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HARTMAN–WINTNER INEQUALITY FOR A CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEM

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ABSTRACT. In this work we derive a Hartman–Wintner type inequality for a fractional boundary value problem depending on the Caputo derivative and with Dirichlet boundary conditions. We explicitly show how this inequality strenghtens previously known results in the literature, in particular the Lyapunov inequality.

1. Introduction

In the theory of second order differential equations, the Hartman–Wintner inequality may be stated as follows (cf. [7]):

Theorem 1.1. *Suppose that $q \in C[a,b]$. If the boundary value problem,*

$$\begin{aligned} y''(t) &= -q(t)y(t), \quad a<t<b, \\ y(a) &= 0,\ y(b)=0, \end{aligned}$$

has a nontrivial solution $y \in C^2[c,b]$, then

$$\int_a^b (s-a)(b-s)|q(s)|ds > b-a. \tag{1}$$

The inequality (1) generalizes the classical Lyapunov inequality, i.e.,

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{2}$$

Recently, boundary value problems depending on fractional derivatives were studied and generalizations of (2) and (1) were obtained in [2, 3, 4] and [8], respectively. We note in passing that several *fractional* Lyapunov-type inequalities were derived for several different boundary value problems (cf. [1, 11, 12, 13, 14] and the references therein). However, we could not find in the literature a generalization of the Hartman–Wintner inequality (1) when the differential equation depends on the Caputo fractional derivative (see (3)–(4) below). That turned out to be our main motivation to complete this work. In the meantime we found some mistakes in a paper dealing with such problems, that we will expose in the next section.

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We trust that the reader is familiar with the fractional calculus concepts. Nevertheless, we refer the reader to the book [10] in order to get acquainted with the theory.

In the next section we present and prove the main results of this manuscript.

2. Main results

Consider the following fractional boundary value problem (FBVP).

$$ {}^{C}D_{a}^{\alpha}y(t) = -q(t)y(t), \quad a < t < b,\ 1 < \alpha \leq 2, \tag{3} $$

$$ y(a) = 0,\ y(b) = 0. \tag{4} $$

The Green function for the FBVP (3)–(4) is defined by

$$ G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)(b-s)^{\alpha-1}}{b-a}, & a \leq t \leq s \leq b. \end{cases} $$

It means that, for $y \in C^1[a,b]$, (3)–(4) may be written equivalently as (cf. [3])

$$ y(t) = \int_a^b G(t,s)q(s)y(s)ds. \tag{5} $$

Now, let us define two auxiliary functions,

$$ x(s) = \frac{s-a}{b-a}(b-s)^{\alpha-1}, \quad s \in [a,b], \tag{6} $$

and

$$ z(s) = \begin{cases} \frac{2-\alpha}{[(\alpha-1)(b-a)]^{\frac{\alpha-1}{\alpha-2}}}(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}} - \frac{s-a}{b-a}(b-s)^{\alpha-1}, & s \in [a, b-(\alpha-1)(b-a)], \\ 0, & s \in [b-(\alpha-1)(b-a), b]. \end{cases} \tag{7} $$

From [3, Lemma 2] we know that x and z are nonnegative functions, x is strictly increasing and z is strictly decreasing on $(a, b-(\alpha-1)(b-a))$. Since $z(a) - x(a) > 0$ and $z(b-(\alpha-1)(b-a)) - x(b-(\alpha-1)(b-a)) < 0$, then there is a unique $\bar{s} \in (a, b-(\alpha-1)(b-a))$ such that

$$ z(s) > x(s) \text{ on } (a,\bar{s}) \text{ and } z(s) < x(s) \text{ on } (\bar{s}, b-(\alpha-1)(b-a)). \tag{8} $$

It follows the Hartman–Wintner type inequality for the FBVP (3)–(4), which is the main result of this work.

