# On the existence of subharmonic and homoclinic solutions for a class of second order quasilinear Schrödinger equations*

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**Abstract.** Using critical point theory, we study the existence of subharmonic and homoclinic solutions for a class of second order quasilinear Schrödinger equations

$$u''(t) - V(t)u(t) + \lambda(u^2(t))''u(t) + g(u(t)) = h(t),$$

where $g(t,u)$ depends periodically on $t$ and is superquadratic.

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## 1 Introduction

The purpose of this paper is to study the existence of subharmonic and homoclinic solutions for the following equation

$$u''(t) - V(t)u(t) + \lambda(u^2(t))''u(t) + g(t,u(t)) = h(t), \tag{\textbf{HS}}$$

where $t \in \mathbf{R}$, $\lambda \geq 0$ is a parameter, $u \in \mathbf{R}^n$, $g(t,u) \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$, $g(t,0) = 0$ and is $T-$periodic in $t$, $V(t) > 0$ is a real continuous functions defined on $\mathbf{R}$ with period $T$.

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In this paper we will consider the existence of subharmonic and homoclinic solutions when $V(t) = \alpha$ ($\alpha > 0$ a constant). However we will keep the general $V(t)$ to set up the variational structure.

When $n = 1$, $\lambda = 1$ and $h(t) \equiv 0$, we obtain the following equation from (**HS**)

$$u^{''}(t) - V(t)u(t) + (u^2(t))^{''}u(t) + g(t, u(t)) = 0, \tag{\textbf{SC}}$$

which is a quasilinear Schrӧdinger equation with dimension 1.

In the literature, many authors studied soliton solutions or ground state solutions for quasilinear Schrӧdinger equations via critical point theory and the Pohožaev manifold method [9, 15, 16, 18-21, 23, 29, 31-35]. Without the nonlinearity term $(u^2(t))^{''}u(t)$, equation (**HS**) becomes the Hamiltonian system, the homoclinic orbits of which has been studied by several authors via critical point theory, see [1, 3-8, 10-12, 22, 25-28, 30].

In this paper, using critical point theory, we will establish the existence of subharmonic and homoclinic solutions for a class of second order quasilinear Schrodinger equations. To this end, let us first introduce some basic concepts on these equations.

As usual, a solution $u$ of (**HS**) is said to be homoclinic (to 0) if $u(t) \to 0$ as $t \to \infty$. In addition, if $u \not\equiv 0$ then $u$ is called a nontrivial homoclinic solution of (**HS**).

Let $h(t) \equiv 0$, we have from (**HS**)

$$u^{''}(t) - V(t)u(t) + \lambda(u^2(t))^{''}u(t) + g(t, u(t)) = 0. \tag{1.1}$$

A solution $u$ of (1.1) is said to be subharmonic if $u$ is $kT-$periodic for any positive integer $k$ (see [25]).

This study is motivated mainly by [23] and [25]. In [25], Rabinowitz obtained the existence of nontrivial homoclinic solutions for the second order Hamiltonian system

$$\ddot{q} + V_q(t, q) = 0, \tag{1.2}$$

where the homoclinic orbit $q$ is obtained as the limit as $k \to \infty$ of $2kT$-periodic solutions (i.e. subharmonic) $q_k$ of (1.2).

In [23], Poppenberg, Schmitt and Wang proved the existence of soliton solutions for the following quasilinear Schrӧdinger equations

$$-\triangle u + V(x)u - \kappa(\triangle|u|^2)u = \nu|u|^{p-1}u. \tag{1.3}$$

The solution of (1.3) is related to the existence of standing wave solutions for the quasilinear Schrӧdinger equation

$$i\partial_t z = -\triangle z + V(x)z - f(x, |z|^2)z - \kappa\triangle\varphi(|z|^2)\varphi^{'}(|z|^2)z, \tag{1.4}$$

where $V = V(x), x \in \mathbf{R^N}$ is a given potential, $\kappa$ is a real constant and $f$, $\varphi$ are real functions. The quasilinear equation (1.4) arises in several models of different physical phenomena corresponding to various types of $\varphi$. For instance, the superfluid film equation in plasma physics

$$i\partial_t z = -\triangle z + V(x)z - f(x, |z|^2)z - \kappa(\triangle|z|^2)z$$

has this structure with $\varphi(s) = s$.

Seeking solutions of the type of stationary waves, namely, the solutions of the form

$$z(t, u) = exp(-iFt)u(x), \quad F \in \mathbf{R},$$

we get an equation of elliptic type from (**1.4**) which has the formal structure

$$-\triangle u + \tilde{V}(x)u - (\triangle|u|^2)u = g(x, u) \quad x \in \mathbf{R^N} \tag{1.5}$$

with $\varphi(s) = s$ and $\kappa = 1$, where $\tilde{V}(x) = V(x) - F$ is the new potential function and $g(x, u) = f(x, u^2)u$.

In this paper, by using the idea of [25] and [13, 20], we will study subharmonic and homoclinic solutions of equation (**1.1**) and the general one, i.e., equation (**HS**). As the main tools in our study, three lemmas will be stated here. First, let us recall the Palais-Smale condition. Let $E$ be a real Banach space, $I \in C^1(E, \mathbf{R})$, i.e., $I$ is a continuously Fréchet-differentiable functional defined on $E$. Now $I$ is said to satisfy the Palais-Smale condition (**PS** condition for short) if any sequence $\{u_n\} \subset E$ for which $I(u_n)$ is bounded and $I'(u_n) \to 0 (n \to \infty)$ possesses a convergent subsequence in $E$.

Let $B_\rho(0)$ denote the open ball in E with radius $\rho$ and with center 0 and let $\partial B_\rho$ denote its boundary.

**Lemma 1.1(Mountain Pass lemma)([2,24]).** Let $E$ be a real Banach space and $I \in C^1(E, \mathbf{R})$ satisfies the **PS** condition. If further $I(0) = 0$, and
$(G_1)$ there exist constants $\rho$, $\alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$, and
$(G_2)$ there exists $e \in E \backslash \overline{B_\rho(0)}$ such that $I(e) \leq 0$,
then $I$ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{\eta \in \Gamma} \max{}_{s \in [0,1]} I(\eta(s)),$$

where

$$\Gamma = \{\eta \in C([0, 1], E) | \eta(0) = 0, \ \eta(1) = e\}.$$

**Lemma 1.2(Symmetric Mountain Pass lemma)([24]).** Let E be an infinite dimensional Banach space and let $I \in C^1(E, \mathbf{R})$ be even, satisfying the **PS** condition and $I(0) = 0$. If

$E = V \oplus X$, where $V$ is finite dimensional, and

$(G_3)$ there exist constants $\rho$, $\alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$, and

$(G_4)$ for each finite dimensional subspace $\widetilde{E} \subset E$, there is a $\gamma = \gamma(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \backslash B_\gamma$,

then $I$ possesses an unbounded sequence of critical values.

**Lemma 1.3 ([15]).** Let $(X, \|\cdot\|)$ be a Banach space and $J \in \mathbf{R}_+$ an interval. Consider the family of $C^1$ functionals on $X$

$$I_\mu(u) = A(u) - \mu B(u), \quad \mu \in J,$$

with $B$ nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $I_\mu(0) = 0$.

For any $\mu \in J$ we set

$$\Gamma_\mu = \{\gamma \in C([0,1], X) : \quad \gamma(0) = 0, \quad I_\mu(\gamma(1)) < 0\}.$$

If for every $\mu \in J$ the set $\Gamma_\mu$ is nonempty and

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max{}_{t \in [0,1]} I_\mu(\gamma(t)) > 0,$$

then for almost every $\mu \in J$ there is a sequence $\{u_n\} \subset X$ such that

$(i)$ $\{u_n\}$ is bounded;

$(ii)$ $I_\mu(u_n) \to c_\mu$;

$(iii)$ $I_\mu^{'}(u_n) \to 0$ in the dual $X^{-1}$ of $X$.

The rest of this paper is organized as follows. In Section 2, we establish a variational structure for (**HS**) with periodic boundary value condition, and give some preliminary results. In Section 3 we prove a first existence result for equation (**1.1**) without the Ambrosetti-Rabinowitz growth condition. A cut-off functional is utilized to obtain the bounded Palais-Smale sequences. In Section 4, by employing the Mountain Pass lemma and the symmetric one, we show a second and a third existence result for (**HS**) under suitable assumptions.

