^2 - 1)}{(x_0 + y_0)(4na^2c^2 - 1)}.$$

Since $n, a^2, b^2, c^2, x_0, y_0, z_0 \in \mathbb{Z}^+$,

$$64n^2a^6b^2c^4 > 48na^4b^2c^2 + 8a^2b^2,\ 2na^6c^2 > a^2c^2,\ 2na^2c^2 > 1.$$

Therefore $u_0 < 0$. Hence $(u_0, v_0) \in S$, which is impossible since S is empty. Thus equation (3) has no solutions in the positive integers.

Remark 3.7. *The method in this paper allows one to study the results in* [1, 2, 5, 8, 9, 10] *within the Brauer-Manin obstruction framework. A major part in* [1, 2, 5, 8, 9, 10] *is to show the nonexistence of rational points on the bounded component $x < 0$ on certain elliptic curves $\mathcal{E}$ of the form*

$$y^2 = x(x^2 + Ax + B)$$

with $A, B \in \mathbb{Q}$. Then Lemma 3.3 is used to construct an Azumaya algebra $\mathcal{M}$ in $\mathrm{Br}(\mathbb{Q}(\mathcal{E}))$ such that for all $P = (x, y) \in \mathcal{E}(\mathbb{Q})$ with $x < 0$,

$$\mathrm{inv}_v(\mathcal{M}(P_v)) = \begin{cases} 0 & \text{if } v \neq \infty, \\ \frac{1}{2} & \text{if } v = \infty, \end{cases} \tag{22}$$

where (P_v) is the image of P in $\mathcal{E}(\mathbb{A}_{\mathbb{Q}})$. And the rest follows exactly as in the proof of Theorem 1.2.

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