Models and homogeneity degree of hyperspaces of a simple closed curve

Alejandro Illanes and Verónica Martínez-de-la-Vega

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Abstract

Given a continuum X and $n \in \mathbb{N}$, let $C_n(X)$ (resp., $F_n(X)$) be the hyperspace of nonempty closed sets with at most n-components (resp., n-points). Let S^1 denote the unit circle in the plane. Given $1 \leq m \leq n$, we consider the quotient space $C_n(S^1)/F_m(S^1)$. The *homogeneity degree of* X, $\mathrm{hd}(X)$, is the number of orbits of the group of homeomorphisms of X. In this paper we discuss the known models for the hyperspaces of S^1, we construct a new model for a hyperspace of S^1 by proving that $C_2(S^1)/F_2(S^1)$ is homeomorphic to the topological suspension of a solid torus and we show that $\mathrm{hd}(C_2(X)/F_2(X)) = 3$, and $\mathrm{hd}(C_2(X)/F_1(X)) = 4$.

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1 Inroduction

A *continuum* is a compact connected metric space with more than one point. A *subcontinuum* of a continuum X is a nonempty compact connected subspace of X, so one-point sets are also subcontinua.

If $1 \leq m \leq n$, in this paper we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}, \text{ and the quotient space}$$
$$C_n(X)/F_m(X).$$

The hyperspace 2^X is considered with the Hausdorff metric [11, Theorem 2.2]. A mapping is a continuous function. Given a topological space Z, the *homogeneity degree* of Z, denoted by $\mathrm{hd}(Z)$, is the number of orbits of the group of homeomorphisms of Z. So Z is homogeneous when $\mathrm{hd}(Z) = 1$. Spaces Z for which $\mathrm{hd}(Z) = n$ ($n \in \mathbb{N}$) are also known as $\frac{1}{n}$-*homogeneous*.

A *finite graph* is a continuum which is a finite union of arcs such that the intersection of each two of them is a finite set.

As usual we denote the unit circle in the plane by S^1 and the unit disk by D^1. The *solid torus* is the space $D^1 \times S^1$.

The following are the known results about models and homogeneity degree of the hyperspaces of S^1.

A. Since for every locally connected continuum X, 2^X is homeomorphic to the Hilbert cube [11, Theorem 11.3], 2^{S^1} is homeomorphic to the Hilbert cube and $\mathrm{hd}(2^{S^1}) = 1$.

B. $C(S^1)$ is a 2-cell [10, p. 41], so $\mathrm{hd}(C(S^1)) = 2$.

C. $C_2(S^1)$ is homeomorphic to the cone over a solid torus [9], so $\mathrm{hd}(C_2(S^1)) = 3$.

D. $F_2(S^1)$ is homeomorphic to the Möebius strip [10, pp. 53 and 54], so $\mathrm{hd}(F_2(S^1)) = 2$.

E. $F_3(S^1)$ is homeomorphic to the 3-dimensional sphere [1], so $\mathrm{hd}(F_3(S^1)) = 1$.

F. If $n \geq 4$, then $\mathrm{hd}(F_n(S^1)) = n$ [5].

G. $\mathrm{hd}(C_2(S^1)/F_1(S^1)) > 2$ [15, Theorem 3.18].

H. $\mathrm{hd}(C_2(S^1)/F_2(S^1)) > 2$ [12, Theorem 4.3].

Models for hyperspaces can be very complicated. In [10] it was discussed almost all the possible models that have been constructed. In [9] it was shown that $C^2(S^1)$ is homeomorphic to the solid torus. As can be seen, the proof is complicated and it does not allow to see what is the result of making the identifications to obtain the spaces $C_2(S^1)/F_2(S^1)$ and $C_2(S^1)/F_1(S^1)$. The main results of this paper are:

-The construction of a model for $C_2(S^1)/F_2(S^1)$. We prove that this space is homeomorphic to the suspension of the solid torus. As a consequence, we obtain that $\mathrm{hd}(C_2(X)/F_2(X)) = 3$.

-The proof that $\mathrm{hd}(C_2(S^1)/F_1(S^1)) = 4$. This answers Question 3.19 of [15], where it was asked whether $\mathrm{hd}(C_2(S^1)/F_1(S^1)) = 3$.

With the results we are presenting in this paper, and some other known results, in [6] the following theorem is proved.

Theorem 1 *[6, Theorem 2] Let X be a finite graph and $1 \leq m \leq n$. Then*
(a) hd$(C_n(X)/F_m(X)) = 1$ if and only if X is homeomorphic to S^1 and $n = m = 1$,
(b) hd$(C_n(X)/F_m(X)) = 2$ if and only if X is an arc and either $n = m = 1$ or $n = 2$ and $m \in \{1, 2\}$,
(c) hd$(C_n(X)/F_m(X)) = 3$ if and only if X is homeomorphic to S^1 and $n = m = 2$, and
(d) hd$(C_n(X)/F_m(X)) = 4$ if and only if X is homeomorphic to S^1, $n = 2$ and $m = 1$.

The geometric ideas behind the arguments in this paper are similar as those used in [9]. However we make important adjustments to the formulas.

2 A model for the hyperspace $C_2(S^1)/F_2(S^1)$

In this section we show a model for the hyperspace $C_2(S^1)/F_2(S^1)$ by proving that this hyperspace is homeomorphic to the suspension over the solid torus.

Let $\mathcal{T}$ be the solid torus, let $\mathcal{C}^1$ be the cone over $\mathcal{T}$ and v its vertex. By the main result of [9], $C_2(S^1)$ is homeomorphic to $\mathcal{C}^1$. Given $z, w \in S^1$ and $A \subset S^1$, denote the complex product of w and z by $w \cdot z$ and $w \cdot A = \{w \cdot a : a \in A\}$.

We consider the *exponential mapping* $e : \mathbb{R} \to S^1$ given by

$$e(t) = (cos(t), sin(t)).$$

Given $\alpha, \beta \in \mathbb{R}$, we will use the following properties of the mapping e.
(1) $e(\alpha + \beta) = e(\alpha) \cdot e(\beta)$,
(2) $e(\alpha) = e(\beta)$ if and only if $\alpha = \beta + 2k\pi$ for some integer k,
(3) if $\alpha \leq \beta$ and $\beta - \alpha < 2\pi$, then the length of the arc $e([\alpha, \beta])$ is $\beta - \alpha$,
(4) if $\alpha \leq \beta$ and $\beta - \alpha < 2\pi$, then the middle points of the arcs joining $e(\alpha)$ and $e(\beta)$ are $e(\frac{\alpha+\beta}{2})$ and $e(\frac{\alpha+2\pi+\beta}{2})$,
(5) $-e(\alpha) = e(\alpha \pm \pi)$, and
(6) $e(t) \cdot A$ is the image of A under a translation, so length$(e(t) \cdot A) =$ length(A).

Define $\sigma : C_2(S^1) \to [0, 2\pi]$ by $\sigma(A) =$ length(A), if A is connected; and $\sigma(A) =$ length$(A_1)+$ length(A_2), where A_1 and A_2 are the components of A, if A is not connected. Observe that σ is continuous, $\sigma^{-1}(0) = F_2(S^1)$ and $\sigma^{-1}(2\pi) = \{S^1\}$.

Define

$$\mathcal{A} = \{A \in C_2(S^1) : \sigma(A) = \pi\}.$$

Theorem 2 *$\mathcal{A}$ is homeomorphic to the solid torus.*

Proof. Define

$$K = \{(r, \beta) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] : -\tfrac{1}{2}(|r| + \tfrac{\pi}{2}) \leq \beta \leq \tfrac{1}{2}(|r| + \tfrac{\pi}{2})\}.$$

Then $K = K^- \cup K^+$, where K^- is the convex quadrilateral in the plane with vertices $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(0, -\frac{\pi}{4})$ and $(0, \frac{\pi}{4})$ and K^+ is the convex quadrilateral with vertices $(0, -\frac{\pi}{4})$, $(0, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{2})$, and $(\frac{\pi}{2}, -\frac{\pi}{2})$.

Define $\varrho : K \to \mathcal{A}$ by

$$\varrho(r, \beta) = e([\beta(\tfrac{2r-\pi}{2r+\pi}) - \tfrac{3\pi}{4} + \tfrac{r}{2}, \beta - \tfrac{\pi}{4} + \tfrac{r}{2}] \cup [\beta + \tfrac{\pi}{4} - \tfrac{r}{2}, \beta(\tfrac{2r-\pi}{2r+\pi}) + \tfrac{3\pi}{4} - \tfrac{r}{2}]), \text{ if } r \in [0, \tfrac{\pi}{2}], \text{ and}$$
$$\varrho(r, \beta) = -\varrho(-r, -\beta), \text{ if } r \in [-\tfrac{\pi}{2}, 0].$$

We check that the following properties hold for each $(r, \beta) \in K$,

Property A. $\varrho(\frac{\pi}{2}, \beta) = e([-\frac{\pi}{2}, \frac{\pi}{2}])$, $\varrho(-\frac{\pi}{2}, \beta) = e([\frac{\pi}{2}, \frac{3\pi}{2}])$,

Property B. If $-\frac{\pi}{2} < r < \frac{\pi}{2}$, then $\varrho(r, \beta)$ is the union of two disjoint subcontinua of S^1. Moreover, the lengths of the components of $S^1 \setminus \varrho(r, \beta)$ are $\frac{\pi}{2} + r$ and $\frac{\pi}{2} - r$; if $r \geq 0$, then the respective middle points of these components are $e(\beta(\frac{2r-\pi}{2r+\pi}) + \pi)$ and $e(\beta)$; and if $r \leq 0$, then the middle points are $e(\beta(\frac{2r+\pi}{2r-\pi}))$ (with length $\frac{\pi}{2} - r$) and $e(-\beta + \pi)$ (with length $\frac{\pi}{2} + r$).

Property C. $\sigma(\varrho(r,\beta)) = \pi$,

Property D. ϱ is well defined and continuous,

Property E. If $-\frac{\pi}{2} < r < \frac{\pi}{2}$, then for every $-\frac{1}{2}(|r| + \frac{\pi}{2}) \leq \beta \leq \frac{1}{2}(|r| + \frac{\pi}{2})$, $\varrho^{-1}(\varrho(r,\beta)) = \{(r,\beta)\}$. Moreover, $\varrho^{-1}(e([-\frac{\pi}{2}, \frac{\pi}{2}])) = \{\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\varrho^{-1}(e([\frac{\pi}{2}, \frac{3\pi}{2}])) = \{-\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$.

First, we prove properties A, B, C and D. Property A is immediate. For proving B and C, first suppose that $r \geq 0$. The inequalities $\beta \geq -\frac{1}{2}(r + \frac{\pi}{2})$ and $\beta \leq \frac{1}{2}(r + \frac{\pi}{2})$ are equivalent to the respective inequalities: $\beta - \frac{\pi}{4} + \frac{r}{2} \geq \beta(\frac{2r-\pi}{2r+\pi}) - \frac{3\pi}{4} + \frac{r}{2}$ and $\beta(\frac{2r-\pi}{2r+\pi}) + \frac{3\pi}{4} - \frac{r}{2} \geq \beta + \frac{\pi}{4} - \frac{r}{2}$. Observe that $\beta + \frac{\pi}{4} - \frac{r}{2} - (\beta - \frac{\pi}{4} + \frac{r}{2}) = \frac{\pi}{2} - r \geq 0$ and the inequality holds if and only $\frac{\pi}{2} > r$. Moreover $\beta(\frac{2r-\pi}{2r+\pi}) + \frac{3\pi}{4} - \frac{r}{2} - (\beta(\frac{2r-\pi}{2r+\pi}) - \frac{3\pi}{4} + \frac{r}{2}) = \frac{3\pi}{2} - r < 2\pi$. Hence $\varrho(r,\beta)$ is the union of two nonempty subcontinua. Clearly, the sum of the lengths of these subcontinua is π. Furthermore, in the case that $r < \frac{\pi}{2}$ these subcontinua are disjoint and $\varrho(r,\beta)$ is the union of two disjoint subcontinua. Observe that the lengths of the components of $S^1 \setminus \varrho(r,\beta)$ are $\frac{\pi}{2} + r$ and $\frac{\pi}{2} - r$. The shortest (resp., largest) component of $S^1 \setminus \varrho(r,\beta)$ is the open subarc $e(\beta - \frac{\pi}{4} + \frac{r}{2}, \beta + \frac{\pi}{4} - \frac{r}{2})$ (resp., $e(\beta(\frac{2r-\pi}{2r+\pi}) + \frac{3\pi}{4} - \frac{r}{2}, \beta(\frac{2r-\pi}{2r+\pi}) + 2\pi - \frac{3\pi}{4} + \frac{r}{2})$). Thus, the respective middle points are $e(\beta)$ and $e(\beta(\frac{2r-\pi}{2r+\pi}) + \pi)$.