## 2 Variational structure

For each $k \in \mathbf{N}$, let $E_k := W^{2,2}_{2kT}(\mathbf{R}, \mathbf{R}^n)$, the Hilbert space of $2kT-$periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|u\|_{E_k} := \left(\int_{-kT}^{kT} [|u(t)|^2 + |u^{'}(t)|^2] dt\right)^{\frac{1}{2}}.$$

Furthermore, let $L^\infty_{[-kT,kT]}(\mathbf{R}, \mathbf{R}^n)$ denote the space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbf{R}$ into $\mathbf{R}^n$ equipped with the norm

$$\|u\|_{L^\infty_{[-kT,kT]}} := ess \sup\{|u(t)| : t \in [-kT, kT]\},$$

and $L^2_{[-kT,kT]}(\mathbf{R},\mathbf{R}^n)$ denotes the Hilbert space of $2kT-$periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|u\|_{L^2_{[-kT,kT]}} = (\textstyle\int_{-kT}^{kT} |u(t)|^2 dt)^{\frac{1}{2}}.$$

As in [13], where the homoclinic solution of (**HS**) is obtained as a limit of a certain sequence of functions $\{u_k\} \subset E_k$, we consider a sequence of systems of functional differential equations

$$u^{''}(t) - V(t)u(t) + \lambda(u^2(t))^{''}u(t) + g(t,u(t)) = h_k(t), \qquad (\mathbf{HS_k})$$

where for each $k \in \mathbf{N}$, $h_k : \mathbf{R} \to \mathbf{R}^n$ is a $2kT-$periodic extension of the restriction of $h$ to the interval $[-kT, kT]$. Denote by $u_k$ the $2kT-$periodic solution of ($\mathbf{HS_k}$) obtained via Mountain Pass Lemma.

Throughout the whole paper we impose the following assumption:

$(G_0)$ there exists a continuously differentiable function $G(t,u) \in C(\mathbf{R}\times\mathbf{R}^n, \mathbf{R})$ being $T-$periodic with respect to $t$, such that $\nabla_u G = g$.

Let

$$\Phi_k(u) = \left(\textstyle\int_{-kT}^{kT}[|u^{'}(t)|^2 + V(t)u^2(t)]dt\right)^{\frac{1}{2}}. \tag{2.1}$$

It is easy to see that there exist $M_1, M_2 > 0$ such that

$$M_1\|u\|^2_{E_k} \le \Phi^2_k(u) \le M_2\|u\|^2_{E_k}. \tag{2.2}$$

Indeed, let $\widehat{V} = \max_{t\in[0,T]} V(t)$, $\overline{V} = \min_{t\in[0,T]} V(t)$, $M_1 := \min\{1, \overline{V}\}$ and $M_2 := \max\{1, \widehat{V}\}$. Then we have

$$\Phi^2_k(u) \le M_2 \int_{-kT}^{kT} [|u(t)|^2 + |u^{'}(t)|^2]dt = M_2\|u\|^2_{E_k}$$

and

$$\Phi^2_k(u) = \int_{-kT}^{kT} [|u^{'}(t)|^2 + V(t)|u(t)|^2]dt \ge \min\{1, \overline{V}\} \int_{-kT}^{kT} [|u^{'}(t)|^2 + |u(t)|^2]dt = M_1\|u\|^2_{E_k}.$$

Define $I_k : E_k \to \mathbf{R}$ as follows

$$I_k(u) = \tfrac{1}{2}\Phi^2_k(u) + \lambda \textstyle\int_{-kT}^{kT} |u^{'}(t)|^2 u^2(t)dt - \int_{-kT}^{kT} G(t,u(t)) + (h_k(t), u(t))\, dt. \tag{2.3}$$

Then $I_k \in C^1(E_k, \mathbf{R})$. From $(G_0)$, it is easy to check that

$$(I^{'}_k(u), v) = \textstyle\int_{-kT}^{kT} \left(-u^{''}(t) + V(t)u(t) - \lambda(u^2(t))^{''}u(t) - g(t,u(t)) - h_k(t), v(t)\right) dt \tag{2.4}$$

and the corresponding Euler equation of functional $I_k$ is Eq.($\mathbf{HS_k}$). Moreover, it is clear that the critical points of $I_k$ are classical $2kT-$periodic solutions of ($\mathbf{HS_k}$).



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In the end of this section, we shall state an important result which will be used in the proof of our main results.

**Proposition 2.1 ([25, pg 36]).** There is a positive constant $\gamma$ such that for each $k \in \mathbf{N}$ and $u \in E_k$, the following inequality holds:

$$\|u\|_{L^\infty_{[-kT,kT]}} \leq \gamma \|u\|_{E_k}. \tag{2.5}$$

## 3 Existence result (I)

In this section we prove that equation (1.1) has a nonconstant homoclinic solution without the Ambrosetti-Rabinowitz growth condition, and a cut-off functional is utilized to obtain the bounded Palais-Smale sequences.

We assume the following conditions:

$(V_1)$ $V(t) = \alpha > 0$, where $\alpha$ is a constant;

$(V_2)$ $g(t,u) = g(u)$ and $|g(u)| \leq C(|u| + |u|^{p-1})$ for some $p \in (2, \infty)$, where C is a positive constant;

$(V_3)$ $\lim\limits_{u \to 0} |\frac{g(u)}{u}| = 0$;

$(V_4)$ $\lim\limits_{u \to \infty} |\frac{g(u)}{u}| = \infty$.

Clearly, under assumption $(V_2)$, $(G_0)$ becomes:

$(\widetilde{G}_0)$ there exists a continuously differentiable function $G \in C(\mathbf{R}, \mathbf{R})$ such that $G'(u) = g$.

**Theorem 3.1.** Suppose that conditions $(V_1) - (V_4)$ and $(\widetilde{G}_0)$ are satisfied. Then equation (1.1) with $\lambda$ sufficiently small possesses a nontrivial homoclinic solution which is the limit as $k \to \infty$ of a sequence of solutions of $(\mathbf{HS}_k)$ under the periodic boundary condition

$$u^{(i)}(kT) = u^{(i)}(-kT) = 0, \text{ for all } \imath = 0, 1.$$

To overcome the difficulty of finding bounded Palais-Smale sequences for the associated functional $I_k$, following [16-18], we use a cut-off function $\psi \in C^\infty(\mathbf{R}_+, \mathbf{R})$ satisfying

$$\begin{cases} \psi(t) = 1, \quad t \in [0,1], \\ 0 \leq \psi(t) \leq 1, \quad t \in (1,2), \\ \psi(t) = 0, \quad t \in [2, \infty), \\ \|\psi'\|_\infty \leq 2, \end{cases}$$

and study the following modified functional $I_k^{\varpi,\mu}: E_k \to \mathbf{R}$ defined by

$$\begin{aligned} I_k^{\varpi,\mu}(u) &= \frac{1}{2}\int_{-kT}^{kT}[|u'(t)|^2 + V(t)u^2(t)]dt + \lambda\psi\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)\int_{-kT}^{kT}|u'(t)|^2u^2(t)dt \\ &\quad - \int_{-kT}^{kT}(h_k(t), u(t))\,dt - \mu\int_{-kT}^{kT}G(u(t))dt \\ &:= A_k(u) - \mu B_k(u), \end{aligned} \tag{3.1}$$

where $\varpi > 0$ is a constant,

$$A_k(u) = \frac{1}{2}\int_{-kT}^{kT}[|u'(t)|^2 + V(t)u^2(t)]dt + \lambda\psi\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)\int_{-kT}^{kT}|u'(t)|^2u^2(t)dt - \int_{-kT}^{kT}(h_k(t), u(t))\,dt$$

and

$$B_k(u) = \int_{-kT}^{kT}G(u(t))dt.$$

It follows that

$$\begin{aligned} ((I_k^{\varpi,\mu}(u))', v) &= \int_{-kT}^{kT}\left(-u''(t) + V(t)u(t) - \lambda\psi\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)(u^2(t))''u(t) - h_k(t) - \mu g(u(t)), v(t)\right)dt \\ &\quad + \frac{2\lambda}{\varpi^2}\psi'\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)\left(\int_{-kT}^{kT}|u'(t)|^2u^2(t)dt\right)\int_{-kT}^{kT}[u'(t)v'(t) + u(t)v(t)]dt, \end{aligned} \tag{3.2}$$

For any $\mu \in J$, we set

$$\Gamma_\mu^{(k)} = \{\gamma \in C([0,1], E_k): \quad \gamma(0) = 0, \quad I_k^{\varpi,\mu}(\gamma(1)) < 0\}$$

and

$$c_\mu^{(k)} = \inf_{\gamma\in\Gamma_\mu^{(k)}} \max{}_{t\in[0,1]} I_k^{\varpi,\mu}(\gamma(t)).$$

With this penalization $\psi$, for $\varpi > 0$ sufficiently large and for $\lambda$ sufficiently small, we are able to find a critical point $u$ of $I_k^{\varpi,\mu}$ such that $\|u\|_{E_k} \le \varpi$ and thus $u$ is also a critical point of $I_k$.

