ON THE LINEAR TYPE PROPERTY OF THE JACOBIAN IDEAL OF AFFINE PLANE CURVES

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Abstract. An ideal I in a Noetherian ring R is called of linear type if the Rees algebra of I is isomorphic to the symmetric algebra of I. In this paper, we prove that the Jacobian ideal of any reduced plane curve with singular points of multiplicity 2 is of linear type. We characterize plane curves with singular points of multiplicity 3 such that its Jacobian ideal is of linear type.

Introduction

Let R be a commutative Noetherian ring and I an ideal of R. The two most common and important commutative algebras related to these data are the *symmetric algebra* $\mathcal{S}_R(I)$ and the Rees algebra $\mathcal{R}_R(I)$ which are known as the blowup algebras. There is natural surjective R-algebra homomorphism $\alpha\colon \mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_R(I)$. The ideal I is called of linear type if α is injective. In this case, the defining equations of the Rees algebra of I arise from the first syzygies of I.

In this work, we focus on the Rees algebra of the Jacobian ideal of an affine plane curve. The basic notion that motivated us for studying the Rees algebra of the Jacobian ideal is the *Aluffi algebra* which is an algebraic version of the characteristic cycle of a hypersurface in intersection theory [10, 11]. Let $X \subseteq M$ be a hypersurface in a smooth variety M and let $Y \subseteq X$ stand for the singular subscheme of X. We denote by $\mathcal{J}_{Y,X}$ the ideal sheaf of Y in X and by $\mathcal{J}_{Y,M}$ the ideal sheaf of Y in M. The Aluffi algebra of $\mathcal{J}_{Y,X}$ is a graded $\mathcal{O}_X$-algebra is defined by

$$\mathcal{A}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) = \mathcal{S}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \otimes_{\mathcal{S}_{\mathcal{O}_M}(\mathcal{J}_{Y,M})} \mathcal{R}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}),$$

where $\mathcal{S}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ and $\mathcal{S}_{\mathcal{O}_M}(\mathcal{J}_{Y,M})$ are the *symmetric algebras* of $\mathcal{J}_{Y,X}$ and $\mathcal{J}_{Y,M}$, respectively and $\mathcal{R}_{\mathcal{O}_M}(\mathcal{J}_{Y,M})$ is the *Rees algebra* of $\mathcal{J}_{Y,M}$. Aluffi [1, Theorem 3.2] proved that the characteristic cycle of X in M equals

$$(-1)^{\dim X}\left[\mathrm{Proj}\mathcal{A}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})\right]. \tag{1}$$

From the definition, the Aluffi algebra is squeezed as

$$\mathcal{S}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \twoheadrightarrow \mathcal{A}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \twoheadrightarrow \mathcal{R}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}). \tag{2}$$

Note that if the ideal sheaf $\mathcal{J}_{Y,M}$ is of *linear type*, then $\mathcal{S}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \simeq \mathcal{A}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ and by (1), the characteristic cycle of X in M is defined in terms of the principal cycles of the naive blowup $\mathrm{Proj}\mathcal{S}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ which simplifies matters computationally. The second author and A. Simis proved that the ideal sheaf $\mathcal{J}_{Y,M}$ is of linear type if and only if the Aluffi algebra of $\mathcal{J}_{Y,X}$ is isomorphic to the corresponding symmetric algebra [11, Proposition 1.4]. In view of Aluffi's results and of applications to characteristic cycles it is natural to pose the problem of *characterizing hypersurface with linear type singular subscheme*. It

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is proven that locally quasi-homogeneous free divisors have linear type Jacobian ideals [2]. Also, a free divisor whose Jacobian ideal is of linear type satisfies a certain Chern class identity [7]. The results and examples in [4,9] indicate that the answer to this problem is difficult in general.

Let $\mathbb{A}_k^2$ be an affine space over a field k of dimension n where k is an algebraically closed field of characteristic zero. Let $X = V(f) \subseteq \mathbb{A}_k^2$ be a reduced curve defined by a reduced polynomial $f \in R = k[x, y]$. By the Jacobian criterion, the singular subscheme of X is defined by the *Jacobian ideal* $I(f) = (f, J(f)) \subseteq R$ where $J(f) = (\partial f/\partial x, \partial f/\partial y)$ is the *gradient* or *critical* ideal of X.

The main goal of this paper is to investigate the linear type property of the Jacobian ideal of affine plane curves. Because we do not know whether the linear type property is an analytic invariant, we will use the ring-theoretic methods to prove our results.

A plane curve is said to be of *Jacobian linear type* if its Jacobian ideal is of linear type. Let $p \in X$ be a non-singular point of X. Denote by $\mathfrak{m}_p$ the maximal ideal of R corresponding to the point p. Then the local ring of X at p, $\mathcal{O}_{X,p} \simeq R_{\mathfrak{m}_p}/I(f)_{\mathfrak{m}_p}$, is a regular local ring and hence $I(f)_{\mathfrak{m}_p}$ is generated by a regular system of parameters which is of linear type. Since the linear type property is local, it follows that $I(f)$ is of linear type if and only if for each singular point p of X the ideal $I(f)_{\mathfrak{m}_p}$ is of linear type. Then the linear type property of the Jacobian ideal $I(f)$ depends on the singularities of $X = V(f)$. Therefore, we characterize Jacobian linear type plane curves in terms of the *multiplicity* of singular points.