Now, suppose that $r \leq 0$. Then $\varrho(r,\beta) = -\varrho(-r,-\beta)$. Since $-r \geq 0$, by the previous paragraph, the lengths of the components of $S^1 \setminus \varrho(r,\beta) = S^1 \setminus -\varrho(-r,-\beta)$ are $\frac{\pi}{2} - r$ and $\frac{\pi}{2} + r$ with middle points $-e(-\beta) = e(-\beta + \pi)$ and $-e(-\beta(\frac{-2r-\pi}{-2r+\pi}) + \pi) = e(\beta(\frac{2r+\pi}{2r-\pi}))$.

For proving D, note that if $r = 0$, then with the first definition (when $r \geq 0$) we obtain $\varrho(r,\beta) = e([-\beta - \frac{3\pi}{4}, \beta - \frac{\pi}{4}] \cup [\beta + \frac{\pi}{4}, -\beta + \frac{3\pi}{4}])$. With the second one ($0 \leq r$), we obtain $\varrho(r,\beta) = -\varrho(-r,-\beta) = -e([\beta - \frac{3\pi}{4}, -\beta - \frac{\pi}{4}] \cup [-\beta + \frac{\pi}{4}, \beta + \frac{3\pi}{4}]) = e([\beta - \frac{3\pi}{4} + \pi, -\beta - \frac{\pi}{4} + \pi] \cup [-\beta + \frac{\pi}{4} - \pi, \beta + \frac{3\pi}{4} - \pi])$. This shows that both ways of defining $\varrho(0,\beta)$ coincide. By A, B and C, we have that $\varrho(r,\beta) \in \mathcal{A}$. Therefore ϱ is well defined, and clearly ϱ is continuous.

We prove E. First suppose that $r \in [0, \frac{\pi}{2})$. Take $(r,\beta), (r_1,\beta_1) \in K$ such that $\varrho(r,\beta) = \varrho(r_1,\beta_1)$. We are going to prove that $(r,\beta) = (r_1,\beta_1)$. By B, $\varrho(r,\beta)$ is not connected. Then A implies that $r_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By B the length of the shortest component of $S^1 \setminus \varrho(r,\beta)$ is equal to $\frac{\pi}{2} - |r|$ and to $\frac{\pi}{2} - |r_1|$. Then $|r| = |r_1|$. We consider three cases.

Case 1. $r_1 > 0$.

In this case $r = r_1$. Since the length of the component $J_1 = e(\beta_1 - \frac{\pi}{4} + \frac{r_1}{2}, \beta_1 + \frac{\pi}{4} - \frac{r_1}{2})$ of $S^1 \setminus \varrho(r_1,\beta_1)$ is $\frac{\pi}{2} - r_1$, we have that J_1 is the shortest component. Since the same happens with $J = e(\beta - \frac{\pi}{4} + \frac{r}{2}, \beta + \frac{\pi}{4} - \frac{r}{2})$, we obtain that $J = J_1$. In particular, the middle point of J_1 is also the middle point of J. This implies that $e(\beta) = e(\beta_1)$. Since $\beta \in [-\frac{1}{2}(r + \frac{\pi}{2}), \frac{1}{2}(r + \frac{\pi}{2})] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\beta_1 \in [-\frac{1}{2}(r_1 + \frac{\pi}{2}), \frac{1}{2}(r_1 + \frac{\pi}{2})] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, we obtain that $\beta = \beta_1$, and we are done in this case.

Case 2. $r_1 = 0$.

In this case $r = 0$. Then $\varrho(r,\beta) = e([-\beta - \frac{3\pi}{4}, \beta - \frac{\pi}{4}] \cup [\beta + \frac{\pi}{4}, -\beta + \frac{3\pi}{4}]) = e([-\beta_1 - \frac{3\pi}{4}, \beta_1 - \frac{\pi}{4}] \cup [\beta_1 + \frac{\pi}{4}, -\beta_1 + \frac{3\pi}{4}])$. Thus the middle points of the components of $S^1 \setminus \varrho(0,\beta)$ are $e(\beta)$ and $e(-\beta + \pi)$. Then $\{\beta, \beta + \pi\} = \{\beta_1, \beta_1 + \pi\}$.

Since $-\frac{\pi}{4} \leq \beta, \beta_1 \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \beta + \pi, \beta_1 + \pi \leq \frac{5\pi}{4}$, we conclude that $\beta = \beta_1$. This case is finished.

Case 3. $r_1 < 0$.

In this case, $r_1 = -r$. By B., the middle point of the shortest component of $S^1 \setminus \varrho(r_1, \beta_1) = S^1 \setminus \varrho(r, \beta)$ is $e(-\beta_1 + \pi)$ and $e(\beta)$. Thus $e(-\beta_1 + \pi) = e(\beta)$. This contradicts the facts that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and $\frac{\pi}{2} \leq -\beta_1 + \pi \leq \frac{3\pi}{2}$. Hence this case is impossible.

Now we analyze the case that $r \in (-\frac{\pi}{2}, 0)$. By symmetry and Case 3, it is impossible that $r_1 \in [0, \frac{\pi}{2})$. Thus suppose that $r_1 \in (-\frac{\pi}{2}, 0)$. In this case, $r = r_1$, and the middle point of the shortest component of $S^1 \setminus \varrho(r, \beta)$ is $-e(-\beta) = e(-\beta + \pi) = e(-\beta_1 + \pi) = -e(-\beta_1)$. Since $-\frac{\pi}{2} < \beta, \beta_1 < \frac{\pi}{2}$, we have that $\beta = \beta_1$. This ends the proof of the first part of E. The second part follows from A and B.

Let L be the continuum obtained from K by identifying the set $\{-\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ to a one-point set $\{p^-\}$ and the set $\{\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ to another one-point set $\{p^+\}$. Let $\eta^* : K \to L$ be the quotient mapping. Observe that L is a 2-cell. By Property E, the mapping ϱ preserves the fibers of the mapping η^* and the mapping η^* preserves the fibers of the mapping ϱ. By the Transgression Theorem [3, Theorem 3.2], there exists a homeomorphism $\eta : L \to \varrho(K)$ such that $\varrho = \eta \circ \eta^*$.

Let $\omega : K \times [0, \pi] \to \mathcal{A}$ be defined by

$$\omega(r, \beta, t) = e(t) \cdot \varrho(r, \beta).$$

We will need the following property.

Property F. Suppose that $(r, \beta, t), (s, \gamma, u) \in K \times [0, \pi]$, $\omega(r, \beta, t) = \omega(s, \gamma, u)$ and $(r, \beta, t) \neq (s, \gamma, u)$. Then:

(i) if $-\frac{\pi}{2} < r < \frac{\pi}{2}$, then $|t - u| = \pi$ and $(r, \beta) = (-s, -\gamma)$; and

(ii) if $\{r, s\} \cap \{-\frac{\pi}{2}, \frac{\pi}{2}\} \neq \emptyset$, then $\{r, s\} \subset \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and either $(t, r) = (u, s)$ or $\{r, s\} = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and $|t - u| = \pi$.

We prove F. Set $A = \omega(r, \beta, t) = \omega(s, \gamma, u)$. We analyze two cases.

Case 1. $-\frac{\pi}{2} < r < \frac{\pi}{2}$.

In this case, by B, A is not connected. This implies that $-\frac{\pi}{2} < s < \frac{\pi}{2}$. Let I and J be the respective largest and shortest components of $S^1 \setminus A$ (there is the possibility that I and J have the same length). By B, length$(J) = \frac{\pi}{2} - |r| = \frac{\pi}{2} - |s|$. Thus $|r| = |s|$.

Subcase 1.1. $r, s > 0$.

In this subcase, $r = s$ and by B, length$(J) <$ length(I). Then the middle point of J is distinct from the middle point of I. Since the middle point of J is $e(t + \beta)$ and $e(u + \gamma)$, we have that $e(t + \beta) = e(u + \gamma)$. Since $0 \leq s, t \leq \pi$ and $-\frac{\pi}{2} < \beta, \gamma < \frac{\pi}{2}$, we obtain that $-\frac{\pi}{2} < t + \beta, u + \gamma < \frac{3\pi}{2}$. Thus

$$t + \beta = u + \gamma.$$

Taking the middle point of I, we obtain that $e(t+\beta(\frac{2r-\pi}{2r+\pi})+\pi) = e(u+\gamma(\frac{2r-\pi}{2r+\pi})+\pi)$. The inequality $r>0$, implies that $-1 < \frac{2r-\pi}{2r+\pi} < 1$. Then $\frac{\pi}{2} < t+\beta(\frac{2r-\pi}{2r+\pi})+\pi, u+\gamma(\frac{2r-\pi}{2r+\pi})+\pi < \frac{5\pi}{2}$. Hence $t+\beta(\frac{2r-\pi}{2r+\pi})+\pi = u+\gamma(\frac{2r-\pi}{2r+\pi})+\pi$. Then $\gamma-\beta = t-u = (\gamma-\beta)(\frac{2r-\pi}{2r+\pi})$. Since $\frac{2r-\pi}{2r+\pi} \neq 1$, we obtain that $\beta=\gamma$ and $t=u$. This contradicts our assumption and shows that this case is impossible.

Subcase 1.2. $r=0$ or $s=0$.

Since $|r|=|s|$, we obtain that $r=s=0$. Then, by B, the middle points of the open intervals I and J are the points $e(t+\beta)$ and $e(t-\beta+\pi)$ in some order and the same happens with $e(u+\gamma)$ and $e(u-\gamma+\pi)$. Hence either:
(I) $e(t+\beta)=e(u+\gamma)$ and $e(t-\beta+\pi)=e(u-\gamma+\pi)$, or
(II) $e(t+\beta)=e(u-\gamma+\pi)$ and $e(t-\beta+\pi)=e(u+\gamma)$.

Recall that $-\frac{\pi}{2} < t+\beta, u+\gamma < \frac{3\pi}{2}$ and $\frac{\pi}{2} < t-\beta+\pi, u-\gamma+\pi < \frac{5\pi}{2}$ ... (*)

If (I) holds, then $t+\beta = u+\gamma$ and $t+\pi-\beta = u+\pi-\gamma$. This implies that $\beta=\gamma$ and $t=u$. This contradicts our assumption and proves that (I) does not hold.

If (II) holds, by (*), either

$$t+\beta = u-\gamma+\pi \text{ or } t+\beta+2\pi = u-\gamma+\pi.$$

Similarly,

$$u+\gamma = t-\beta+\pi \text{ or } u+\gamma+2\pi = t-\beta+\pi.$$

Thus we need to consider four possibilities:
(1.) $t+\beta = u-\gamma+\pi$ and $u+\gamma = t-\beta+\pi$,
(2.) $t+\beta = u-\gamma+\pi$ and $u+\gamma+2\pi = t-\beta+\pi$,
(3.) $t+\beta+2\pi = u-\gamma+\pi$ and $u+\gamma = t-\beta+\pi$,
(4.) $t+\beta+2\pi = u-\gamma+\pi$ and $u+\gamma+2\pi = t-\beta+\pi$.

If (1.) holds, then $\pi-(\beta+\gamma) = t-u = -\pi+\beta+\gamma$. This implies that $\pi = \beta+\gamma$. Which contradicts the fact that $\beta,\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence (1.) does not hold.

If (2.) holds, then $\pi-(\beta+\gamma) = t-u = \pi+\beta+\gamma$. This implies that $\beta=-\gamma$, $t=u+\pi$ and $r=0=-s$.

If (3.) holds, as in the previous paragraph, $\beta=-\gamma$, $t=u-\pi$ and $r=0=-s$.

If (4.) holds, as in the case (1.), $\beta+\gamma=-\pi$, a contradiction. So (4.) does not hold.

Subcase 1.3. $r,s<0$.

Since $\omega(r,\beta,t)=\omega(s,\gamma,u)$, we have that $e(t)\cdot\varrho(-r,-\beta) = e(u)\cdot\varrho(-s,-\gamma)$. Then $\omega(-r,-\beta,t)=\omega(-s,-\gamma,u)$. By Subcase 1.1., this equality does not hold. Thus, this subcase is also impossible.