**Lemma 3.2.** $\Gamma_\mu^{(k)} \ne \emptyset$ for all $\mu \in J = [\xi, 1]$, where $\xi \in (0,1)$ is a positive constant.

**Proof.** We choose $\phi \in E_k$ with $\phi(u) \ge 0$, $\|\phi\|_{E_k} = 1$ and $supp(\phi) \subset B(0,R)$ for some $0 < R < kT$. By $(V_4)$, we have that for any $C_1 > 0$ with $C_1\xi\int_{-R}^{R}\phi^2(t)dt > \frac{M_2}{2}$, there exists $C_2 > 0$ such that

$$G(\theta) \ge C_1|\theta|^2 - C_2, \quad \theta \in \mathbf{R}. \tag{3.3}$$

Then for $\theta^2 > 2\varpi^2$ we have from (2.5) and (3.3) that

$$\begin{aligned} I_k^{\varpi,\mu}(\theta\phi) &= \frac{\theta^2}{2}\int_{-kT}^{kT}[|\phi'(t)|^2 + V(t)\phi^2(t)]dt - \mu\int_{-kT}^{kT}G(\theta\phi(t))dt \\ &\quad + \lambda\psi\left(\frac{\theta^2\|\phi\|_{E_k}^2}{\varpi^2}\right)\int_{-kT}^{kT}\theta^4|\phi'(t)|^2\phi^2(t)dt \\ &\le \frac{M_2\theta^2}{2} - \xi\int_{-kT}^{kT}G(\theta\phi(t))dt \\ &\le \frac{M_2\theta^2}{2} - \theta^2C_1\xi\int_{-kT}^{kT}\phi^2(t)dt + C_3. \end{aligned} \tag{3.4}$$

Then we can choose $\theta$ large such that $I_k^{\varpi,\mu}(\theta\phi) < 0$. The proof is completed. $\square$

**Lemma 3.3.** For any given $\xi \in (0,1)$, there exists a constant $c > 0$ such that $c_\mu^{(k)} \geq c > 0$ for all $\mu \in J$.

**Proof.** For any $u \in E_k$ and $\mu \in J$, using $(V_2)$ and $(V_3)$, for any $\epsilon \in (0, \frac{M_1}{2})$, we have

$$
\begin{aligned}
I_k^{\varpi,\mu}(u) &\geq \tfrac{M_1\|u\|_{E_k}^2}{2} + \lambda\psi\left(\tfrac{\|u\|_{E_k}^2}{\varpi^2}\right)\int_{-kT}^{kT}|u^{'}(t)|^2u^2(t)dt - \int_{-kT}^{kT}(\tfrac{\epsilon}{2}u^2(t) + C_\epsilon|u(t)|^p)dt \\
&\geq \tfrac{M_1\|u\|_{E_k}^2}{4} - \int_{-kT}^{kT}C_\epsilon|u(t)|^p dt \\
&\geq \tfrac{M_1\|u\|_{E_k}^2}{4} - C_\epsilon\|u\|_{L^\infty_{[-kT,kT]}}^{p-2}\int_{-kT}^{kT}|u(t)|^2dt \\
&\geq \tfrac{M_1\|u\|_{E_k}^2}{4} - C_\epsilon\varrho^{p-2}\|u\|_{E_k}^p,
\end{aligned}
\tag{3.5}
$$

where $C_\epsilon$ is a positive number. Since $p > 2$, we conclude that there exists $\rho > 0$ such that $I_k^{\varpi,\mu}(u) > 0$ for any $\mu \in J$ and $u \in E_k$ with $0 < \|u\|_{E_k} \leq \rho$. In particular, for $\|u\|_{E_k} = \rho$ we have $I_k^{\varpi,\mu}(u) \geq c > 0$. Fix $\mu \in J$ and $\eta \in \Gamma_\mu^{(k)}$. By the definition of $\Gamma_\mu^{(k)}$, $\|\eta(1)\| > \rho$. By the continuity, we deduce that there exists $t_\eta \in (0,1)$ such that $\|\eta(t_\eta)\| = \rho$. Therefore, for any $\mu \in J$,

$$
c_\mu^{(k)} \geq \inf_{\eta\in\Gamma_\mu^{(k)}} I_k^{\varpi,\mu}(\eta(t_\eta)) \geq c > 0.
$$

The proof is completed. $\square$

**Lemma 3.4.** For any $\mu \in J$ and $16\lambda\gamma^2\varpi^2 < M_1$, each bounded Palais-Smale sequence of the functional $I_k^{\varpi,\mu}$ admits a convergent subsequence.

**Proof.** Let $\mu \in J$ and $\{u_n\}$ be a bounded **PS** sequence of $I_k^{\varpi,\mu}$, namely $\{u_n\}$ and $\{I_k^{\varpi,\mu}(u_n)\}$ are bounded, $(I_k^{\varpi,\mu})^{'}(u_n) \to 0$ in $H^{'}$, where $H^{'}$ is the dual space of $H = E_k$. Subject to a subsequence, we can assume that there exists $u \in E_k$ such that

$$
\begin{aligned}
&u_n \hookrightarrow u \quad in \quad E_k \\
&u_n \to u \quad in \quad L^p(\mathbf{R}), \\
&u_n \to u \quad a.e. \quad in \quad \mathbf{R}.
\end{aligned}
$$

From $(V_2)$ and $(V_3)$, for $\epsilon^* \in (0, \frac{M_1}{2})$, there exists $C_{\epsilon^*} > 0$ such that

$$
|g(u(t))| \leq \epsilon^*|u(t)| + C_{\epsilon^*}|u(t)|^{p-1}, \quad u \in E_k, \tag{3.6}
$$

hence,

$$
\begin{aligned}
&|\int_{-kT}^{kT} g(u_n(t))(u_n(t)-u(t))dt| \\
\leq &\int_{-kT}^{kT}|g(u_n(t))||u_n(t)-u(t)|dt \\
\leq &\epsilon^*\|u_n\|_{L^2_{[-kT,kT]}}\|u_n-u\|_{L^2_{[-kT,kT]}} + C_{\epsilon^*}\|u_n\|_{L^p_{[-kT,kT]}}^{p-1}\|u_n-u\|_{L^p_{[-kT,kT]}}.
\end{aligned}
\tag{3.7}
$$

It follows that

$$\int_{-kT}^{kT} g(u_n(t))(u_n(t)-u(t))dt \to 0. \tag{3.8}$$

Similarly, we have

$$\int_{-kT}^{kT} V(t)u_n(t)(u_n(t)-u(t))dt \to 0 \quad \text{as} \quad n\to\infty, \tag{3.9}$$

$$\int_{-kT}^{kT}[(u_n^{'})^2(t)u(t)(u_n(t)-u(t))+(u_n(t))^2u^{'}(t)(u_n^{'}(t)-u^{'}(t))]dt \to 0 \quad \text{as} \quad n\to\infty \tag{3.10}$$

and

$$\int_{-kT}^{kT}[(u^{'})^2(t)u_n(t)(u_n(t)-u(t))+(u(t))^2u_n^{'}(t)(u_n^{'}(t)-u^{'}(t))]dt \to 0 \quad \text{as} \quad n\to\infty. \tag{3.11}$$