Theorem 2.1. *Suppose that $q \in C[a,b]$ and $1 < \alpha < 2$. If the boundary value problem (3)–(4) has a nontrivial solution $y \in C^1[a,b]$, then*

$$ \int_a^{\hat{s}} z(s)|q(s)|ds + \int_{\hat{s}}^b x(s)|q(s)|ds > \Gamma(\alpha), \tag{9} $$

where x and z are given by (6) and (7) and $\hat{s}$ is the unique solution of $x(s) = z(s)$ on $(a, b-(\alpha-1)(b-a))$.

Proof. By (5), we have

$$\frac{\Gamma(\alpha)|y(t)|}{\|y\|} \leq \int_a^t \left| \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1} \right| |q(s)| ds + \int_t^b \frac{(t-a)(b-s)^{\alpha-1}}{b-a} |q(s)| ds.$$

By [3, Lemma 2] we know that,

$$\max_{t\in[s,b]} \left| \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1} \right| < \max\{x(s), z(s)\}, \quad s \in (a,b),$$

whence,

$$\frac{\Gamma(\alpha)|y(t)|}{\|y\|} < \int_a^t \max\{x(s), z(s)\} |q(s)| ds + \int_t^b \frac{(s-a)(b-s)^{\alpha-1}}{b-a} |q(s)| ds := F(t).$$

Now observe that $F'(t) = (\max\{x(t), z(t)\} - x(t))|q(t)|$, which is nonnegative on (a,b). Therefore, $\max_{t\in[a,b]} F(t) = F(b)$, and

$$\Gamma(\alpha) < \int_a^b \max\{x(s), z(s)\} |q(s)| ds,$$

from which the desired result follows. □

Remark 2.2. *In* [9, Theorem 3.1] *the authors present a Hartman–Wintner type inequality for a certain sequential fractional boundary value problem. In order to obtain it, the authors consider the Green function,*

$$G(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} \frac{(t-a)^{\beta}(b-s)^{\alpha+\beta-1}}{(b-a)^{\beta}} - (t-s)^{\alpha+\beta-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\beta}(b-s)^{\alpha+\beta-1}}{(b-a)^{\beta}}, & a \leq t \leq s \leq b, \end{cases}$$

with $0 < \alpha, \beta \leq 1$ *and* $1 < \alpha + \beta \leq 2$*. They write (without a proof)* $G(t,s) \leq G(s,s)$ *for* $t,s \in [a,b]$ *(cf.* [9, ineq. (23)]*), and, three lines after, they use*

$$|G(t,s)| \leq G(s,s), \tag{10}$$

which would be true if G *was a nonnegative function. However, the nonnegativity does not hold in general. Indeed, with* $\beta = 1$ *and* $0 < \alpha < 1$*, it was explicitly observed in* [6] *that* G *is not a nonnegative function. In any case, it is easily seen that* (10) *does not hold in general; indeed, consider* $a = 0$*,* $b = 1$*,* $\alpha = 1/2$ *and* $\beta = 1$*. Then,*

$$|G(1/2, 1/10)| = \frac{2}{\sqrt{\pi}} \left[\frac{1}{2} \left(1 - \frac{1}{10}\right)^{\frac{1}{2}} - \left(\frac{2}{5}\right)^{\frac{1}{2}} \right] = \frac{2}{\sqrt{\pi}} \left(\sqrt{\frac{2}{5}} - \frac{3}{2\sqrt{10}} \right),$$

and

$$G(1/10, 1/10) = \frac{2}{\sqrt{\pi}} \frac{1}{10} \left(1 - \frac{1}{10}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \frac{3}{10\sqrt{10}},$$

from which we easily conclude that $|G(1/2, 1/10)| > G(1/10, 1/10)$*.*

The Green function defined above was considered for the first time in [4] *and with it, it was possible to deduce a Lyapunov-type inequality for a certain sequential fractional boundary value problem (cf.*

[4] *for the details). We have shown here that the Hartman–Wintner inequality given in* [9, ineq. (21)] *is not justified. With* $\beta = 1$*, that inequality follows from Theorem* 2.1*, but with* $\beta < 1$ *what such an inequality should be still eludes us.*

Corollary 2.2.1 (Fractional Lyapunov's inequality)**.** *Under the conditions of Theorem* 2.1 *it holds the following inequality:*

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\frac{\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{11}$$

Proof. In [3] it was shown that both the functions x and z are bounded above by $\frac{[(\alpha-1)(b-a)]^{\alpha-1}}{\alpha^\alpha}$. Hence the result follows by Theorem 2.1. □

Remark 2.3. *The Lyapunov inequality* (2) *follows from the previous result by letting* $\alpha \to 2$.