Subcase 1.4. $r<0<s$ or $s<0<r$.

We analyze the case $r<0<s$, the other one is similar. Since $\omega(r,\beta,t) = \omega(s,\gamma,u)$, we have $-e(t)\cdot\varrho(-r,-\beta) = e(u)\cdot\varrho(s,\gamma)$. Let I and J be the respective largest and shortest components of $S^1\setminus A$. By B, length$(J) = \frac{\pi}{2}-|r| = \frac{\pi}{2}-|s|$. This implies that $s=-r$. Moreover, the middle point of J coincides with the points $-e(t-\beta) = e(t-\beta+\pi)$ and $e(u+\gamma)$. Since $\frac{\pi}{2} < t-\beta+\pi < \frac{5\pi}{2}$ and $-\frac{\pi}{2} < u+\gamma < \frac{3\pi}{2}$, we have that either

$$t-\beta+\pi=u+\gamma \text{ or } t-\beta+\pi=u+\gamma+2\pi.$$

On the other hand, the middle point of I coincides with the points $e(t+\beta(\frac{2r+\pi}{2r-\pi}))$ and $e(u+\gamma(\frac{2s-\pi}{2s+\pi})+\pi)$. Since $-1<\frac{2r+\pi}{2r-\pi},\frac{2s-\pi}{2s+\pi}<1$ and $-\frac{\pi}{2}<\beta,\gamma<\frac{\pi}{2}$ we have that $-\frac{\pi}{2}<\beta(\frac{2r+\pi}{2r-\pi}),\gamma(\frac{2s-\pi}{2s+\pi})<\frac{\pi}{2}$, $-\frac{\pi}{2}<t-\beta(\frac{2r+\pi}{2r-\pi})<\frac{3\pi}{2}$ and $\frac{\pi}{2}<u+\gamma(\frac{2s-\pi}{2s+\pi})+\pi<\frac{5\pi}{2}$. Then either

$$t-\beta(\tfrac{2r+\pi}{2r-\pi})=u+\gamma(\tfrac{2s-\pi}{2s+\pi})+\pi \text{ or } t-\beta(\tfrac{2r+\pi}{2r-\pi})+2\pi=u+\gamma(\tfrac{2s-\pi}{2s+\pi})+\pi.$$

Hence we have to consider four possibilities.

(5.) $t-\beta+\pi=u+\gamma$ and $t-\beta(\frac{2r+\pi}{2r-\pi})=u+\gamma(\frac{2s-\pi}{2s+\pi})+\pi$,

(6.) $t-\beta+\pi=u+\gamma$ and $t-\beta(\frac{2r+\pi}{2r-\pi})=u+\gamma(\frac{2s-\pi}{2s+\pi})-\pi$

(7.) $t-\beta+\pi=u+\gamma+2\pi$ and $t-\beta(\frac{2r+\pi}{2r-\pi})=u+\gamma(\frac{2s-\pi}{2s+\pi})+\pi$,

(8.) $t-\beta+\pi=u+\gamma+2\pi$ and $t-\beta(\frac{2r+\pi}{2r-\pi})=u+\gamma(\frac{2s-\pi}{2s+\pi})-\pi$.

If (5.) holds, then $\beta+\gamma-\pi=t-u=\beta(\frac{2r+\pi}{2r-\pi})+\gamma(\frac{2s-\pi}{2s+\pi})+\pi>0$. This implies that $\beta+\gamma>\pi$, a contradiction. Therefore (5.) does not hold.

If (6.) holds, then $\beta+\gamma-\pi=t-u=\beta(\frac{2r+\pi}{2r-\pi})+\gamma(\frac{2s-\pi}{2s+\pi})-\pi$. This implies that $\gamma(\frac{2\pi}{2s+\pi})=\beta(\frac{2\pi}{2r-\pi})$. Since $s=-r$, we infer that $\beta=-\gamma$ and $t=u-\pi$, so in this case, we are done.

If (7.) holds, then $\beta+\gamma+\pi=t-u=\beta(\frac{2r+\pi}{2r-\pi})+\gamma(\frac{2s-\pi}{2s+\pi})+\pi$. As in (6.) we obtain that $\beta=-\gamma$ and $t=u+\pi$, so in this case, we are done.

If (8.) holds, then $\beta+\gamma+\pi=t-u=\beta(\frac{2r+\pi}{2r-\pi})+\gamma(\frac{2s-\pi}{2s+\pi})-\pi<0$. This implies that $\beta+\gamma<-\pi$, a contradiction. Hence (8.) does not hold.

This completes the proof for Case 1.

Case 2. $r=\frac{\pi}{2}$ or $r=-\frac{\pi}{2}$.

We analyze the case $r=\frac{\pi}{2}$, the other case is similar. By A, $\omega(r,\beta,t)=e(t)\cdot\varrho(r,\beta)=e([t-\frac{\pi}{2},t+\frac{\pi}{2}])=\omega(s,\gamma,u)$. Since $\omega(s,\gamma,u)$ is connected, by B, $s\in\{-\frac{\pi}{2},\frac{\pi}{2}\}$. If $s=\frac{\pi}{2}$, then $\omega(s,\gamma,u)=e([u-\frac{\pi}{2},u+\frac{\pi}{2}])$. Thus $e([t-\frac{\pi}{2},t+\frac{\pi}{2}])=e([u-\frac{\pi}{2},u+\frac{\pi}{2}])$ and the middle point of this set is $e(t)$ and $e(u)$, so $e(t)=e(u)$ and $t=u$. Hence $(t,r)=(u,s)$.

Now suppose that $s=-\frac{\pi}{2}$. Then $\omega(s,\gamma,u)=e([u+\frac{\pi}{2},u+\frac{3\pi}{2}])$. Since $e([t-\frac{\pi}{2},t+\frac{\pi}{2}])=e([u+\frac{\pi}{2},u+\frac{3\pi}{2}])$, taking the middle points of these arcs, we obtain that $e(t)=e(u+\pi)$. Since $t\in[0,\pi]$ and $\pi+u\in[\pi,2\pi]$, we conclude that either $t=\pi=\pi+u$ or $t=0$ and $\pi+u=2\pi$. In both cases, $|t-u|=\pi$. Therefore $\{r,s\}=\{-\frac{\pi}{2},\frac{\pi}{2}\}$ and $|t-u|=\pi$. This finishes the proof of F.

Property G. ω is onto.

We prove G. Take an element $A\in\mathcal{A}$. First we consider the case that A is not connected. Then $A=e([a,b]\cup[c,d])$, with $b<c$ and $d-a<2\pi$. Since $A\in\mathcal{A}$, $\pi=b-a+d-c$. We assume that the open interval $J=e((b,c))$ is the shortest component of $S^1\setminus A$. That is, $c-b\leq a+2\pi-d$. Since $\sigma(A)=\pi$, the sum of the lengths of the components of $S\setminus A$ is equal to π. That is, $(c-b)+(a+2\pi-d)=\pi$. Then $c-b\leq\frac{\pi}{2}$. We consider two cases.

Case 1. $(b+c)(\pi+a-d)=(a+d)(\pi+b-c)$.

Define

$$\beta = \tfrac{b+c}{2},\ \alpha = \tfrac{a+d}{2} + \pi \text{ and } r = a - d + \tfrac{3\pi}{2}.$$

Since $\pi = b - a + d - c$, we have that

$$r = b - c + \tfrac{\pi}{2}.$$

Then $r \in [0, \frac{\pi}{2})$, observe that $2r - \pi = 2(\pi + a - d) = 2(b - c)$ and $2r + \pi = 2(2\pi + a - d) = 2(\pi + b - c)$. Hence $\beta(\frac{2r-\pi}{2r+\pi}) = (\frac{b+c}{2})(\frac{2(\pi+a-d)}{2(\pi+b-c)}) = \frac{a+d}{2}$, so $\beta(\frac{2r-\pi}{2r+\pi}) = \frac{a+d}{2}$.

Let $r_0 = \frac{\pi}{2} + r \in [\frac{\pi}{2}, \pi)$, $s_0 = \pi - r_0 \in (0, \frac{\pi}{2}]$ and $w = bs_0 + ar_0$. Then $r_0 = \pi + b - c = 2\pi + a - d$, $s_0 = c - b$, and

$$(b+c)s_0 + (a+d)r_0 = (b+c)(-\pi - a + d) + (a+d)(2\pi + a - d) = \\ -(b+c)(\pi + a - d) + (a+d)(\pi + b - c) = 0.$$

This implies that $-w = cs_0 + dr_0$. Then

$$2w = bs_0 + ar_0 - cs_0 - dr_0 = (b-c)s_0 + (a-d)r_0 = \\ (\pi + a - d)s_0 + (-\pi + b - c)r_0 = (-\pi + 2\pi + a - d)s_0 + (-\pi + b - c)r_0 = \\ (-\pi + r_0)s_0 + (-\pi - s_0)r_0 = -\pi(r_0 + s_0) = -\pi^2.$$

Thus $\frac{w}{\pi} = -\frac{\pi}{2}$, $a\frac{r_0}{\pi} + b\frac{s_0}{\pi} = -\frac{\pi}{2}$ and $c\frac{s_0}{\pi} + d\frac{r_0}{\pi} = \frac{\pi}{2}$. Since $\frac{s_0}{\pi} + \frac{r_0}{\pi} = 1$, we have that $-\frac{\pi}{2}$ (respectively, $\frac{\pi}{2}$) is a convex combination of a and b (respectively, c and d). Therefore $a \leq -\frac{\pi}{2} \leq b$ and $c \leq \frac{\pi}{2} \leq d$.

Then $2c \leq \pi$, $b + c \leq \pi + b - c = \frac{\pi}{2} + r$ and $\beta \leq \frac{1}{2}(r + \frac{\pi}{2})$. On the other hand $-\pi \leq 2b$, $-r - \frac{\pi}{2} = -\pi - b + c \leq b + c$ and $-\frac{1}{2}(r + \frac{\pi}{2}) \leq \beta$.

Since $0 \leq r < \frac{\pi}{2}$, we have that $(r, \beta) \in K$. Moreover,

$$\omega(r, \beta, 0) = \\ \varrho(r,\beta) = e([\beta(\tfrac{2r-\pi}{2r+\pi}) - \tfrac{3\pi}{4} + \tfrac{r}{2}, \beta - \tfrac{\pi}{4} + \tfrac{r}{2}] \cup [\beta + \tfrac{\pi}{4} - \tfrac{r}{2}, \beta(\tfrac{2r-\pi}{2r+\pi}) + \tfrac{3\pi}{4} - \tfrac{r}{2}]) = \\ e([\tfrac{a+d}{2} + \tfrac{a-d}{2}, \tfrac{b+c}{2} + \tfrac{b-c}{2}] \cup [\tfrac{b+c}{2} - \tfrac{b-c}{2}, \tfrac{a+d}{2} - \tfrac{a-d}{2}]) = A.$$

Therefore, $\omega(r, \beta, 0) = A$ and $A \in \mathrm{Im}\ \omega$.

Case 2. $(b+c)(\pi + a - d) \neq (a+d)(\pi + b - c)$.

Since $d - c + b - a = \pi$, we have that $b - c = \pi + a - d$ and $d - a = \pi + c - b$. Define

$$t = \tfrac{(b+c)(\pi+a-d)-(a+d)(\pi+b-c)}{2(b-c+d-a)}.$$

Then $(a+d)(\pi + b - c) + 2t(b - c) = (b+c)(\pi + a - d) - 2t(d - a)$. Thus $(b+c)(\pi + a - d) + 2t(\pi + a - d) = (a+d)(\pi + b - c) + 2t(\pi + b - c)$. This implies that $(b + t + c + t)(\pi + a + t - (d + t)) = (a + t + d + t)(\pi + b + t - (c + t))$. Set

$$a_0 = a + t,\ b_0 = b + t,\ c_0 = c + t,\ d_0 = d + t \text{ and } A_0 = e([a_0, b_0] \cup [c_0, d_0]).$$

Observe that $(b_0 + c_0)(\pi + a_0 - d_0) = (a_0 + d_0)(\pi + b_0 - c_0)$, $A_0 \in \mathcal{A}$ and $e((b_0, c_0))$ is the shortest component of $S^1 \setminus A_0$.

By Case 1, there exists $(r, \beta) \in K$ such that $\varrho(r, \beta) = A_0$. Observe that $A_0 = e(t) \cdot A$ and $e(-t) \cdot \varrho(r, \beta) = A$.