We have from (2.2) and (3.8)-(3.11) that

$$\begin{aligned}
&0 \leftarrow \left((I_k^{\varpi,\mu})'(u_n), u_n(t)-u(t)\right)\\
&= \int_{-kT}^{kT}\left(-u_n^{''}(t)+V(t)u_n(t)-\lambda\psi(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2})(u^2(t))^{''}u(t)-\mu g((u_n(t)),u_n(t)-u(t)\right)dt\\
&+\tfrac{2\lambda}{\varpi^2}\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)(\textstyle\int_{-kT}^{kT}|u_n^{'}(t)|^2u_n^2(t)dt)[u_n^{'}(t)(u_n(t)-u(t))'+u_n(t)(u_n(t)-u(t))]dt\\
&= \Phi_k^2((u_n(t)-u(t))+2\lambda\psi\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)\textstyle\int_{-kT}^{kT}[(u_n^{'})^2(t)(u_n(t)-u(t))^2+(u_n(t))^2(u_n^{'}(t)-u^{'}(t))^2]dt\\
&+\tfrac{2\lambda}{\varpi^2}\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)\textstyle\int_{-kT}^{kT}|u_n^{'}(t)|^2u_n^2(t)dt))[\int_{-kT}^{kT}[|u_n^{'}(t)-u^{'}(t)|^2+|u_n(t)-u(t)|^2]dt+o(1)\\
&\geq 2\lambda\psi\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)\textstyle\int_{-kT}^{kT}[(u_n^{'})^2(t)(u_n(t)-u(t))^2+(u_n(t))^2(u_n^{'}(t)-u^{'}(t))^2]dt\\
&+(M_1-\tfrac{2\lambda\gamma^2}{\varpi^2}|\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)|\||u_n\|_{E_k}^4)\|u_n-u\|_{E_k}^2+o(1),
\end{aligned} \tag{3.12}$$

and then

$$\left(M_1-\tfrac{2\lambda\gamma^2}{\varpi^2}|\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)|\||u_n\|_{E_k}^4\right)\|u_n-u\|_{E_k}^2 \to 0. \tag{3.13}$$

Since $\left|\psi'\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)\|u_n\|_{E_k}^4\right| \leq 8\varpi^4$ and $16\lambda\varpi^2\gamma^2 < M_1$, we obtain that $\|u_n-u\|_{E_k}^2 \to 0$ as $n\to\infty$.

The proof is completed. $\square$

**Lemma 3.5.** Let $16\lambda\gamma^2\varpi^2 < M_1$. For almost every $\mu \in J$, there exists $u^\mu \in E_k\backslash\{0\}$ such that $(I_k^{\varpi,\mu})'(u^\mu)=0$ and $I_k^{\varpi,\mu}(u^\mu)=c_\mu^{(k)}$.

**Proof.** By Lemma 1.3, for almost every $\mu\in J$, there exists a bounded sequence $\{u_n^\mu\}\subset E_k$ such that

$$I_k^{\varpi,\mu}(u_n^\mu)\to c_\mu, \quad (I_k^{\varpi,\mu})'(u_n^\mu)\to 0.$$

From Lemma 3.4, we can suppose that there exists $u^\mu\in E_k$ such that $u_n^\mu \to u^\mu$ in $E_k$. Then the assertion follows from Lemma 3.3. $\square$

According to Lemma 3.5, there exists sequences $\{\mu_n\} \subset J$ with $\mu_n \to 1^-$ and $\{u_n\} \subset E_k$ as $n \to \infty$ such that

$$I_k^{\varpi,\mu_n}(u_n) = c_{\mu_n}^{(k)}, \quad (I_k^{\varpi,\mu_n})'(u_n) \to 0.$$

The Pohozaev identity is important for many problems and in this section we also use this identity to obtain $\|u_n\|_{E_k} \le \varpi$.

**Lemma 3.6.** Let $16\lambda\gamma^2\varpi^2 < M_1$. If $u \in E_k$ is a weak solution of $(\mathbf{HS_k})$, then the following Pohozaev type identity holds

$$\begin{aligned}&\tfrac{1}{2}\int_{-kT}^{kT}[|u'(t)|^2 + 2\lambda\psi\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)(u'(t))^2|u(t)|^2 + 2\mu G(u(t)) - (V(t)+tV'(t))|u(t)|^2 \\ &+\tfrac{2\lambda}{\varpi^2}\psi'\left(\frac{\|u\|_{E_k}^2}{\varpi^2}\right)(\int_{-kT}^{kT}|u'(t)|^2|u(t)|^2dt)(|u'(t)|^2 - |u(t)|^2)]dt = 0.\end{aligned} \tag{3.14}$$

**Proof.** The proof is standard, thus we only sketch the proof here briefly. From the fact that $(u^2(t))''u(t) = 2((u'(t))^2u(t) + u''(t)u^2(t))$, the problem $(\mathbf{HS_k})$ can be rewritten as

$$\begin{aligned}&-u''(t) + V(t)u(t) - 2\lambda\psi(\tfrac{\|u\|^2}{\varpi^2})((u'(t))^2u(t) + u''(t)u^2(t)) - \mu g(u(t)) - h_k(t) \\ &+\tfrac{2\lambda}{\varpi^2}\psi'\left(\tfrac{\|u\|^2}{\varpi^2}\right)[\int_{-kT}^{kT}|u'(t)|^2u^2(t)dt][-u''(t)+u(t)] = 0.\end{aligned} \tag{3.15}$$

Integrating by parts in $[-kT, kT]$, we obtain that

$$\begin{aligned}\int_{-kT}^{kT}u''(t)(tu'(t))dt &= \tfrac{t(u'(t))^2}{2}\Big|_{-kT}^{kT} - \tfrac{1}{2}\int_{-kT}^{kT}|u'(t)|^2dt \\ &= \tfrac{-1}{2}\int_{-kT}^{kT}|u'(t)|^2dt,\end{aligned} \tag{3.16}$$

here we used the periodic boundary condition which yields that

$$\begin{aligned}\int_{-kT}^{kT}V(t)u(t)(tu'(t))dt &= \tfrac{1}{2}tV(t)u^2(t)\Big|_{-kT}^{kT} - \tfrac{1}{2}\int_{-kT}^{kT}V(t)|u(t)|^2dt - \tfrac{1}{2}\int_{-kT}^{kT}(tV'(t))|u(t)|^2dt \\ &= -\tfrac{1}{2}\int_{-kT}^{kT}V(t)|u(t)|^2dt - \tfrac{1}{2}\int_{-kT}^{kT}(tV'(t))|u(t)|^2dt\end{aligned} \tag{3.17}$$

and

$$\int_{-kT}^{kT}g(u(t))(tu'(t))dt = tG(u(t))\Big|_{-kT}^{kT} - \int_{-kT}^{kT}G(u(t))dt = -\int_{-kT}^{kT}G(u(t))dt. \tag{3.18}$$

We have from (3.16) that

$$\int_{-kT}^{kT}\left[-u''(t)+u(t)\right](tu'(t))dt = \tfrac{1}{2}\int_{-kT}^{kT}\left[[|u'(t)|^2 - |u(t)|^2\right]dt, \tag{3.19}$$

and

$$\begin{aligned}\int_{-kT}^{kT}(u^2(t))''u(t)(tu'(t))dt &= 2\int_{-kT}^{kT}((u'(t))^2u(t) + u''(t)u^2(t))(tu'(t))dt \\ &= t(u'(t))^2u^2(t)\Big|_{-kT}^{kT} - \int_{-kT}^{kT}(u'(t))^2|u(t)|^2dt \\ &= -\int_{-kT}^{kT}(u'(t))^2|u(t)|^2dt.\end{aligned} \tag{3.20}$$

Multiplying (3.15) by $tu'(t)$ and integrating in $[-kT, kT]$, by (3.16)-(3.20) we get (3.14). $\square$

As we will now assume $(V_1)$ then (3.14) reduces to

$$\begin{aligned}&\tfrac{1}{2}\int_{-kT}^{kT}[|u'(t)|^2 + 2\lambda\psi\left(\tfrac{\|u\|_{E_k}^2}{\varpi^2}\right)(u'(t))^2|u(t)|^2 + 2\mu G(u(t)) - \alpha|u(t)|^2\\ &+\tfrac{2\lambda}{\varpi^2}\psi'\left(\tfrac{\|u\|_{E_k}^2}{\varpi^2}\right)(\int_{-kT}^{kT}|u'(t)|^2|u(t)|^2dt)(|u'(t)|^2 - |u(t)|^2)]dt = 0.\end{aligned}$$

**Lemma 3.7.** Let $u_n$ be a critical point of $I_k^{\varpi,\mu_n}$ at level $c_{\mu_n}^{(k)}$. Then for $\varpi > 0$ sufficiently large, there exists $\lambda_0 = \lambda_0(\varpi)$ with $16\lambda_0\gamma^2\varpi^2 < M_1$ such that for any $\lambda \in [0, \lambda_0)$, there is a subsequence $\{u_n\}$ subject to $\|u_n\|_{E_k} \le \varpi$ for all $n \in \mathbf{N}$.