A plane curve $X = V(f) \subseteq \mathbb{A}_k^2$ is called *locally Eulerian* if for each singular point p of X the polynomial f belongs to the localization of the gradient ideal $J(f)_{\mathfrak{m}_p}$ at maximal ideal $\mathfrak{m}_p$. By its very definition, the plane curve X is locally Eulerian if and only if the *Milnor number* and the *Tjurina number* of X at each singular point are equal. We prove that a reduced plane curve X is of Jacobian linear type if and only if X is locally Eulerian, see Proposition 1.4. We show that any reduced plane curve with singular points of multiplicity 2 is of Jacobian linear type (Theorem 1.5).

In Theorem 1.7, we characterize the reduced plane curves with singular points of multiplicity 3. More precisely, a reduced plane curve X with a point of multiplicity 3 at the origin of $\mathbb{A}_k^2$ is not of Jacobian linear type if and only if X is defined by the polynomial $x^3 + xy^{k-1} + h(x, y)$ with $k \geq 6$, where $h(x, y)$ has the initial degree $k+1$ such that y^{k+1} belongs to the support of $h(x, y)$. We give examples of the plane curves of multiplicity≥ 4, which are not of Jacobian linear type. We close the paper with a conjecture related to the Jacobian linear type plane curve with points of multiplicity 4.

1. Jacobian linear type plane curves

1.1. **Preliminaries and notations.** Let $R = k[x, y]$ be a polynomial ring over an algebraically closed field k of characteristic zero. Let $X = V(f)$ be a reduced affine plane curve defined by the reduced polynomial $f = f_m + f_{m+1} + \ldots + f_d$, where each f_i is a homogeneous polynomial of degree i and $f_m \neq 0$. The *multiplicity* of X at the origin $p = (0, 0)$ is defined to be $\mathrm{mult}_p(f) = m$. For a point $p = (a, b) \neq (0, 0)$, let T be a linear change of coordinates on $\mathbb{A}_k^2$ such that $T(0, 0) = p$. Setting $f^T = f(x + a, y + b)$. Define $\mathrm{mult}_p(f) = \mathrm{mult}_{(0,0)}(f^T)$. Note that $p \in X$ if and only if $\mathrm{mult}_p(f) > 0$. A point p is called a simple point on X if and only if $\mathrm{mult}_p(f) = 1$. If $\mathrm{mult}_p(f) = r > 1$, then we say that p is singular point (or singularity) of multiplicity r on X.

6 Feb 2023 20:03:49 PST
220821-Nasrollah Version 4 - Submitted to Rocky Mountain J. Math.

The singular subscheme of $X = V(f) \subseteq \mathbb{A}_k^2$ is defined by the Jacobian ideal $I(f) = (f, J(f))$, where $J(f) = (\partial f/\partial x, \partial f/\partial y) \subseteq R$ is the gradient ideal of X. Since X is reduced, it follows that the singular locus of X consists of finitely many points. The *Milnor algebra* and the *Tjurina algebra* of X are defined by

$$M(f) := R/J(f) \quad , \quad T(f) := R/I(f),$$

which are related to the singularities of X. Denote by $\mathfrak{m}_p$, the maximal ideal of R corresponding to the point $p \in X$. For each singular point $p \in \mathrm{Sing}(X)$, the local Milnor algebra $M(f)_{\mathfrak{m}_p}$ is an Artinian Gorenstein ring [4, Lemma 1.1] and $T(f)_{\mathfrak{m}_p}$ is Artinian ring . The numbers

$$\mu_{\mathfrak{m}_p}(f) := \dim_k(M(f)_{\mathfrak{m}_p}) \quad , \quad \tau_{\mathfrak{m}_p}(f) := \dim_k(T(f)_{\mathfrak{m}_p}),$$

are called the *Milnor number* and *Tjurina number* of the plane curve X at singular point p, respectively. When the base field is $k = \mathbb{C}$, it is well-known that the Milnor number and the multiplicity are topological invariants while the Tjurina number is an analytic invariant.

Definition 1.1. A plane singular curve $X = V(f)$ is called *locally Eulerian* if $f \in J(f)_{\mathfrak{m}_p}$ for every singular point p.

Clearly, a plane curve X is locally Eulerian if and only if $\mu_{\mathfrak{m}_p}(f) = \tau_{\mathfrak{m}_p}(f)$ for each singular point p. Also, recall that a polynomial $f \in R$ is *quasi-homogeneous* of degree d and weight r_1, r_2 if it is satisfies $f = (r_1/d)x\partial f/\partial x + (r_2/d)y\partial f/\partial y$. Clearly, any quasi-homogeneous plane curve is locally Eulerian but the converse is not true in general. Note that by K. Saito's famous result[12], locally Eulerian is equivalent to quasi-homogeneous in the analytic category.