Let $t_0 \in [0, 2\pi]$ be such that $e(t_0) = e(-t)$. In the case that $t_0 \in [0, \pi]$, we conclude that $(r, \beta, t_0) \in K \times [0, \pi]$, $\omega(r, \beta, t_0) = A$ and $A \in \text{Im } \omega$. In the case that $t_0 \in [\pi, 2\pi]$, let $t_1 = t_0 - \pi$. Then $t_1 \in [0, \pi]$ and $e(t_1) = -e(t_0)$. Hence $(-r, -\beta, t_1) \in K \times [0, \pi]$ and, by the definition of ϱ, $\omega(-r, -\beta, t_1) = e(t_1) \cdot \varrho(-r, -\beta) = e(t_1) \cdot (-\varrho(r, \beta)) = e(t_0) \cdot \varrho(r, \beta) = A$. Therefore $A \in \text{Im } \omega$.

This completes the proof that $A \in \text{Im } \omega$ for each $A \in \mathcal{A}$ such that A is disconnected.

Now, we suppose that $A \in \mathcal{A}$ and A is connected. Then $A = e([a, b])$ for some $a, b \in \mathbb{R}$ such that $b - a = \pi$. We may assume that the number $t_0 = \frac{a+b}{2} \in [0, 2\pi]$. Then $e(t_0) \cdot e([-\frac{\pi}{2}, \frac{\pi}{2}]) = e([t_0 - \frac{\pi}{2}, t_0 + \frac{\pi}{2}]) = e([a, b]) = A$. In the case that $t_0 \in [0, \pi]$, we have that $\omega(\frac{\pi}{2}, 0, t_0) = A$. In the case that $t_0 \in [\pi, 2\pi]$, the number $t_1 = t_0 - \pi$ belongs to $[0, \pi]$ and by A, $\omega(-\frac{\pi}{2}, 0, t_1) = e(t_1) \cdot \varrho(-\frac{\pi}{2}, 0) = -e(t_0) \cdot e([\frac{\pi}{2}, \frac{3\pi}{2}]) = e(t_0) \cdot e([-\frac{\pi}{2}, \frac{\pi}{2}]) = A$. This ends the proof that $A \in \text{Im } \omega$.

This completes the proof of G.

Let D_0 be the unit disk in the plane given by

$$D_0 = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Since K is a 2-cell and the arcs $L^+ = \{\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $L^- = \{-\frac{\pi}{2}\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ are contained in the manifold boundary of K, it is possible to find an onto mapping $\lambda : K \to D_0$ such that $\lambda(L^+) = \{(1, 0)\}$, $\lambda(L^-) = \{(-1, 0)\}$, the mapping $\lambda|_{K \setminus (L^+ \cup L^-)} : K \setminus (L^+ \cup L^-) \to D_0 \setminus \{(-1, 0), (1, 0)\}$ is a homeomorphism, and since both sets K and D_0 are symmetric with respect to the origin $(0, 0)$, it is possible to ask that for every $(r, \beta) \in K$, the following equality holds:

$$\lambda(r, \beta) = -\lambda(-r, -\beta).$$

By A, ϱ preserves the fibers of λ. Then Theorem 3.2, Ch. VI of [3], implies that there exists a mapping $\varrho^* : D_0 \to \mathcal{A}$ such that $\varrho = \varrho^* \circ \lambda$.

Define $\omega^* : D_0 \times [0, \pi] \to \mathcal{A}$ by

$$\omega^*(r, \beta, t) = e(t) \cdot \varrho^*(r, \beta).$$

Property H. Suppose that $(r^*, \beta^*, t), (s^*, \gamma^*, u) \in D_0 \times [0, \pi]$, $\omega^*(r^*, \beta^*, t) = \omega^*(s^*, \gamma^*, u)$ and $(r^*, \beta^*, t) \neq (s^*, \gamma^*, u)$. Then $(r^*, \beta^*) = (-s^*, -\gamma^*)$ and $\pi = |t - u|$.

We prove H. Let $(r, \beta), (s, \gamma) \in K$ be such that $\lambda(r, \beta) = (r^*, \beta^*)$ and $\lambda(s, \gamma) = (s^*, \gamma^*)$. Then $\omega^*(r^*, \beta^*, t) = e(t) \cdot \varrho^*(r^*, \beta^*) = e(t) \cdot \varrho(r, \beta) = \omega(r, \beta, t)$ and $\omega^*(s^*, \gamma^*, u) = \omega(s, \gamma, u)$. Thus $\omega(r, \beta, t) = \omega(s, \gamma, u)$.

In the case that $-\frac{\pi}{2} < r < \frac{\pi}{2}$, by Property F, $|t - u| = \pi$ and $(r, \beta) = (-s, -\gamma)$. By the choice of λ, $(r^*, \beta^*) = \lambda(r, \beta) = \lambda(-s, -\gamma) = -\lambda(s, \gamma) = -(s^*, \gamma^*)$. So, in this case we are done.

In the case that $\{r,s\} \cap \{-\frac{\pi}{2}, \frac{\pi}{2}\} \neq \emptyset$, by Property F, $\{r,s\} \subset \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and either $(t,r) = (u,s)$ or $\{r,s\} = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and $|t-u| = \pi$. We consider three cases.

Case 1. $r = s = \frac{\pi}{2}$.

In this case, $(t,r) = (u,s)$ and $(r,\beta), (s,\gamma) \in L^+$. Then $(r^*, \beta^*) = \lambda(r,\beta) = (1,0)$, so $r^* = 1$ and $\beta^* = 0$. Similarly, $s^* = 1$ and $\gamma^* = 0$. Therefore $(r*, \beta*, t) = (s^*, \gamma^*, u)$, a contradiction. Thus this case is impossible.

Case 2. $r = s = -\frac{\pi}{2}$.

This case is similar to Case 1.

Case 3. $\{r,s\} = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and $|t-u| = \pi$.

We may suppose that $r = \frac{\pi}{2}$ and $s = -\frac{\pi}{2}$. Then $(r,\beta) \in L^+$ and $(s,\gamma) \in L^-$. Thus $(r^*, \beta^*) = \lambda(r,\beta) = (1,0)$ and $(s^*, \gamma^*) = \lambda(s,\gamma) = (-1,0)$. So $r^* = 1$, $\beta^* = 0$, $s^* = -1$ and $\gamma^* = 0$. Therefore $(r*, \beta*) = (-s^*, -\gamma^*)$. This completes the proof of H.

In the space $D_0 \times [0,\pi]$ define the relation $(r^*, \beta^*, t) \simeq (s^*, \gamma^*, u)$ if and only if either $(r^*, \beta^*, t) = (s^*, \gamma^*, u)$ or $(r^*, \beta^*) = (s^*, \gamma^*)$ and $\{t,u\} = \{0, 2\pi\}$ or $(r^*, \beta^*) = (-s^*, -\gamma^*)$ and $|t-u| = \pi$. Clearly, $\simeq$ is an equivalence relation, so it is possible to consider the quotient space

$$(D_0 \times [0,\pi])/\simeq.$$

Let $\zeta : D_0 \times [0,\pi] \to (D_0 \times [0,\pi])/\simeq$ be the quotient mapping. By Property H, the fibers of ζ coincide with the fibers of ω^*.

Given $A \in \mathcal{A}$, by G, there exists $(r,\beta,t) \in K \times [0,\pi]$ such that $A = \omega(r,\beta,t) = e(t) \cdot \varrho(r,\beta) = e(t) \cdot (\varrho^*(\lambda(r,\beta)) = e(t) \cdot \varrho^*(r^*, \beta^*) = w^*(r^*, \beta^*, t)$, where $(r^*, \beta^*) = \lambda(r,\beta)$. This proves that ω^* is onto.

The Transgression Theorem [3, Theorem 3.2, Ch. VI] implies that $\mathcal{A}$ is homeomorphic to $(D_0 \times [0,\pi])/\simeq$.

Observe that $(D_0 \times [0,\pi])/\simeq$ can be obtained by taking the convex cilinder $D_0 \times [0,\pi]$ and identifying its top and its bottom making a rotation of 180^0. It is easy to see that after this identification we obtain a solid torus. Therefore, $\mathcal{A}$ is homeomorphic to the solid torus. ■

Theorem 3 *The continuum $C_2(S^1)/F_2(S^1)$ is homeomorphic to the suspension of $\mathcal{A}$.*

Proof. Set $\mathcal{P} = C_2(S^1) \setminus (F_1(S^1) \cup \{S^1\})$. We are going to define a mapping $f : \mathcal{P} \times (0, 2\pi) \to \mathcal{P}$.

In order to define f, take $(A,t) \in \mathcal{P} \times (0,2\pi)$. We consider two cases.

If A is connected, A is a subarc of S^1. Let $I = S^1 \setminus A$ and $p = e(\alpha)$ be the middle point of I. Then

$$f(A,t) = e([\alpha + \pi - \tfrac{t}{2}, \alpha + \pi + \tfrac{t}{2}]).$$

In the case that A is not connected, the complement of A in S^1 consists of two open subarcs I and J. Let p and q be the respective middle points of I and J, and let r and s be the respective lengths of I and J, we choose I and J in such a way that $s \leq r$. Let α and β be real numbers such that $|\alpha - \beta| \leq \pi$, $p = e(\alpha)$ and $q = e(\beta)$.

Define

$$\mathcal{P}^+ = \{A \in \mathcal{P} : \alpha \leq \beta\}, \text{ and}$$
$$\mathcal{P}^- = \{A \in \mathcal{P} : \alpha \geq \beta\}.$$

For defining $f(A,t)$ (when A is not connected), we consider two subcases:
Case 1. $A \in \mathcal{P}^+$.
In this case, $\alpha \leq \beta$. Observe that

$$A = e([\alpha + \tfrac{r}{2}, \beta - \tfrac{s}{2}] \cup [\beta + \tfrac{s}{2}, \alpha + 2\pi - \tfrac{r}{2}]).$$

Define

$$\alpha' = \tfrac{1}{r+s}(\alpha r + \beta s - \tfrac{s}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))),$$
$$\beta' = \tfrac{1}{r+s}(\alpha r + \beta s + \tfrac{r}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))),$$
$$r' = \tfrac{r(2\pi - t)}{r+s} \text{ and } s' = \tfrac{s(2\pi - t)}{r+s}.$$

Then define

$$f(A,t) = e([\alpha' + \tfrac{r'}{2}, \beta' - \tfrac{s'}{2}] \cup [\beta' + \tfrac{s'}{2}, \alpha' + 2\pi - \tfrac{r'}{2}]).$$

Claim 1.
(a) $\sigma(A) = 2\pi - (r+s)$, $\beta - \alpha \geq \frac{r+s}{2}$ and $2\pi + \alpha - \beta \geq \frac{r+s}{2}$,
(b) $\alpha r + \beta s = \alpha' r + \beta' s$,
(c) $\alpha r' + \beta s' = \alpha' r' + \beta' s'$,
(d) $\beta' - \alpha' = \frac{1}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))$,
(e) $\beta - \alpha = \frac{1}{t}((\beta' - \alpha')\sigma(A) - \pi(\sigma(A) - t))$,
(f) $\alpha = \frac{1}{r+s}(\alpha' r + \beta' s - \frac{s}{t}((\beta' - \alpha')\sigma(A) - \pi(\sigma(A) - t)))$,
(g) $\beta = \frac{1}{r+s}(\alpha' r + \beta' s + \frac{r}{t}((\beta' - \alpha')\sigma(A) - \pi(\sigma(A) - t)))$,
(h) $0 \leq \beta' - \frac{s'}{2} - \alpha' - \frac{r'}{2}$,
(i) $0 \leq \alpha' + 2\pi - \frac{r'}{2} - \beta' - \frac{s'}{2}$,
(j) $\beta' - \alpha' \leq \pi$,
(k) $f(A,t) \in \mathcal{P}^+$,
(l) $\sigma(f(A,t)) = t$,
(m) $f(f(A,t), \sigma(A)) = A$.

We prove Claim 1. The first part of (a) is immediate. The two inequalities in (a) follow from the fact that the arc-length distance from p to q is greater than or equal to $\frac{r+s}{2}$.

(b) and (d) are immediate. (c) follows from (b). (e) follows from (d). (f) and (g) are obtained as solutions of the system of linear equations given in (b) and (e).