**Proof.** We can obtain from (2.5), $(V_1)$, (3.1) and (3.14) that

$$\begin{aligned}&\tfrac{1}{2}\int_{-kT}^{kT}|u_n'(t)|^2dt\\ &\le \tfrac{1}{2}\int_{-kT}^{kT}\left[|u_n'(t)|^2 + 4\lambda\psi\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)(u_n'(t))^2|u_n(t)|^2\right]dt\\ &\le c_{\mu_n} + (\tfrac{\lambda}{\varpi^2}|\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)|\int_{-kT}^{kT}|u_n'(t)|^2u_n^2(t)dt)\int_{-kT}^{kT}(|u'(t)|^2 - |u(t)|^2)\Big]dt\\ &\le c_{\mu_n}^{(k)} + \tfrac{(2m-1)\lambda\gamma_2^2}{\varpi^2}|\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)|\|u_n\|_{E_k}^6.\end{aligned}\tag{3.21}$$

We estimate the right hand side of (3.21). By the min $-$max definition of the mountain pass level, Lemma 3.2 and (3.3), we have

$$\begin{aligned}c_{\mu_n}^{(k)} &\le \max{}_\theta I_k^{\varpi,\mu_n}(\theta\phi)\\ &\le \max{}_\theta\left\{\tfrac{M_2\theta^2}{2} - \mu_n\int_{-kT}^{kT}G(\theta\phi(t))dt\right\} + \max{}_\theta\lambda\psi(\tfrac{\theta^2}{\varpi^2})\theta^4\\ &\le \max{}_\theta\left\{\tfrac{M_2\theta^2}{2} - \delta C_1\theta^2\int_{-R}^{R}\phi^2(t)dt + C_3\right\} + \max{}_\theta\lambda\psi(\tfrac{\theta^2}{\varpi^2})\theta^4\\ &= C_3 + \Re(\varpi).\end{aligned}\tag{3.22}$$

If $\theta^2 \ge 2\varpi^2$, then $\psi(\frac{\theta^2}{\varpi^2}) = 0$. Thus, we have that

$$\Re(\varpi) \le 4\lambda\gamma_2^2\varpi^4.$$

By the definition of $\psi$, we have also that

$$\psi\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)\int_{-kT}^{kT}(u_n'(t))^2|u_n(t)|^2dt \le 4\gamma_2^2\varpi^4$$

and

$$\tfrac{1}{\varpi^2}|\psi'\left(\tfrac{\|u_n\|_{E_k}^2}{\varpi^2}\right)|\|u_n\|_{E_k}^6 \le 16\varpi^4.$$

Then we have

$$\tfrac{1}{2}\int_{-kT}^{kT}|u_n'(t)|^2dt \le C_3 + 20\lambda\gamma^2\varpi^4.\tag{3.23}$$

On the other hand, by (3.2) and (3.6), we have that

$$
\begin{aligned}
&M_1\|u_n\|^2_{E_k} \le \int_{-kT}^{kT}[|u_n^{'}(t)|^2 + \alpha u_n^2(t) + 4\lambda\psi\left(\frac{\|u_n\|^2_{E_k}}{\varpi^2}\right)|u_n^{'}(t)|^2|u_n(t)|^2]dt \\
&= \int_{-kT}^{kT}[g(u_n(t))u_n(t) - \tfrac{2\lambda}{\varpi^2}\psi^{'}\left(\frac{\|u_n\|^2_{E_k}}{\varpi^2}\right)(\int_{-kT}^{kT}|u_n^{'}(t)|^2|u_n(t)|^2dt)(|u_n(t)|^2 + |u_n^{'}(t)|^2)]dt \\
&\le \epsilon^*\|u_n\|^2_{L^2_{[-kT,kT]}} + C_{\epsilon^*}\|u_n\|^p_{L^p_{[-kT,kT]}} + \tfrac{2\lambda}{\varpi^2}|\psi^{'}\left(\frac{\|u_n\|^2_{E_k}}{\varpi^2}\right)|\|u_n\|^6_{E_k} \\
&\le \epsilon^*\|u_n\|^2_{E_k} + C_4\|u_n^{'}\|^p_{L^2_{[-kT,kT]}} + 32\lambda\gamma_2^2\varpi^4.
\end{aligned}
\tag{3.24}
$$

We have from (3.23) and (3.24) that

$$
\begin{aligned}
&(M_1-\epsilon^*)\|u_n\|^2_{E_k} \le C_4\|u_n^{'}\|^p_{L^2_{[0,2kT]}} + 32\lambda\gamma^2\varpi^4 \\
&\le C_5(C_3 + 20\lambda\gamma^2\varpi^4)^{\frac{p}{2}} + 32\lambda\gamma^2\varpi^4.
\end{aligned}
\tag{3.25}
$$

Choose $\varpi > 0$ with $\varpi^2 > C_6(C_3 + \frac{5}{4}M_1)^{\frac{p}{2}} + C_7M_1$ and $16\lambda\gamma^2\varpi^4 < M_1$, where $C_6 = \frac{C_5}{M_1-\epsilon^*}$ and $C_7 = \frac{2}{M_1-\epsilon^*}$.

From (3.25) since $16\lambda\gamma^2\varpi^4 < M_1$ we have

$$
\|u_n\|^2_{E_k} \le C_6(C_3 + 20\lambda\gamma^2\varpi^4)^{\frac{p}{2}+16C_7\lambda\gamma^2\varpi^4} < C_6(C_3 + \tfrac{5}{4}M_1)^{\frac{p}{2}} + C_7M_1.
$$

Thus, by setting $\lambda_0 < M_1/16\gamma^2\varpi^2$, we obtain the conclusion. □

Consequently, let $\varpi$ be defined as in Lemma 3.7, and $u_{k_n}$ be a critical point for $I_k^{\varpi,\mu_n}$ at level $c^{(k)}_{\mu_n}$. Then from Lemma 3.7 we may assume that

$$
\|u_{k_n}\|_{E_k} \le \varpi.
$$

Hence

$$
I_k^{\varpi,\mu_n}(u_{k_n}) = \tfrac{1}{2}\Phi_k^2(u_{k_n}) + \int_{-kT}^{kT}[\lambda|u_{k_n}^{'}(t)|^2|u_{k_n}(t)|^2 - \mu_nG(u_{k_n}(t)) + (h_k(t), u_{k_n}(t))]dt. \tag{3.26}
$$

Since $\mu_n \to 1$, we can show that $\{u_n\}$ is a **PS** sequence of $I_k$. Indeed, the boundedness of $\{u_{k_n}\}$ implies that $\{I_k(u_{k_n})\}$ is bounded. Also

$$
(I_k^{'}(u_{k_n}), v) = ((I_k^{\varpi,\mu_n})^{'}(u_{k_n}), v) + (\mu_n - 1)\int_{-kT}^{kT}\int_{-kT}^{kT} g(u_{k_n}(t))vdt, \quad v \in E_k. \tag{3.27}
$$

Thus $I_k^{'}(u_{k_n}) \to 0$, and then $u_{k_n}$ is a bounded **PS** sequence of $I_k$. By Lemma 3.4 $\{u_{k_n}\}$ has a convergent subsequence. We may assume that $u_{k_n} \to u_k$. Consequently $I_k^{'}(u_k) = 0$. According to Lemma 3.3, it follows that $I_k(u_k) = \lim\limits_{n\to\infty} I_k(u_{k_n}) = \lim\limits_{n\to\infty} I_k^{\varpi,\mu_n}(u_{k_n}) \ge c > 0$ and $u_k$ is a solution of ($\mathbf{HS_k}$).

**Lemma 3.8.** Let $\{u_k\}_{k\in\mathbf{N}}$ be the sequence given by Lemma 3.7. Then there exists a solution $u$ of (**HS**) such that $u_k \to u$ in $C^1_{loc}(\mathbf{R}, \mathbf{R}^n)$ as $k \to +\infty$.

**Proof.** Let us start with showing boundedness of the sequences $\{c_k\}_{k\in\mathbf{N}}$. Obviously there exists a $\widetilde{u}_1 \in E_1$ with $\widetilde{u}_1(\pm T) = 0$ such that

$$c_1 \le I_1(\widetilde{u}_1) = \inf_{\widetilde{g}\in\Gamma_1} \max{}_{t\in[0,1]} I_1(\widetilde{g}(t)). \tag{3.28}$$

For every $k \in \mathbf{N}$, let

$$\widetilde{u}_k(t) = \begin{cases} \widetilde{u}_1(t) & \text{for} \quad |t| \le T \\ 0 & \text{for} \quad T < |t| \le kT \end{cases} \tag{3.29}$$

and $\widetilde{g}_k : [0,1] \to E_k$ be a curve given by

$$\widetilde{g}_k(s) = s\widetilde{u}_k.$$

Therefore, from (3.28) and (3.29),

$$c_k \le \max{}_{t\in[0,1]} I_k(\widetilde{g}_k(t)) = \max{}_{t\in[0,1]} I_1(\widetilde{g}_1(t)) \equiv M_0 \tag{3.30}$$

independently of $k \in \mathbf{N}$.