One of the features of Lyapunov or Hartman-Wintner type inequalities is that they provide lower bounds for the eigenvalues of the respective boundary value problems. Then, it is possible to obtain intervals where certain Mittag-Leffler functions possess no real zeros (see, e.g., [1, 2, 3]). So, let us now consider the inequality (9) with $q(t) = \lambda \in \mathbb{R}$. We get,

$$|\lambda|\left[\int_a^{\hat{s}} z(s)ds + \int_{\hat{s}}^b x(s)ds\right] > \Gamma(\alpha). \tag{12}$$

In general it is impossible to determine explicitly $\hat{s}$. Nevertheless, observe that

$$\begin{aligned} x(s) = z(s) &\Longleftrightarrow 2\frac{s-a}{b-a}(b-s)^{\alpha-1} = \frac{2-\alpha}{[(\alpha-1)(b-a)]^{\frac{\alpha-1}{\alpha-2}}}(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}}, \\ &\Longleftrightarrow 2\frac{s-a}{b-a}(b-s)^{\frac{1-\alpha}{\alpha-2}} = \frac{2-\alpha}{[(\alpha-1)(b-a)]^{\frac{\alpha-1}{\alpha-2}}}. \end{aligned} \tag{13}$$

We see that we may at least obtain $\hat{s}$ analytically for when the left hand side of the previous equality is a polynomial of degree at most 4. For the sake of simplicity we now consider $\alpha = 3/2$. Then, from (13), it is not hard to conclude that

$$\hat{s} = \frac{a+b}{2} - \frac{(b-a)\sqrt{2}}{4}.$$

If we now let $a = 0$ and $b = 1$, then from (12) we find that

$$|\lambda| > \frac{\sqrt{\pi}}{2}\frac{1}{\frac{1}{120}\sqrt{482+329\sqrt{2}} + \frac{1}{120}(28 - \sqrt{482+89\sqrt{2}})} \approx \frac{\sqrt{\pi}}{2} \times 3.5167. \tag{14}$$

If we now use the same parameters in (11), we find that

$$|\lambda| > \frac{\sqrt{\pi}}{2}\frac{(\frac{3}{2})^{\frac{3}{2}}}{(\frac{1}{2})^{\frac{1}{2}}} \approx \frac{\sqrt{\pi}}{2} \times 2.5981. \tag{15}$$

As expected we see that (14) is better than (15). In particular, we may follow the same steps of the proof of [3, Theorem 2] to conclude that the Mittag-Leffler function $E_{3/2,2}(x)$ has no real zeros for

$$x \in \left[-\frac{\sqrt{\pi}}{2} \frac{1}{\frac{1}{120}\sqrt{482+329\sqrt{2}} + \frac{1}{120}(28-\sqrt{482+89\sqrt{2}})}, 0 \right).$$

Remark 2.4. *In [5] it was shown that $E_{\alpha,2}(x)$ has no real zeros for*

$$x \in [-\Gamma(\alpha)(1+\alpha), 0) \supset \left[-\Gamma(\alpha)\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}, 0 \right),$$

with $1 < \alpha < \bar{\alpha}$, where $\bar{\alpha}$ is such that $\frac{\bar{\alpha}^{\bar{\alpha}}}{(\bar{\alpha}-1)^{\bar{\alpha}-1}} = \bar{\alpha} + 1$. We wish to emphasize that $\bar{\alpha} \approx 1.447 < 3/2$. Therefore, we cannot use this result with $\alpha = 3/2$.

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