In order to prove (h), note that (a) and (d) imply that:

$$\begin{gathered}\beta' - \alpha' - \tfrac{r'+s'}{2} = \\ \tfrac{1}{\sigma(A)}((\beta-\alpha)t + \pi(\sigma(A)-t)) - \tfrac{2\pi-t}{2} \geq \\ \tfrac{1}{\sigma(A)}(\tfrac{r+s}{2}t + \pi(\sigma(A)-t) - \tfrac{\sigma(A)(2\pi-t)}{2}) = \\ \tfrac{1}{2\sigma(A)}((2\pi-\sigma(A))t + 2\pi(\sigma(A)-t) - \sigma(A)(2\pi-t)) = 0.\end{gathered}$$

In order to prove (i), note that (a) and (d) imply that:

$$\begin{gathered}\alpha' + 2\pi - \tfrac{r'}{2} - \beta' - \tfrac{s'}{2} = \\ \tfrac{1}{\sigma(A)}((\alpha-\beta)t + \pi(t-\sigma(A))) + 2\pi - \tfrac{2\pi-t}{2} = \\ \tfrac{1}{\sigma(A)}((2\pi+\alpha-\beta)t - 2\pi t + \pi t - \pi\sigma(A)) + 2\pi - \tfrac{2\pi-t}{2} \geq \\ \tfrac{1}{\sigma(A)}(\tfrac{r+s}{2}t - \pi t - \pi\sigma(A)) + 2\pi - \tfrac{2\pi-t}{2} = \\ \tfrac{1}{2\sigma(A)}((2\pi-\sigma(A))t - 2\pi t - 2\pi\sigma(A) + 4\pi\sigma(A) - 2\pi\sigma(A) + t\sigma(A)) = 0.\end{gathered}$$

We prove (j). Since $\beta - \alpha \leq \pi$, we have that $(\beta-\alpha)t \leq \pi t$. This implies that $(\beta-\alpha)t + \pi(\sigma(A)-t) \leq \pi\sigma(A)$. By (d), we conclude that $\beta' - \alpha' \leq \pi$.

We prove (k) and (l). Note that by (h) and (i), the sets $e([\alpha' + \frac{r'}{2}, \beta' - \frac{s'}{2}])$ and $e([\beta' + \frac{s'}{2}, \alpha' + 2\pi - \frac{r'}{2}])$ are nonempty subsets of S^1. Since $2\pi + \alpha' + \frac{r'}{2} > \alpha' + 2\pi - \frac{r'}{2}$ and $\beta' + \frac{s'}{2} > \beta' - \frac{s'}{2}$, we have that $f(A,t)$ is the union of two disjoint subcontinua of S^1. Hence $f(A,t) \in C_2(S^1) \setminus C(S^1)$. Moreover, $\sigma(f(A,t))$ is the sum of the lengths of the two intervals. Then $\sigma(f(A,t)) = 2\pi - (r'+s') = t$. Let $I' = e((\alpha' - \frac{r'}{2}, \alpha' + \frac{r'}{2}))$ and $J' = ((\beta' - \frac{s'}{2}, \beta' + \frac{s'}{2}))$. Then the components of $S^1 \setminus f(A,t)$ are the two open subarcs I' and J'. The respective middle points of I' and J' are the points $p' = e(\alpha')$ and $q' = e(\beta')$. Observe that the respective lengths of I' and J' are r' and s'. Note that $s' \leq r'$. By (h) and (j), we have that $0 \leq \beta' - \alpha' \leq \pi$. Thus $f(A,t) \in \mathcal{P}^+$.

Finally, we prove (m). By definition,

$$f(f(A,t),\sigma(A)) = e([\alpha'' + \tfrac{r''}{2}, \beta'' - \tfrac{s''}{2}] \cup [\beta'' + \tfrac{s''}{2}, \alpha'' + 2\pi - \tfrac{r''}{2}]),$$

where

$$\begin{gathered}\alpha'' = \tfrac{1}{r'+s'}(\alpha' r' + \beta' s' - \tfrac{s'}{t}((\beta'-\alpha')\sigma(A) + \pi(t-\sigma(A))), \\ \beta'' = \tfrac{1}{r'+s'}(\alpha' r' + \beta' s' + \tfrac{r'}{t}((\beta'-\alpha')\sigma(A) + \pi(t-\sigma(A))), \\ r'' = \tfrac{r'(2\pi-\sigma(A))}{r'+s'} \text{ and } s'' = \tfrac{s'(2\pi-\sigma(A))}{r'+s'}.\end{gathered}$$

Observe that $r'' = \frac{\frac{r(2\pi-t)}{r+s}(2\pi-\sigma(A))}{\frac{r(2\pi-t)}{r+s} + \frac{s(2\pi-t)}{r+s}} = \frac{(2\pi-\sigma(A))r}{r+s}$. Since $\sigma(A) + r + s = 2\pi$, we conclude that $r'' = r$. Similarly, $s'' = s$. Since r' and s' have the same factor $(\frac{2\pi-t}{r+s})$, we can change r' by r and s' by s in the expression for α''. Then

$$\alpha'' = \tfrac{1}{r+s}(\alpha' r + \beta' s - \tfrac{s}{t}((\beta'-\alpha')\sigma(A) + \pi(t-\sigma(A))).$$

By (f), we conclude that $\alpha'' = \alpha$. Similarly, $\beta'' = \beta$. Thus $f(f(A,t),\sigma(A)) = A$.

Case 2. $A \in \mathcal{P}^-$.

In this case, $\alpha \geq \beta$. Observe that

$$A = e([\beta + \tfrac{s}{2}, \alpha - \tfrac{r}{2}] \cup [\alpha + \tfrac{r}{2}, \beta + 2\pi - \tfrac{s}{2}]).$$

Define

$$\begin{gathered}\alpha' = \tfrac{1}{r+s}(\alpha r + \beta s + \tfrac{s}{\sigma(A)}((\alpha - \beta)t + \pi(\sigma(A) - t))),\\ \beta' = \tfrac{1}{r+s}(\alpha r + \beta s - \tfrac{r}{\sigma(A)}((\alpha - \beta)t + \pi(\sigma(A) - t))),\\ r' = \tfrac{r(2\pi - t)}{r+s} \text{ and } s' = \tfrac{s(2\pi - t)}{r+s}.\end{gathered}$$

Then define

$$f(A,t) = e([\beta' + \tfrac{s'}{2}, \alpha' - \tfrac{r'}{2}] \cup [\alpha' + \tfrac{r'}{2}, \beta' + 2\pi - \tfrac{s'}{2}]).$$

Proceeding as in Case 1, interchanging the roles of α and β in properties (a)-(m), it is possible to prove that for every $A \in \mathcal{P}^-$, $f(A,t) \in \mathcal{P}^-$, $\sigma(f(A,t)) = t$ and $f(f(A,t), \sigma(A)) = A$.

This finishes the definition of f.

Claim 2. f is well defined.

In order to prove this claim, we only need to consider the case that A is not connected.

First we see that the definition of $f(A,t)$ does not change if we add the same integer multiple of 2π to both α and β. In order to see this, it is enough to show that $f(A,t)$ does not change when we take $\alpha + 2\pi$ and $\beta + 2\pi$ instead of α and β, respectively. We consider the case that $\alpha \leq \beta$, the other case is similar. In this case,

$$\begin{gathered}(\alpha + 2\pi)' = \tfrac{1}{r+s}((\alpha + 2\pi)r + (\beta + 2\pi)s - \tfrac{s}{\sigma(A)}((\beta + 2\pi - \alpha - 2\pi)t + \pi(\sigma(A) - t))) =\\ 2\pi + \tfrac{1}{r+s}(\alpha r + \beta s - \tfrac{s}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))) = 2\pi + \alpha'.\end{gathered}$$

Similarly, $(\beta + 2\pi)' = 2\pi + \beta'$. Since r and s are the lengths of the components of $S^1 \setminus A$, they do not depend on the choice of α and β. Therefore, the value of $f(A,t)$ does not change under adding integer multiples of 2π.

In the case that $r = s$, both components of $S^1 \setminus A$ have the same length, so it is possible to interchange the roles of I and J. We check that in this case the set $f(A,t)$ does nto change. Take $I_0 = J$ and $J_0 = I$. Then $r_0 = s = r = s_0$, $\alpha_0 = \beta$ and $\beta_0 = \alpha$. Consider the case that $\alpha \leq \beta$. Then $\beta_0 \leq \alpha_0$.

So,

$$\begin{gathered}\alpha_0' = \tfrac{1}{r_0+s_0}(\alpha_0 r_0 + \beta_0 s_0 + \tfrac{s_0}{\sigma(A)}((\alpha_0 - \beta_0)t + \pi(\sigma(A) - t))) =\\ \tfrac{1}{2}(\beta + \alpha + \tfrac{1}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))).\end{gathered}$$

On the other hand,

$$\begin{gathered}\beta' = \tfrac{1}{r+s}(\alpha r + \beta s + \tfrac{r}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))) =\\ \tfrac{1}{2}(\alpha + \beta + \tfrac{1}{\sigma(A)}((\beta - \alpha)t + \pi(\sigma(A) - t))).\end{gathered}$$

Therefore $\alpha_0' = \beta'$. Similarly, $\beta_0' = \alpha'$. This implies that the value $f(A,t)$ does not change when we take I_0 and J_0 instead of I and J.

Now, we see that the definition of $f(A,t)$ does not change in the case that it is possible to take numbers $\alpha_1 \leq \beta_1$ and $\beta_2 \leq \alpha_2$, for the same A. So suppose that there exist α_1, α_2, β_1 and β_2 such that $p = e(\alpha_1) = e(\alpha_2)$, $q = e(\beta_1) = e(\beta_2)$, $|\alpha_1 - \beta_1| \leq \pi$, $|\alpha_2 - \beta_2| \leq \pi$, $\alpha_1 \leq \beta_1$, $\beta_2 \leq \alpha_2$ and either $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Since the definition of $f(A,t)$ does not change by adding the same integer multiple of 2π to both α and β, we may assume that $\alpha_1 = \alpha_2$. Then $\beta_2 \in [\alpha_1 - \pi, \alpha_1)$, $\beta_1 \in (\alpha_1, \alpha_1 + \pi]$ and $\beta_1, \beta_2 \in [\alpha_1 - \pi, \alpha_1 + \pi]$. Since $e(\beta_1) = e(\beta_2)$, we have that $\beta_2 = \alpha_1 - \pi$ and $\beta_1 = \alpha_1 + \pi$.

The value of α_1' (using α_1 and β_1) is

$$\begin{aligned} \alpha_1' = \tfrac{1}{r+s}(\alpha_1 r + \beta_1 s - \tfrac{s}{\sigma(A)}((\beta_1 - \alpha_1)t + \pi(\sigma(A) - t))) = \\ \tfrac{1}{r+s}(\alpha_1(r+s) + \pi s - \tfrac{s}{\sigma(A)}(\pi t + \pi(\sigma(A) - t))) = \alpha_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha_2' = \tfrac{1}{r+s}(\alpha_2 r + \beta_2 s + \tfrac{s}{\sigma(A)}((\alpha_2 - \beta_2)t + \pi(\sigma(A) - t))) = \\ \tfrac{1}{r+s}(\alpha_1(r+s) - \pi s + \tfrac{s}{\sigma(A)}(\pi t + \pi(\sigma(A) - t))) = \alpha_1. \end{aligned}$$

Therefore, $\alpha_1' = \alpha_2'$. Similarly, $\beta_1' = \beta_1$ and $\beta_2' = \beta_2$. Thus, using α_1 and β_1, we have that $f(A,t) = e([\alpha_1 + \frac{r'}{2}, \beta_1 - \frac{s'}{2}] \cup [\beta_1 + \frac{s'}{2}, \alpha_1 + 2\pi - \frac{r'}{2}]) = e([\alpha_1 + \frac{r'}{2}, \alpha_1 + \pi - \frac{s'}{2}] \cup [\alpha_1 + \pi + \frac{s'}{2}, \alpha_1 + 2\pi - \frac{r'}{2}])$, while using α_2 and β_2, we have $f(A,t) = e([\beta_2 + \frac{s'}{2}, \alpha_2 - \frac{r'}{2}] \cup [\alpha_2 + \frac{r'}{2}, \beta_2 + 2\pi - \frac{s'}{2}]) = e([\alpha_2 - \pi + \frac{s'}{2}, \alpha_2 - \frac{r'}{2}] \cup [\alpha_2 + \frac{r'}{2}, \alpha_2 + \pi - \frac{s'}{2}]) =$ (adding 2π to both end-points of the first interval) $e([\alpha_1 + \pi + \frac{s'}{2}, \alpha_1 + 2\pi - \frac{r'}{2}] \cup [\alpha_1 + \frac{r'}{2}, \alpha_1 + \pi - \frac{s'}{2}])$. Therefore the value of $f(A,t)$ using α_1 and β_1 is the same as the value of $f(A,t)$ using α_2 and β_2.