By (2.6) and Lemma 3.7, we have the existence of a constant $D_0$ (independent of $k$) with

$$\|u_k\|_{L^\infty_{[-T,T]}} \le \gamma_2\varpi := D_0 \quad \text{for every} \quad k \in \mathbf{N}. \tag{3.31}$$

Now we will obtain some estimates for $u_k^{'}(t)$.

By $(\mathbf{HS_k})$, we have that for $t \in [-kT, kT]$,

$$\begin{aligned} &(u_k^{''}(t) - V(t)u_k(t) + \lambda(u_k^2(t))^{''}u_k(t) - g(u_k(t)), u_k^{''}(t)) \\ &= ((1+2\lambda u_k^2(t))u_k^{''}(t) - V(t)u_k(t) + 2\lambda(u_k^{'}(t))^2u_k(t) - g(u_k(t)), u_k^{''}(t)) = 0. \end{aligned} \tag{3.32}$$

This implies that there is a constant $C_0 > 0$ being independent of $k$ such that

$$\begin{aligned} &(1+2\lambda D_0^2)\textstyle\int_{-kT}^{kT} |u_k^{''}(t)|^2dt \\ &\le \widehat{V}(\textstyle\int_{-kT}^{kT} |u_k(t)|^2dt)^{\frac{1}{2}}(\textstyle\int_{-kT}^{kT}(u_k^{''}(t))^2dt)^{\frac{1}{2}} + \frac{2\lambda}{3}\textstyle\int_{-kT}^{kT}(u_k^{'}(t))^3u_k(t)dt \\ &\quad +[\textstyle\int_{-kT}^{kT} |g(u_k(t))|^2dt)^{\frac{1}{2}}|^2dt)^{\frac{1}{2}}](\textstyle\int_{-kT}^{kT} |u_k^{''}(t)|^2dt)^{\frac{1}{2}} \\ &\le [\widehat{V}\varpi + (\textstyle\int_{-kT}^{kT} |g(u_k(t))|^2dt)^{\frac{1}{2}}](\textstyle\int_{-kT}^{kT}(u_k^{''}(t))^2dt)^{\frac{1}{2}} \\ &\quad +C_0(\textstyle\int_{-kT}^{kT} |u_k^{'}(t)|^6dt)^{\frac{1}{2}}(\textstyle\int_{-kT}^{kT} |u_k(t)|^2dt)^{\frac{1}{2}} \\ &\le [\widehat{V}\varpi + (\textstyle\int_{-kT}^{kT} |g(u_k(t))|^2dt)^{\frac{1}{2}}](\textstyle\int_{-kT}^{kT}(u_k^{''}(t))^2dt)^{\frac{1}{2}} + C_0(\varpi)^4. \end{aligned} \tag{3.33}$$

Therefore, (3.31), (3.33) and assumption $(V_3)$ imply that there are $d_1 > 0$ and $\overline{D}_1 > 0$ being independent of $k$ such that

$$\|u_k^{''}\|_{L^2_{[0,2k\tau]}} \le d_1 \tag{3.34}$$

and

$$\|u_k^{'}\|_{L^\infty_{[0,2k\tau]}} \le \gamma(\textstyle\int_{-kT}^{kT} |u_k^{''}(t)|^2dt + \int_{-kT}^{kT} |u_k^{'}(t)|^2dt) \le \gamma(d_1+\varpi)^2 = \overline{D}_1. \tag{3.35}$$

Indeed, from the definition of $\|u_k\|_{E_k}$, (2.5) and ($V_3$), we have $\int_{-kT}^{kT}|g(u_k(t))|^2dt < +\infty$.

By (3.31) and the Arzelà-Ascoli theorem, we obtain that a subsequence of $\{u_k\}_{k\in\mathbf{N}}$ (again we call it $\{u_k\}_{k\in\mathbf{N}}$) which converges in $C^1_{loc}(\mathbf{R},\mathbf{R}^n)$ to a solution $u$ of (**HS**) satisfying

$$\int_{-\infty}^{\infty}[|u^{'}(t)|^2+V(t)|u(t)|^2]dt<\infty. \tag{3.36}$$

Indeed,

$$\begin{aligned}&\int_{-\infty}^{\infty}[|u^{'}(t)|^2+V(t)|u(t)|^2]dt\\ &=\lim_{k\to\infty}\int_{-kT}^{kT}[|u_k^{'}(t)|^2+V(t)|u_k(t)|^2]dt\\ &\le M_2\lim_{k\to\infty}\|u\|^2_{E_k}<\infty.\end{aligned}$$

The proof is finished. $\square$

**Lemma 3.9.** The function $u$ determined by Lemma 3.8 is the desired homoclinic solution of (**HS**).

**Proof.** The proof will be divided into three steps.

Step 1: We prove that $u(t)\to 0$, as $t\to\pm\infty$.

From $\|u_k\|^2_{E_k}\le D_1$, we have

$$\int_{-\infty}^{\infty}[|u(t)|^2+|u^{'}(t)|^2]dt\le\lim_{k\to\infty}\|u_k\|^2_{E_k}<\infty.$$

This implies

$$\lim_{j\to\infty}\int_{|t|\ge j}[|u(t)|^2+|u^{'}(t)|^2]dt=0. \tag{3.37}$$

Now (3.37) show that

$$\lim_{j\to\infty}\max{}_{|t|\ge j}|u(t)|\le 2\lim_{j\to\infty}\left(\int_{|t|\ge j}[|u(t)|^2+|u^{'}(t)|^2]dt\right)^{\frac{1}{2}}=0.$$

Hence our claim holds.

Step 2: We next show that $u^{'}(t)\to 0$, as $t\to\pm\infty$.

If

$$\int_j^{j+1}|u^{''}(t)|^2dt\to 0,\quad \text{as}\quad j\to+\infty, \tag{3.38}$$

then $u^{'}(t)\to 0$, as $t\to\pm\infty$.

We obtain from (**HS**) that

$$\int_j^{j+1}|u^{''}(t)|^2dt\ =\int_j^{j+1}|-V(t)u(t)+\lambda(u^2(t))^{''}u(t)+g(u(t))|^2dt.$$

Since $g(0)=0$ and $u(t)\to 0$, as $t\to\pm\infty$, (3.38) follows. Here use the fact that $(u^2(t))^{''}u(t)=2((u^{'}(t))^2u(t)+u^{''}(t)u^2(t))$.

Step 3: We show that $u \not\equiv 0$ when $h(t) \equiv 0$. Let $Y : [0, +\infty) \to [0, +\infty)$ be defined as follows : $Y(0) = 0$ and

$$Y(s) = \max{}_{t\in[0,T],0<|u|\le s} \frac{(g(u),u)}{|u|^2} \quad \text{for} \quad s > 0.$$

Then Y is continuous and nondecreasing, $Y(s) > 0$ for $s > 0$ and $Y(s) \to +\infty$ as $s \to +\infty$. It is easy to verify this fact applying $(V_4)$. By the definition of $Y$ we obtain

$$2Y(\|u_k\|_{L^\infty_{[0,2kT]}})\|u_k\|^2_{E_k} \quad \ge \int_{-kT}^{kT}(g(u_k(t)), u_k(t))dt \tag{3.39}$$

for every $k \in \mathbf{N}$. Since $I_k'(u_k)u_k = 0$, (2.4) gives

$$\int_{-kT}^{kT}(g(u_k(t)), u_k(t))dt = \int_{-kT}^{kT}[|u_k'(t)|^2 + V(t)|u_k(t)|^2 + 4|u_k'(t)|^2|u_k(t)|^2]dt. \tag{3.40}$$

By (2.3), (3.39) and (3.40), we have

$$Y(\|u_k\|_{L^\infty_{[0,2kT]}})\|u_k\|^2_{E_k} \ge M_1\|u_k\|^2_{E_k},$$

and hence

$$Y(\|u_k\|_{L^\infty_{[0,2kT]}}) \ge M_1. \tag{3.41}$$

Consequently the properties of $Y$ imply there is a $\kappa > 0$(being independent of $k$), such that

$$\|u_k\|_{L^\infty_{[0,2kT]}} \ge \kappa. \tag{3.42}$$

To complete the proof, observe that by the $T-$periodicity of $G$, whenever $u(t)$ is a $2kT-$periodic solution of (**HS**), so is $u(t + jT)$ for all $j \in \mathbf{Z}$. Hence by replacing $u_k(t)$ earlier if necessary by $u_k(t+jT)$ for some $j \in [-k, k]\cap\mathbf{Z}$ we can assume that the maximum of $u_k(t)$ occurs in $[-T, T]$.