This completes the proof of Claim 2.

Claim 3. *f is continuous.*

Clearly, f is continuous at the elements $(A,t) \in (\mathcal{P} \setminus C(S^1)) \times (0,1)$, and f restricted to $(\mathcal{P} \cap C(S^1)) \times (0,1)$ is also continuous. In order to complete the proof of the continuity of f, take a sequence $\{(A_n, t_n)\}_{n=1}^{\infty}$ in $(\mathcal{P} \setminus C(S^1)) \times (0,1)$ such that $\lim_{n\to\infty}(A_n, t_n) = (A,t)$ for some $(A,t) \in (\mathcal{P} \cap C(S^1)) \times (0,1)$.

Set $I = S^1 \setminus A$. Let $p = e(\alpha)$ be the middle point of I and r the length of I. Then $A = e([\alpha + \frac{r}{2}, \alpha + 2\pi - \frac{r}{2}])$. For each $n \in \mathbb{N}$, let $I_n, J_n, p_n = e(\alpha_n), q_n = e(\beta_n), r_n$ and s_n be as in the definition of $f(A_n, t_n)$. Since $\lim_{n\to\infty} A_n = A$, we may assume that $\lim_{n\to\infty} \mathrm{cl}_{S^1}(I_n) = \mathrm{cl}_{S^1}(I)$. Then $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} r_n = r$. We may assume that $\lim_{n\to\infty} \alpha_n = \alpha$ and we consider the case that for each $n \in \mathbb{N}$, $\alpha_n \leq \beta_n$ (the case $\beta_n \leq \alpha_n$ for each $n \in \mathbb{N}$ is similar). So $\beta_n \in [\alpha_n, \alpha_n + \pi]$ and the sequence $\{\beta_n\}_{n=1}^{\infty}$ is bounded. Since A is connected, we have that $\lim_{n\to\infty} s_n = 0$. Then

$$\begin{gathered} \lim_{n\to\infty} \alpha_n' = \\ \lim_{n\to\infty} \tfrac{1}{r_n + s_n}(\alpha_n r_n + \beta_n s_n - \tfrac{s_n}{\sigma(A_n)}((\beta_n - \alpha_n)t_n + \pi(\sigma(A_n) - t_n))) = \alpha, \end{gathered}$$

$$\lim_{n\to\infty} r'_n = \lim_{n\to\infty} \frac{r_n(2\pi - t_n)}{r_n + s_n} = 2\pi - t \text{ and } \lim_{n\to\infty} s'_n = \lim_{n\to\infty} \frac{s_n(2\pi - t_n)}{r_n + s_n} = 0.$$

Hence,

$$\lim_{n\to\infty} f(A_n, t_n) = \lim_{n\to\infty} e([\alpha'_n + \tfrac{r'_n}{2}, \beta'_n - \tfrac{s'_n}{2}] \cup [\beta'_n + \tfrac{s'_n}{2}, \alpha'_n + 2\pi - \tfrac{r'_n}{2}]) = e([\alpha + \pi - \tfrac{t}{2}, \alpha + \pi + \tfrac{t}{2}]) = f(A, t).$$

Therefore f is continuous.

In the following claim, we resume the properties that we will use of f.

Claim 4. For every $(A, t) \in \mathcal{P} \times (0, 1)$, $f(A, t) \in \mathcal{P}$, $\sigma(f(A, t)) = t$ and $f(f(A, t), \sigma(A)) = A$.

We prove Claim 4. In the case that $A \notin C(S^1)$, these properties follow from properties (k), (l) and (m) in Claim 1, and the corresponding properties when $\beta \leq \alpha$.

In the case that $A \in C(S^1)$, let $I = S^1 \setminus A$ and $p = e(\alpha)$ be the middle point of I. Then $f(A, t) = e([\alpha + \pi - \frac{t}{2}, \alpha + \pi + \frac{t}{2}])$. Thus $\sigma(f(A, t)) = t$ and $f(A, t) \in \mathcal{P}$. Note that the middle point of $S^1 \setminus f(A, t)$ is $e(\alpha)$ and the middle point of A is $e(\alpha + \pi)$. Then $f(f(A, t), \sigma(A)) = e([\alpha + \pi - \frac{\sigma(A)}{2}, \alpha + \pi + \frac{\sigma(A)}{2}]) = A$. This ends the proof of Claim 4.

Consider the suspension of $\mathcal{A}$ defined as the space obtained from $\mathcal{A} \times [0, 2\pi]$ by identifying the set $\mathcal{A} \times \{2\pi\}$ to a one-point set and $\mathcal{A} \times \{0\}$ to another one-point set. We denote this suspension by $\mathcal{B}$. Let $\psi : \mathcal{A} \times [0, 2\pi] \to \mathcal{B}$ and $\varphi : C_2(S^1) \to C_2(S^1)/F_2(S^1)$ be the quotient mappings, and denote by q_0 the point in $C_2(S^1)/F_2(S^1)$ such that $q_0 = \varphi(\{p, q\})$ for every $p, q \in X$.

Define $g : \mathcal{A} \times [0, 2\pi] \to C_2(S^1)/F_2(S^1)$ by

$$g(A, t) = \begin{cases} \varphi(S^1), & \text{if } t = 2\pi, \\ q_0, & \text{if } t = 0, \\ \varphi(f(A, t)), & \text{if } t \in (0, 2\pi). \end{cases}$$

Claim 5. $C_2(S^1)/F_2(S^1)$ is homeomorphic to $\mathcal{B}$.

First we prove that g is continuous. Take $(A, t) \in \mathcal{A} \times [0, 2\pi]$. If $t \in (0, 2\pi)$, clearly, g is continuous at (A, t).

Now suppose that $t = 0$. Take an open subset $\mathcal{U}$ of $C_2(S^1)/F_2(S^1)$ such that $q_0 \in \mathcal{U}$. Then $F_2(S^1)$ is contained in the open subset $\varphi^{-1}(\mathcal{U})$ of $C_2(S^1)$. Thus, there exists $\delta > 0$ such that if $B \in C_2(S^1)$ and $\sigma(B) < \delta$, then $B \in \varphi^{-1}(\mathcal{U})$. Given $u < \delta$ and $B \in \mathcal{A}$, by (l), $\sigma(f(B, u)) = u < \delta$, then $\varphi(f(B, u)) \in \mathcal{U}$. Hence the open subset $\mathcal{A} \times [0, \delta)$ of $\mathcal{A} \times [0, 2\pi]$ is a neighborhood of $(A, 0)$ contained in $g^{-1}(\mathcal{U})$. Therefore g is continuous at $(A, 0)$.

The continuity of g in the case that $t = 2\pi$ can be proved in a similar way. Observe that g preserves the fibers of ψ. We check that ψ preserves the fibers of g. Take $(A, t), (B, u) \in \mathcal{A} \times [0, 2\pi]$ such that $g(A, t) = g(B, u)$. If

$t \in (0, 2\pi)$, $g(A, t) = \varphi(f(A, t))$. By Claim 4, $\sigma(f(A, t)) = t$, so $f(A, t) \notin F_2(S^1) \cup \{S^1\}$, $g(B, u) = g(A, t) = \varphi(f(A, t)) \notin \{\varphi(S^1)\} \cup \{q_0\}$, so $u \in (0, 2\pi)$. Since $\varphi(f(A, t)) = \varphi(f(B, u))$, we have that $f(A, t) = f(B, u)$. By Claim 4, $t = \sigma(f(A, t)) = \sigma(f(B, u)) = u$. Since $A, B \in \mathcal{A}$, $\sigma(A) = \pi = \sigma(B)$. Applying Claim 4, we obtain that $A = f(f(A, t), \pi) = f(f(B, u), \pi) = B$. Hence $(A, t) = (B, u)$. Similarly, if $u \in (0, 2\pi)$, we obtain that $(A, t) = (B, u)$. In the case that $t = 0$, $g(B, u) = q_0$, so $u = 0$, and $\psi(A, t) = \psi(B, u)$. Similarly, if $t = 2\pi$, then $\psi(A, 2\pi) = \psi(B, 2\pi)$. Thus ψ preserves the fibers of g.

Now, we prove that g is onto. Take $E \in C_2(S^1)$ such that $E \notin F_1(S^1) \cup \{S^1\}$. Set $D = f(E, \pi)$. By Claim 4, $\sigma(D) = \pi$, so $D \in \mathcal{A}$. Moreover, Claim 4 also implies that $f(D, \sigma(E)) = f(f(E, \pi), \sigma(E)) = E$. Then $g(D, \sigma(E)) = \varphi(f(D, \sigma(E))) = \varphi(E)$. Therefore g is onto.

The Transgression Theorem [3, Theorem 3.2, Ch. VI] implies that $\mathcal{B}$ is homeomorphic to $C_2(S^1)/F_2(S^1)$. This finishes the proof of Claim 5. ■

Theorem 4 *$C_2(S^1)/F_2(S^1)$ is homeomorphic to the suspension of the solid torus.*

Corollary 5 *$hd(C_2(S^1)/F_2(S^1)) = 3$.*

3 Homogeneity degree of $C_2(S^1)/F_1(S^1)$

Throughout this section, let $\mathcal{Z} = C_2(S^1)/F_1(S^1)$ and let $\varphi : C_2(S^1) \to \mathcal{Z}$ be the quotient mapping. Let $Z_0 \in \mathcal{Z}$ be such that $\{Z_0\} = \varphi(F_1(S^1))$ and $Z_1 = \varphi(S^1)$. Note that the mapping

$$\varphi_0 = \varphi|_{C_2(S^1)\setminus F_1(S^1)} : C_2(S^1) \setminus F_1(S^1) \to \mathcal{Z} \setminus \{Z_0\}$$

is a homeomorphism.

For each $Z \in \mathcal{Z}$, let

$$o(Z) = \{A \in \mathcal{Z} : \text{there is a homeomorphism } h : \mathcal{Z} \to \mathcal{Z} \text{ such that } h(Z) = A\}.$$

Then $\mathrm{hd}(\mathcal{Z})$ is the cardinality of the pairwise distinct sets of the form $o(Z)$ with $Z \in \mathcal{Z}$.

Lemma 6 *Let X be a continuum such that $C(X)$ is contractible. Then $C_2(X) \setminus F_1(X)$ is contractible.*

Proof. By [13, Lemma 16.5], there exists a homotopy $D : 2^X \times [0, 1] \to 2^X$ such that:

(a) $D(A, 0) = A$ for each $A \in 2^X$,

(b) $D(A, 1) = X$ for each $A \in 2^X$, and

(c) if $A \in 2^X$ and $0 \leq s \leq t \leq 1$, then $D(A, s) \subset D(A, t)$.

By [2, Lemma 2.1], property (c) implies that $D(C_2(X) \times [0, 1]) \subset C_2(X)$, and if $A \notin F_1(X)$, then for every $t \in [0, 1]$, $D(A, t) \notin F_1(X)$. This implies that $D((C_2(X) \setminus F_1(X)) \times [0, 1]) = C_2(X) \setminus F_1(X)$. Therefore $C_2(X) \setminus F_1(X)$ is contractible. ■

Since φ_0 is a homeomorphism, Lemma 6 implies the following.

Lemma 7 *$\mathcal{Z} \setminus \{Z_0\}$ is contractible.*

Lemma 8 *$F_2(S^1)$ is a retract of $C_2(S^1) \setminus \{S^1\}$*

Proof. Define $r : C_2(S^1) \setminus \{S^1\} \to F_2(S^1)$ in the following way.

In the case that A is connected, set $I = S^1 \setminus A$. Let $p = e(\alpha)$ ($\alpha \in \mathbb{R}$) be the middle point of I and r the length of I. Then $A = e([\alpha + \frac{r}{2}, \alpha + 2\pi - \frac{r}{2}])$. Define

$$g(A) = e(\{\alpha + \tfrac{\pi}{2} + \tfrac{r}{4}, \alpha + \tfrac{3\pi}{2} - \tfrac{r}{4}\}).$$

In the case that A is not connected, the complement of A in S^1 consists of two open subarcs I and J. Let p and q be the respective middle points of I and J, and let r and s be the respective lengths of I and J, we choose I and J in such a way that $s \le r$. Let α and β be real numbers such that $|\alpha - \beta| \le \pi$, $p = e(\alpha)$ and $q = e(\beta)$.