If $u = 0$ then using the subsequence from Lemma 3.7 we have

$$\|u_k\|_{L^\infty_{[-kT,kT]}} = \max{}_{t\in[-T,T]}|u_k(t)| \to 0,$$

which contradicts (3.51). $\square$

**Proof of Theorem 3.1:** The result follows from Lemma 3.9. $\square$

# 4 Existence result (II)

In the present section we assume that $g$ and $h$ in (**HS**) satisfy the following conditions:

$(H_1)$ $g$ is odd with respect to u, i.e., for any $u \in R^n, g(t, -u) = -g(t, u)$;

$(H_2)$ there is a constant $\theta > 4$ such that for every $t \in \mathbf{R}$ and $u \in \mathbf{R}^n\backslash\{0\}$

$$0 < \theta G(t, u) \le (g(t, u), u),$$

$(H_3)$ $h : \mathbf{R} \to \mathbf{R}^n$ is a continuous and bounded function and $(\int_{\mathbf{R}} |h(t)|^2 dt)^{\frac{1}{2}} \leq \frac{\sigma}{2\gamma 2}$, where $0 < \sigma < M_1 - 2G^*$ with $G^* := \sup\{G(t,u) : t \in [0,T], |u| = 1\}$ and $M_1$ being the number defined in (2.5).

**Theorem 4.1.** If conditions $(G_0)$ and $(H_2) - (H_3)$ are true, then system (**HS**) possesses a nontrivial homoclinic solution $u \in W^{1,2}(\mathbf{R}, \mathbf{R}^n)$ such that $u^{(i)}(t) \to 0 (i = 0, 1)$ as $t \to \pm\infty$.

If $h(t) \equiv 0$ and $g(t, \cdot)$ is an odd function for any $t \in \mathbf{R}$, we will show that the system (**HS**) has infinitely many subharmonic solutions.

**Theorem 4.2.** If $h(t) \equiv 0$ and the conditions $(G_0)$ and $(H_1) - (H_3)$ are satisfied then the system (**HS**) possesses infinitely many subharmonic solutions.

To make the paper self contained we include the proofs. In order to prove Theorem 4.1 and Theorem 4.2, we have to introduce some necessary preliminaries which were partly motivated by the ideas in [14].

**Proposition 4.3.** For every $t \in [0,T]$ the following inequalities hold:

$$G(t,u) \leq G(t, \tfrac{u}{|u|})|u|^{\theta}, \quad \text{if} \quad 0 < |u| \leq 1 \tag{4.1}$$

and

$$G(t,u) \geq G(t, \tfrac{u}{|u|})|u|^{\theta}, \quad \text{if} \quad |u| \geq 1. \tag{4.2}$$

**Proof.** Let $\Theta : (0, \infty) \to (0, \infty)$ be defined as follows:

$$\Theta(\lambda) = G(t, \lambda^{-1}u, )\lambda^{\theta}.$$

By $(H_2)$, we have

$$\Theta^{'}(\lambda) \quad = \lambda^{\theta-1}(\theta\, G(t, \lambda^{-1}u) - (g(t, \lambda^{-1}u), \lambda^{-1}u)) \leq 0.$$

This shows that the function $G(t, \lambda^{-1}u)\lambda^{\theta}$ is nonincreasing. Hence (4.1) and (4.2) follow. $\square$

**Proposition 4.4.** Let $\iota := \inf\{G(t,u) : t \in [0,T], |u|^2 = 1\}$, $\tau \in \mathbf{R}\backslash\{0\}$ and $u \in E_k\backslash\{0\}$. Then

$$\textstyle\int_{-kT}^{kT} G(t, \tau u)dt \geq \iota|\tau|^{\theta} \int_{-kT}^{kT} |u(t)|^{\theta} dt - 2\iota kT. \tag{4.3}$$

**Proof.** Fix $\tau \in \mathbf{R}\backslash\{0\}$ and $u \in E_k\backslash\{0\}$. Set

$$\Omega_k = \{t \in [-kT, kT] : |\tau u| \leq 1\}$$

and

$$\overline{\Omega}_k = \{t \in [-kT, kT] : |\tau u| \geq 1\}.$$

By (4.2), we get

$$\begin{aligned}
\int_{-kT}^{kT} G(t,\tau u)dt & \geq \int_{\overline{\Omega}_k} G(t,\tau u)dt \geq \int_{\overline{\Omega}_k} G(t,\tfrac{\tau u}{|\tau u|})|\tau u|^{\theta}dt \\
& \geq \iota \int_{\overline{\Omega}_k} |\tau u|^{\theta}dt \\
& \geq \iota \int_{-kT}^{kT} |\tau u|^{\theta}dt - \iota \int_{\Omega_k} |\tau u|^{\theta}dt \\
& \geq \iota |\tau|^{\theta} \int_{-kT}^{kT} |u|^{\theta}dt - 2\iota kT.
\end{aligned}$$

This complete the proof. □

Let $h_k : \mathbf{R} \to \mathbf{R}^n$ be a $2kT-$periodic extension of $h|_{[-kT,kT]}$ on $\mathbf{R}$. From $(H_3)$ it follows that

$$\|h_k\|_{L^2_{[-kT,kT]}} \leq \frac{\sigma}{2\gamma}. \tag{4.4}$$

**Lemma 4.5.** If $g$, $G$ and $h$ satisfy $(G_0)$ and $(H_2)-(H_3)$, then for every $k \in \mathbf{N}$ the system $(\mathbf{HS_k})$ possesses a $2kT-$periodic solution.

**Proof.** It is clear that $I_k(0) = 0$. We show that $I_k$ satisfies the **PS** condition. Assume that $\{u_{kn}\}_{n\in\mathbf{N}}$ in $E_k$ is a sequence such that $\{I_k(u_{kn})\}_{n\in\mathbf{N}}$ is bounded and $I_k'(u_{kn}) \to 0$ as $n \to +\infty$. Then there exists a constant $d_2 > 0$ such that

$$|I_k(u_{kn})| \leq d_2, \quad I_k'(u_{kn})u_{kn} \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.5}$$

We first prove that $\{u_{kn}\}_{n\in\mathbf{N}}$ is bounded.

From $(G_0)$ and (4.5), we have

$$d_2 + o(1) \geq \frac{1}{2}\Phi_k^2(u_{kn}(t)) + \lambda \int_{-kT}^{kT} |u_{kn}'(t)|^2 u_{kn}^2(t)dt - \int_{-kT}^{kT}(h_k(t),u_{kn}(t))dt - \int_{-kT}^{kT} G(t,u_{kn}(t))dt \tag{4.6}$$

and

$$\begin{aligned}
o(\|u_{kn}\|_{E_k}) & = I_k'(u_{kn})u_{kn} = \Phi_k^2(u_{kn}(t)) + 4\lambda \int_{-kT}^{kT} |u_{kn}'(t)|^2 u_{kn}^2(t)dt \\
& \quad - \int_{-kT}^{kT}(h_k(t),u_{kn}(t))dt - \int_{-kT}^{kT}(g(t,u_{kn}(t)),u_{kn}(t))dt.
\end{aligned} \tag{4.7}$$

From $(H_2)$, (2.6), (4.6) and (4.7), we get

$$\begin{aligned}
d_2\theta + o(1) + o(\|u_{kn}\|_{E_k}) & \geq \frac{\theta-2}{2}\Phi_k^2(u_{kn}(t)) + \lambda(\theta-4) \int_{-kT}^{kT} |u_{kn}'(t)|^2 u_{kn}^2(t)dt \\
& \quad + \int_{-kT}^{kT}[(g(t,u_{kn}(t)),u_{kn}(t)) - \theta G(t,u_{kn}(t))]dt \\
& \quad +(1-\theta) \int_{-kT}^{kT}(h_k(t),u_{kn}(t))dt \\
& \geq \frac{M_1(\theta-2)}{2}\|u_{kn}\|_{E_k}^2 - (\theta-1)\|h_k\|_{L^2_{[0,2kT]}}\|u_{kn}\|_{E_k}.
\end{aligned} \tag{4.8}$$

Now (4.8) shows that $\{u_{kn}\}_{n\in\mathbf{N}}$ is bounded in $E_k$. Hence we can extract a subsequence (again we call it $\{u_{kn}\}_{n\in\mathbf{N}}$) such that $\{u_{kn}\}_{n\in\mathbf{N}}$ converges to $u_k$ in $E_k$(weakly). This implies $u_{kn} \to u_k$ uniformly on $[-kT,kT]$. Hence

$$(I_k'(u_{kn}) - I_k'(u_k))(u_{kn}-u_k) \to 0 \quad \text{and} \quad \|u_{kn}-u_k\|_{L^2_{[0,2kT]}} \to 0.$$



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A similar argument as in (3.7)-(3.11) guarantees that $\|u_{kn}-u_k\|_{E_k}\to 0$.

We now show that there exist constants $\rho$, $\alpha>0$ independent of $k$ such that every $I_k$ satisfies the assumption $(G_1)$ of Lemma 1.1 with these constants. Assume that $\|u\|_{L^\infty_{[0,2kT]}}\le 1$.
From (3.1) we have

$$\int_{-kT}^{kT}G(t,u(t))dt \quad \le \int_{-kT}^{kT}G(t,\tfrac{u(t)}{|u(t)|})|u(t)|^\theta dt\le G^*\int_{-kT}^{kT}|u(t)|^\theta dt\le G^*\int_{-kT}^{kT}|u(t)|^2dt\le G^*\|u\|^2_{E_k}. \tag{4.9}$$

By (2.1), (2.4), (4.4) and (4.9), we obtain

$$\begin{aligned} I_k(u)\ge\quad & \tfrac12 M_1\|u\|^2_{E_k}-G^*\|u\|^2_{E_k}-\|h_k\|_{L^2_{[0,2kT]}}\|u\|_{L^2_{[0,2kT]}} \\ &\ge \tfrac12 M_1\|u\|^2_{E_k}-G^*\|u\|^2_{E_k}-\tfrac{\sigma}{2\gamma_2}\|u\|_{E_k} \\ &\ge \tfrac12(M_1-\sigma-2G^*)\|u\|^2_{E_k}+\tfrac{\sigma}{2}\|u\|^2_{E_k}-\tfrac{\sigma}{2\gamma_2}\|u\|_{E_k}. \end{aligned} \tag{4.10}$$

Note that $(H_3)$ implies $(M_1-\sigma-2G^*)>0$. Set

$$\rho=\frac{1}{\gamma},\quad \alpha=\frac{M_1-\sigma-2G^*}{2\gamma}^2.$$

From (2.4), if $\|u\|_{E_k}=\rho$ (note (2.6) and the definition of $\rho$ yields $||u||_{L^\infty}\le 1$), then (4.10) gives

$$I_k(x)\ge\alpha.$$

It remains to prove that for every $k\in\mathbf{N}$ there exists $e_k\in E_k$ such that $\|e_k\|_{E_k}>\rho$ and $I_k(e_k)\le 0$. By (2.1), (2.3), (4.3) and (4.4), we have that for every $\varsigma\in\mathbf{R}\backslash\{0\}$ and $u\in E_k\backslash\{0\}$,

$$\begin{aligned} I_k(\varsigma u)\le\quad & \tfrac12 M_2|\varsigma|^2\|u\|^2_{E_k}+\tfrac{\sigma|\varsigma|}{2\gamma_2}\|u\|_{L^2_{[0,2k\tau]}}+\lambda|\varsigma|^4\|u\|^4_{E_k} \\ & +2k\iota T-\iota|\varsigma|^\theta\int_{-k\tau}^{k\tau}|u(t)|^\theta dt\le 0 \end{aligned} \tag{4.11}$$

provided $\varsigma>0$ is large enough since $\theta>4$. Take $\tilde u\in E_1$ such that $\tilde u(\pm T)=0$. Now (4.11) implies that there exists $\zeta\in\mathbf{R}\backslash\{0\}$ such that $\|\zeta\tilde u\|_{E_1}>\rho$ and $I_1(\zeta\tilde u)<0$.

Set

$$e_1(t)=\zeta\tilde u(t) \tag{4.12}$$

and

$$e_k(t)=\begin{cases} e_1(t), & \text{for}\quad |t|\le T,\\ 0, & \text{for}\quad T<|t|\le kT,\end{cases} \tag{4.13}$$

for $k>0$. Then $e_k\in E_k$, $\|e_k\|_{E_k}=\|e_1\|_{E_1}>\rho$ and $I_k(e_k)=I_1(e_1)<0$ for every $k\in\mathbf{N}$. Thus the conditions $(G_1)$ and $(G_2)$ of Lemma 1.1 are satisfied. It follows that $I_k$ possesses a critical value $c_k$ given by

$$c_k=\inf_{\eta\in\Gamma_k}\max{}_{s\in[0,1]}I_k(\eta(s)), \tag{4.14}$$

where

$$\Gamma_k=\{\eta\in C([0,1],E_k)|\eta(0)=0\quad\text{and}\quad\eta(1)=e_k\}.$$

Hence, for every $k \in \mathbf{N}$, there is a $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \quad I_k'(u_k) = 0. \tag{4.15}$$

The function $u_k$ is a desired classical $2kT-$periodic solution of $(\mathbf{HS_k})$. Since $c_k > 0$, $u_k$ is a nontrivial solution even if $h_k(t) = 0$. $\square$

Using an argument as in Lemma 3.8 and Lemma 3.9, we have the next Lemmas.

**Lemma 4.6.** Let $\{u_k\}_{k\in\mathbf{N}}$ be the sequence given by (4.15). Then there exists a solution $u$ of $(\mathbf{HS})$ and a subsequence of $\{u_k\}_{k\in\mathbf{N}}$ (again we call it $\{u_k\}_{k\in\mathbf{N}}$) such that $u_k \to u$ in $C^1_{loc}(\mathbf{R}, \mathbf{R}^n)$ as $k \to +\infty$.

**Lemma 4.7.** The function $u$ determined by Lemma 4.6 is the desired homoclinic solution of $(\mathbf{HS})$.

**Proof of theorem 4.1.** The result follows from Lemma 4.7.

**Proof of theorem 4.2.** The condition $(H_1)$ implies that $I_k$ is even. We know $I_k \in C^1(E, R)$, $I(0) = 0$ and $I$ satisfies the **PS** condition. In order to prove Theorem 4.2 by using the Symmetric Mountain Pass Lemma, we shall show that both $(G_3)$ and $(G_4)$ are true. From the proof of Theorem 4.1, $(G_1)$ is true, so is $(G_3)$.

Consider the condition $(G_4)$. Let $\widetilde{E}_k \subset E_k$ be a finite dimensional subspace. By $(H_2)$, there exist some constants $\alpha_1 > 0, \alpha_2 > 0$ such that

$$G(t, u(t)) \geq \alpha_1 |u(t)|^\theta - \alpha_2, \quad u \in \widetilde{E_k}. \tag{4.17}$$

Then choosing $u_0 \in \widetilde{E}_k$ with $u_0 \neq 0$ arbitrarily, we have by (2.2) and (3.51) that

$$\begin{aligned} I_k(\varsigma u_0) &= \tfrac{\varsigma^2}{2}\Phi_k(u_0) + \varsigma^4 \textstyle\int_{-kT}^{kT} |u_0'(t)|^2 |u_0(t)|^2 dt - \int_{-kT}^{kT} G(t, \mu u_0(t))dt \\ &\leq \tfrac{M_2\varsigma^2}{2}\|u_0\|_{E_k} + \varsigma^4 \textstyle\int_{-kT}^{kT} |u_0'(t)|^2 |u_0(t)|^2 dt - \alpha_1\varsigma^\theta \int_{-kT}^{kT} |u_0(t)|^\theta dt + 2k\alpha_2 T \leq 0, \end{aligned} \tag{4.18}$$

provided $\varsigma > 0$ is large enough and $\theta > 4$. By Lemma 1.2, $I_k$ possesses an unbounded sequence of critical values. This means that, for any positive integer $k$, system $(\mathbf{HS_k})$ possesses infinitely many solutions. Note that when $h(t) \equiv 0$, system $(\mathbf{HS_k})$ becomes $(\mathbf{HS})$. Consequently, system $(\mathbf{HS})$ possesses infinitely many $2kT-$periodic (i.e. subharmonics) solutions. The proof is complete.

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