In the case that $\alpha \le \beta$, define $a = \frac{\alpha r + (\beta - \pi)s}{r+s}$ and $g(A)$ by:

$$g(A) = e(\{a + \tfrac{\pi}{2} + \tfrac{r}{4} - \tfrac{s}{4}, a + \tfrac{3\pi}{2} + \tfrac{s}{4} - \tfrac{r}{4}\}).$$

In the case that $\beta \le \alpha$, define $a = \frac{(\alpha - \pi)r + \beta s}{r+s}$ and $g(A)$ by:

$$g(A) = e(\{a + \tfrac{\pi}{2} + \tfrac{s}{4} - \tfrac{r}{4}, a + \tfrac{3\pi}{2} + \tfrac{r}{4} - \tfrac{s}{4}\}).$$

Claim 1. g is well defined.

First we see that the definition of $g(A)$ does not change if we add the same integer multiple of 2π to both α and β. In order to do this it is enough to see that $g(A)$ does not change when we take $\alpha + 2\pi$ and $\beta + 2\pi$ instead of α and β, respectively. In the case that $\alpha \le \beta$, the definition of the number a for $\alpha + 2\pi$ and $\beta + 2\pi$ is equal to $\frac{(\alpha+2\pi)r + (\beta + 2\pi - \pi)s}{r+s} = \frac{\alpha r + (\beta - \pi)s}{r+s} + 2\pi$. Since a is the only term in the definition of $g(A)$ that changes when we consider $\alpha + 2\pi$ and $\beta + 2\pi$ instead of α and β, respectively, we conclude that, if $\alpha \le \beta$, then $g(A)$ does not change by adding 2π to both α and β. The case $\beta \le \alpha$ can be treated in a similar way.

In the case that $r = s$, it is possible to interchange the choice of I and J. Observe that in both cases $a = \frac{\alpha + \beta - \pi}{2}$, so α and β play symmetric roles. Moreover, in both cases, $g(A) = e(\{a + \frac{\pi}{2}, a + \frac{3\pi}{2}\})$. This shows that, when $r = s$, the definition of g does not depend on the choice of the open subarcs I and J.

Now, we see that the definition of $g(A)$ does not change in the case that it is possible to take numbers $\alpha_1 \le \beta_1$ and $\beta_2 \le \alpha_2$, for the same A. So suppose that there exist α_1, α_2, β_1 and β_2 such that $p = e(\alpha_1) = e(\alpha_2)$, $q = e(\beta_1) = e(\beta_2)$, $|\alpha_1 - \beta_1| \le \pi$, $|\alpha_2 - \beta_2| \le \pi$, $\alpha_1 \le \beta_1$, $\beta_2 \le \alpha_2$ and either $\alpha_1 \ne \alpha_2$ or $\beta_1 \ne \beta_2$. Since the definition of $g(A)$ does not change by adding the same integer multiple of 2π to both α and β, we may assume that $\alpha_1 = \alpha_2$.

Then $\beta_2 \in [\alpha_1 - \pi, \alpha_1), \beta_1 \in (\alpha_1, \alpha_1 + \pi]$ and $\beta_1, \beta_2 \in [\alpha_1 - \pi, \alpha_1 + \pi]$. Since $e(\beta_1) = e(\beta_2)$, we have that $\beta_2 = \alpha_1 - \pi$ and $\beta_1 = \alpha_1 + \pi$.

If we use α_1 and β_1, we obtain $a_1 = \frac{\alpha_1 r + (\beta_1 - \pi)s}{r+s} = \frac{\alpha_1 r + \alpha_1 s}{r+s} = \alpha_1$, and if we use α_2 and β_2, we obtain $a_2 = \frac{(\alpha_2 - \pi)r + \beta_2 s}{r+s} = \alpha_1 - \pi$. So, if we use α_1 and β_1, we obtain:

$$g(A) = e(\{\alpha_1 + \tfrac{\pi}{2} + \tfrac{r}{4} - \tfrac{s}{4}, \alpha_1 + \tfrac{3\pi}{2} + \tfrac{s}{4} - \tfrac{r}{4}\}).$$

On the other hand, if we use α_2 and β_2, we obtain:

$$g(A) = e(\{\alpha_1 - \pi + \tfrac{\pi}{2} + \tfrac{s}{4} - \tfrac{r}{4}, \alpha_1 - \pi + \tfrac{3\pi}{2} + \tfrac{r}{4} - \tfrac{s}{4}\}) =$$

$$e(\{\alpha_1 + \pi + \tfrac{\pi}{2} + \tfrac{s}{4} - \tfrac{r}{4}, \alpha_1 + \tfrac{\pi}{2} + \tfrac{r}{4} - \tfrac{s}{4}\}).$$

Thus both ways of defining $g(A)$ coincide.

Claim 2. g is continuous.

Clearly, g is continuous at the elements $A \in C_2(S^1) \setminus C(S^1)$, and g restricted to $C(S^1)$ is also continuous. In order to complete the proof of the continuity of g, take a sequence $\{A_n\}_{n=1}^{\infty}$ in $C_2(S^1) \setminus C(S^1)$ such that $\lim_{n\to} A_n = A$ for some $A \in C(S^1)$.

Set $I = S^1 \setminus A$. Let $p = e(\alpha)$ be the middle point of I and r the length of I. Then $A = e([\alpha + \frac{r}{2}, \alpha + 2\pi - \frac{r}{2}])$. For each $n \in \mathbb{N}$, let $I_n, J_n, p_n = e(\alpha_n), q_n = e(\beta_n), r_n$ and s_n be as in the definition of $g(A_n)$. Since $\lim_{n\to\infty} A_n = A$, we may assume that $\lim_{n\to\infty} \mathrm{cl}_{S^1}(I_n) = \mathrm{cl}_{S^1}(I)$. Then $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} r_n = r$. We may assume that $\lim_{n\to\infty} \alpha_n = \alpha$ and for each $n \in \mathbb{N}$, $\alpha_n \leq \beta_n$. So $\beta_n \in [\alpha_n, \alpha_n + \pi]$ and the sequence $\{\beta_n\}_{n=1}^{\infty}$ is bounded. Since A is connected, we have that $\lim_{n\to\infty} s_n = 0$. For each $n \in \mathbb{N}$, let $a_n = \frac{\alpha_n r_n + (\beta_n - \pi)s_n}{r_n + s_n}$

Then $\lim_{n\to\infty} a_n = \frac{\alpha_n r_n + (\beta_n - \pi)s_n}{r_n + s_n} = \alpha$, and

$$\lim_{n\to\infty} g(A_n) =$$
$$\lim_{n\to\infty} e(\{a_n + \tfrac{\pi}{2} + \tfrac{r_n}{4} - \tfrac{s_n}{4}, a_n + \tfrac{3\pi}{2} + \tfrac{s_n}{4} - \tfrac{r_n}{4}\}) = e(\{\alpha + \tfrac{\pi}{2} + \tfrac{r}{4}, \alpha + \tfrac{3\pi}{2} - \tfrac{r}{4}\}) = g(A)$$

Therefore g is continuous.

Claim 3. For each $A \in F_2(S^1)$, $g(A) = A$.

First we prove Claim 3. In the case that $A = \{w\}$ for some $w \in S^1$, set $I = S^1 \setminus A$, let $p = e(\alpha)$ ($\alpha \in \mathbb{R}$) be the middle point of I and r the length of I. Then $r = 2\pi$ and $p = -w$. Then

$$g(A) = e(\{\alpha + \tfrac{\pi}{2} + \tfrac{r}{4}, \alpha + \tfrac{3\pi}{2} - \tfrac{r}{4}\}) = e(\{\alpha + \pi\}) = \{w\} = A.$$

Now suppose that $A = \{w, z\}$, for some $z \neq w$. Let I, J, r, s, p, q, α and β be as in the definition of g. We consider the case that $\alpha \leq \beta$, the other case is similar. Observe that $r + s = 2\pi$, $p = -q$, $\beta = \alpha + \pi$, $e(\alpha + \frac{r}{2}) = e(\beta - \frac{s}{2})$ and $e(\beta + \frac{s}{2}) = e(2\pi + \alpha - \frac{r}{2})$. We may assume that $e(\alpha + \frac{r}{2}) = w$ and $e(2\pi + \alpha - \frac{r}{2}) = z$. Since $\beta = \alpha + \pi$, we have that $a = \alpha$. Observe that $\pi + \frac{r}{2} - \frac{s}{2} = r$, and $3\pi + \frac{s}{2} - \frac{r}{2} = 4\pi - r$. Then

$$g(A) = e(\{a+\tfrac{\pi}{2}+\tfrac{r}{4}-\tfrac{s}{4}, a+\tfrac{3\pi}{2}+\tfrac{s}{4}-\tfrac{r}{4}\}) = e(\{\alpha+\tfrac{r}{2}, \alpha+2\pi-\tfrac{r}{2}\}) = \{w,z\} = A.$$

This finishes the proof of Claim 3 and the proof of the lemma. ■

Let $\mathcal{P} = C_2(S^1) \setminus \{S^1\}$, $\mathcal{Q} = \mathcal{Z} \setminus \{\varphi(S^1)\} = \mathcal{Z} \setminus \{Z_1\}$, $\mathcal{R} = \varphi(F_2(S^1))$ and

$$\psi = \varphi|_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}.$$

Lemma 9 *$\mathcal{R}$ is a retract of $\mathcal{Q}$.*

Proof. Since φ is an identification, $\mathcal{P}$ is open in $C_2(S^1)$ and $\varphi^{-1}(\varphi(\mathcal{P})) = \mathcal{P}$, we have that ψ is an identification.

Observe that Theorem 4.3 of [3, Ch. VI] can be applied to obtain a mapping g_* that makes commutative the following diagram.

$$\begin{array}{ccc} \mathcal{P} & \overset{g}{\rightarrow} & F_2(S^1) \\ \psi \ \downarrow & & \downarrow \ \psi \\ \mathcal{Q} & \overset{g_*}{\rightarrow} & \mathcal{R}. \end{array}$$

Given $A \in \mathcal{R}$, there exists $B \in F_2(S^1)$ such that $A = \psi(B) = \psi(g(B))$. Then $g_*(A) = g_*(\psi(B)) = \psi(g(B)) = \psi(B) = A$. Thus, for each $A \in \mathcal{R}$, $g_*(A) = A$. This proves that g_* is a retraction. ■

Lemma 10 *$\mathcal{Q}$ is not contractible and then $o(Z_0) \neq o(Z_1)$.*

Proof. Suppose to the contrary that $\mathcal{Q}$ is contractible, since $\mathcal{R}$ is a retract of $\mathcal{Q}$, $\mathcal{R}$ is contractible. It is known [10, p. 53 and 54] that $F_2(S^1)$ is a Möbius strip having $F_1(S^1)$ as its manifold boundary. Since $\varphi|_{F_2(S^1)} : F_2(S^1) \to \varphi(F_2(S^1))$ is an identification (it is a closed function), the Transgression Theorem [3, Theorem 3.2, Ch. VI] implies that $\mathcal{R} = \varphi(F_2(S^1))$ is homeomorphic to the quotient space $F_2(S^1)/F_1(S^1)$. Since $F_2(S^1)/F_1(S^1)$ is homeomorphic to the 2-dimensional real projective space $\mathbb{RP}^2$, this implies that $\mathbb{RP}^2$ is contractible. This is a contradiction since the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}_2$. Therefore $\mathcal{Q}$ is not contractible. Thus, Lemma 7 implies that $o(Z_0) \neq o(Z_1)$. ■

Lemma 11 *Z_0 does not have a 4-cell neighborhood in $\mathcal{Z}$.*

Proof. Suppose to the contrary that there exists a 4-cell neighborhood $\mathcal{M}$ of Z_0 in $\mathcal{Z}$. Take a homeomorphism $f : F_1(S^1) \to S^1$. Since S^1 is an ANR, there exist an open subset $\mathcal{U}$ of $C_2(S^1)$, containing $F_1(S^1)$ and a mapping $F : \mathcal{U} \to S^1$ that extends f. We may assume that $S^1 \notin \mathcal{U}$. Since $\varphi^{-1}(\varphi(\mathcal{U})) = \mathcal{U}$, $\varphi(\mathcal{U})$ is open in $\mathcal{Z}$. Then there exists a 4-cell neighborhood $\mathcal{K}$ of Z_0 in $\mathcal{Z}$ such that $\mathcal{K} \subset \mathcal{M} \cap \varphi(\mathcal{U})$. Let $\mathcal{W} = \varphi^{-1}(\mathcal{K}) \subset \mathcal{U}$. Then $F_1(S^1) \subset \mathrm{Int}_{C_2(S^1)}(\mathcal{W})$. Thus there exists $\delta > 0$ such that $\delta < \frac{\pi}{4}$ and for each $t \in [0, 2\pi]$, $F_2(e([t, t+\delta])) \subset \mathcal{W} \subset \mathcal{U}$. Set $\mathcal{S}_0 = \{\{e(t), e(t+\delta)\} : t \in [0, 2\pi]\}$. Observe that $\mathcal{S}_0$ is a simple closed curve contained in $\mathcal{W}$. Let $L : S^1 \times [0,1] \to \mathcal{W}$ be defined by $L(z, s) = \{e(t), e(t+\delta s)\}$

(where $z = e(t)$ and $t \in [0, 2\pi]$). Then L is a homotopy between the mappings L_0 and L_1 given by $L_0(z) = \{z\}$ and $L_1(z) = \{e(t), e(t + \delta)\}$. Since $f \circ L_0$ is a homeomorphism, $F \circ L_0 = f \circ L_0$ is not homotopic to a constant mapping. Therefore $F \circ L_1$ is not homotopic to a constant mapping.

By [4, Théorème 1] does not exist a mapping $\sigma : S^1 \to \mathbb{R}$ such that $e \circ \sigma = F \circ L_1$.

On the other hand, since $\varphi_0 : C_2(S^1) \setminus F_1(S^1) \to \mathcal{Z} \setminus \{Z_0\}$ is a homeomorphism and $\varphi(\mathcal{W}) = \mathcal{K}$, we obtain that $\mathcal{W} \setminus F_1(S^1)$ is homeomorphic to $\mathcal{K} \setminus \{Z_0\}$. Since $\mathcal{K}$ is a 4-cell, $\mathcal{K} \setminus \{Z_0\}$ is unicoherent. Hence $\mathcal{W} \setminus F_1(S^1)$ is also unicoherent. By [4, Théorème 3], there exists a mapping $\Sigma : \mathcal{W} \setminus F_1(S^1) \to \mathbb{R}$ such that $F|_{\mathcal{W} \setminus F_1(S^1)} = e \circ \Sigma$. Since $\mathcal{S}_0 \subset \mathcal{W} \setminus F_1(S^1)$, we obtain that $F \circ L_1 = e \circ \Sigma \circ L_1$. Making $\sigma = \Sigma \circ L_1$, we obtain a contradiction with the previous paragraph. This finishes the proof of the lemma. ■

Set

$$\mathcal{A}_1 = \{A \in \mathcal{Z} : A \text{ has a 4-cell neighborhood } \mathcal{M} \text{ in } \mathcal{Z} \text{ such that } A \text{ is in the manifold boundary of } \mathcal{M}\}, \text{ and}$$
$$\mathcal{A}_2 = \{A \in \mathcal{Z} : A \text{ has a 4-cell neighborhood } \mathcal{M} \text{ in } \mathcal{Z} \text{ such that } A \text{ is in the manifold interior of } \mathcal{M}\}.$$

Lemma 12 *$\mathcal{Z} = \{Z_0, Z_1\} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ and the sets $\{Z_0\}$ and $\{Z_1\}$ are equivalence classes of $\cong$ in $\mathcal{Z}$.*

Proof. By [15, Lemma 3.17], each $A \in C_2(S^1) \setminus \{S^1\}$ has a 4-cell neighborhood in $C_2(S^1)$ and the element $Z_1 = \varphi(S^1)$ does not have a 4-cell neighborhood in $\mathcal{Z}$. Hence $Z_1 \notin \mathcal{A}_1 \cup \mathcal{A}_2$. Recall that we are denoting the cone over the solid torus by $\mathcal{C}^1$ and its vertex by v. Since every point in $\mathcal{C}^1 \setminus \{v\}$ has a 4-cell neighborhood in $\mathcal{C}^1$, we conclude that each homeomorphism from $C_2(S^1)$ onto $\mathcal{C}^1$ sends S^1 to v. Since $F_1(S^1)$ is closed in $C_2(S^1)$, each element $A \in C_2(S^1) \setminus (S^1 \cup F_1(S^1))$ has a 4-cell neighborhood in $\mathcal{P}$. Since $C_2(S^1) \setminus (\{S^1 \cup F_1(S^1)\})$ is homeomorphic to $\mathcal{Z} \setminus \{Z_0, Z_1\}$, each element in $\mathcal{Z} \setminus \{Z_0, Z_1\}$ belongs to $\mathcal{A}_1 \cup \mathcal{A}_2$. Therefore, $\mathcal{Z} = \{Z_0, Z_1\} \cup \mathcal{A}_1 \cup \mathcal{A}_2$.

By Lemma 11, $Z_0 \notin \mathcal{A}_1 \cup \mathcal{A}_2$. Thus Lemma 10 implies that each of the sets $\{Z_0\}$ and $\{Z_1\}$ is an equivalence class of $\cong$. ■

Lemma 13 *$\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalence classes of $\cong$ in $\mathcal{Z}$.*

Proof. Clearly, $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Take two elements $A_1, B_1 \in \mathcal{A}_1$, we are going to show that $A_1 \cong B_1$. Since $A_1, B_1 \notin \{Z_0, Z_1\}$, there exist unique elements $A, B \in \mathcal{P}$ such that $\varphi(A) = A_1$ and $\varphi(B) = B_1$. Since $\varphi|_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}$ is a homeomorphism each of the sets A and B has a 4-cell neighborhood in $\mathcal{P}$ containing it in its manifold boundary.

Since $C_2(S^1) \setminus \{S^1\}$ is homeomorphic to $\mathcal{C}^1 \setminus \{v\}$, we have that $C_2(S^1) \setminus \{S^1\}$ is homeomorphic to $\mathcal{T} \times [0, 1)$ ($\mathcal{T}$ is the solid torus). Since $\mathcal{T}$ is a 3-dimensional manifold with boundary, we obtain that $\mathcal{T} \times [0, 1)$ and $C_2(S^1) \setminus \{S^1\}$ are 4-dimensional manifolds with boundary. Let $\mathcal{F}$ be the manifold boundary of

$C_2(S^1) \setminus \{S^1\}$. Observe that $\mathcal{F}$ is homeomorphic to $(\mathcal{T} \times \{0\}) \cup ((S^1 \times S^1) \times [0, 1)) \subset \mathcal{T} \times [0, 1)$. Thus $\mathcal{F}$ is a connected 3-dimensional manifold (in fact, it is possible to show that $\mathcal{F}$ is homeomorphic to $D_0 \times S^1$, where D_0 is the open disc in the plane given by $D_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$). Since $A, B \in C_2(S^1) \setminus \{S^1\}$, we conclude that $A, B \in \mathcal{F}$. Since $F_1(S^1)$ is homeomorphic to S^1, from Corollary to Theorem IV.4 of [7], it is possible to prove that $\mathcal{F} \setminus F_1(S^1)$ is connected.

Set $\mathcal{G} = (\mathcal{T} \times \{0\}) \cup ((S^1 \times S^1) \times [0, 1))$. Given a point $p \in \mathcal{G}$ and an open neighborhood U of p in $\mathcal{G}$, observe that there exist a 4-cell $M_p \subset \mathcal{T} \times [0, 1)$ and a homeomorphism $\varphi_p : [0, 1]^4 \to M_p$ such that $p = \varphi_p((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0))$, $\varphi_p([0, 1]^3 \times \{0\}) = M_p \cap \mathcal{G} \subset U$, $Fr_{\mathcal{T} \times [0,1)}(M_p) = \varphi_p(($manifold boundary of $[0, 1]^4) \setminus ((0, 1)^3 \times \{0\})))$ and $\varphi_p((0, 1)^3 \times \{0\})$ is open in $\mathcal{G}$. Thus, for each element $D \in \mathcal{F}$ (respectively, $\mathcal{F} \setminus F_1(S^1)$) and each open neighborhood $\mathcal{U}$ of D in $\mathcal{F}$ (respectively, $\mathcal{F} \setminus F_1(S^1)$), there exist a 4-cell $\mathcal{M}_D \subset C_2(S^1) \setminus \{S^1\}$ (respectively, $\mathcal{M}_D \subset \mathcal{P}$) and a homeomorphism $\varphi_D : [0, 1]^4 \to \mathcal{M}_D$ such that $D = \varphi_D((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0))$, $\varphi_D([0, 1]^3 \times \{0\}) = \mathcal{M}_D \cap \mathcal{F} \subset \mathcal{U}$ (respectively, $\mathcal{M}_D \cap \mathcal{F} \setminus F_1(S^1) \subset \mathcal{U}$), $Fr_P(\mathcal{M}_D) = \varphi_D(($manifold boundary of $[0, 1]^4) \setminus ((0, 1)^3 \times \{0\})))$ and $\varphi_D((0, 1)^3 \times \{0\})$ is open in $\mathcal{F}$ (respectively, in $\mathcal{F} \setminus F_1(S^1)$).

Since $\mathcal{V} = \mathcal{F} \setminus F_1(S^1)$ is connected, it is possible to find $m \in \mathbb{N}$ and elements $D_1, \ldots, D_m \in \mathcal{F} \setminus F_1(S^1)$ such that for the 4-cells $\mathcal{M}_{D_1}, \ldots, \mathcal{M}_{D_m} \subset P$ and the homeomorphisms $\varphi_1, \ldots, \varphi_m$, described in the previous paragraph, we have the following properties: $A = D_1$, $B = D_m$, and for each $i \in \{1, \ldots, m-1\}$, there exists an element $E_i \in \varphi_{D_i}((0, 1)^3 \times \{0\}) \cap \varphi_{D_{i+1}}((0, 1)^3 \times \{0\})$.

Since $A, E_1 \in \varphi_{D_1}((0, 1)^3 \times \{0\})$, it is easy to see that there is a homeomorphism $g : \varphi_{D_1}([0, 1]^4) \to \varphi_{D_1}([0, 1]^4)$ such that $g(A) = E_1$ and for every $D \in \varphi_{D_1}(($manifold boundary of $[0, 1]^4) \setminus ((0, 1)^3 \times \{0\}))) = Fr_{\mathcal{P}}(\mathcal{M}_{D_1})$, $g(D) = D$. Since $\psi(A) = A_1$ and $\varphi_{D_1}([0, 1]^4) = \mathcal{M}_{D_1}$, the homeomorphism $\sigma = \psi \circ g \circ ((\psi|_{\mathcal{Q}})^{-1})|_{\psi(\mathcal{M}_{D_1})} : \psi(\mathcal{M}_{D_1}) \to \psi(\mathcal{M}_{D_1})$, satisfies $\sigma(A_1) = \psi(E_1)$ and for every $D \in \psi(Fr_{\mathcal{P}}(\mathcal{M}_{D_1})) = Fr_{\mathcal{Q}}(\psi(\mathcal{M}_{D_1})) = Fr_{\mathcal{Z}}(\psi(\mathcal{M}_{D_1}))$, $\sigma(D) = D$. Thus defining $G : Z \to Z$ by:

$$G(D) = \begin{cases} \sigma(D), & D \in \psi(\mathcal{M}_{D_1}), \\ D, & D \in \mathcal{Z} \setminus \psi(\mathcal{M}_{D_1}), \end{cases}$$

is a homeomorphism such that $G(A_1) = \psi(E_1)$.

We have shown that $A_1 \cong E_1$. Proceeding as in the previous paragraph, using $\varphi_{D_2}, \ldots \varphi_{D_m}$ instead of φ_{D_1}, we obtain that $E_1 \cong E_2 \cong \cdots \cong E_m \cong B_1$. Thus $A_1 \cong B_1$.

This completes the proof that $\mathcal{A}_1$ is an equivalence class of $\cong$ in $\mathcal{Z}$.

In a similar way, it is possible to prove that $\mathcal{A}_2$ is an equivalence class of $\cong$ in $\mathcal{Z}$. ■

Combining Lemmas 12 and 13, we obtain the following.

Theorem 14 $hd(C_2(S^1)/F_1(S^1))) = 4$.

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A. Illanes and Verónica Martínez-de-la-Vega, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México 04510, Ciudad de México.

Email addresses: illanes@matem.unam.mx and vmvm@matem.unam.mx

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