Definition 1.2. A plane curve $X = V(f)$ is said to be of *Jacobian linear type* if the Jacobian ideal $I(f) \subseteq R$ is of linear type.

Remark 1.3. Let $X = V(f) \subseteq \mathbb{A}_k^2$ be a reduced plane curve. Let $p \in X$ be a point. If p is a simple point, then $(k[x, y]/I(f))_{\mathfrak{m}_p}$ is a regular local ring. By [6, Corollary 3.10], the ideal $I(f)_{\mathfrak{m}_p}$ is of linear type. Since the linear type property is local, it follows that $I(f)$ is of linear type if and only if $I(f)_{\mathfrak{m}_p}$ is of linear type for every singular point of X. Therefore, the linear type property of the Jacobian ideal of X depends on the singularities of X.

The non-singular and quasi-homogeneous curves are of Jacobian linear type [9, Proposition 2.1]. An algebraic characterization of Jacobian linear type affine hypersurfaces with isolated singularities is given in [4, Theorem 1.3]. In analytic category, X. Liao in [7] showed that a free divisor whose Jacobian ideal is of linear type must satisfy a certain Chern class identity, and [8] showed that a curve singularity satisfies the Chern identity if and only if it is locally Eulerian. Since singular plane curves are automatically free divisors, it follows that they are locally Eulerian.

In the following result, we characterize Jacobian linear type plane curves which is the particular case of [9, Theorem 2.3]. The proof is different from the latter and it is based only on ring theoretic arguments.

Theorem 1.4. *A reduced singular plane curve $X = V(f)$ is of Jacobian linear type if and only if X is locally Eulerian.*

Proof. Assume that X is of Jacobian linear type. Then the Jacobian ideal $I(f) \subseteq R = k[x, y]$ is of linear type. Let p be a singular point of X. Note that since X is reduced with isolated singularities, it follows that $\mathrm{ht}\,(I(f)) = 2 = \mathrm{ht}\,(\mathfrak{m}_p)$. By [6, Proposition 2.4], one has

$$\mathrm{ht}\,(I(f)_{\mathfrak{m}_p}) \leq \mu(I(f)_{\mathfrak{m}_p}) \leq \mathrm{ht}\,\mathfrak{m}_p = \mathrm{ht}\,(I(f)) \leq \mathrm{ht}\,(I(f)_{\mathfrak{m}_p}),$$

which implies that $I(f)_{\mathfrak{m}_p} \subseteq R_{\mathfrak{m}_p}$ is a complete intersection. By Nakayama's lemma the two generators of $I(f)_{\mathfrak{m}_p}$ may be found among $f, \partial f/\partial x, \partial f/\partial y$. Assume that $I(f)_{\mathfrak{m}_p} = (f, \partial f/\partial y)_{\mathfrak{m}_p}$. We have $\mu_{\mathfrak{m}_p}(f) \geq \tau_{\mathfrak{m}_p}(f) = \dim_k R_{\mathfrak{m}_p}/(f, \partial f/\partial y)_{\mathfrak{m}_p}$. The latter is the intersection multiplicity of the plane curves f and the polar curve $\partial f/\partial y$ at the singular point p. By [5, Lemma 3.37], we have

$$\begin{aligned}
\mu_{\mathfrak{m}_p}(f) &\geq \dim_k R_{\mathfrak{m}_p}/(f, \partial f/\partial y)_{\mathfrak{m}_p} \\
&= \dim_k R_{\mathfrak{m}_p}/(\partial f/\partial x, \partial f/\partial y)_{\mathfrak{m}_p} + \dim_k R_{\mathfrak{m}_p}/(x, \partial f/\partial x)_{\mathfrak{m}_p} \\
&= \mu_{\mathfrak{m}_p}(f) + \dim_k R_{\mathfrak{m}_p}/(x, f)_{\mathfrak{m}_p} - 1.
\end{aligned}$$

Thus $\dim_k R_{\mathfrak{m}_p}/(x, f)_{\mathfrak{m}_p} = 1$. Since the multiplicity does not increase by taking a hyperplane section, it follows that $X = V(f)$ is smooth at p which is a contradiction. Hence $I(f)_{\mathfrak{m}_p} = J(f)_{\mathfrak{m}_p}$, which proves that X is locally Eulerian.

Now assume that X is locally Eulerian. Then $I(f)_{\mathfrak{m}_p} = (\partial f/\partial x, \partial f/\partial y)_{\mathfrak{m}_p}$ for each singular point $\mathfrak{m}_p$. Thus the Jacobian ideal is a complete intersection locally at singular points and hence is of linear type locally at each singular point. The assertion follows from the fact that the linear type property is local. $\square$

1.2. **Plane curves with singular points of multiplicity 2 and 3.** In this part, we characterize the Jacobian linear type affine plane curves in terms of the multiplicity of singular points. Note that in the analytic category, the singular plane curve of multiplicity 2 is analytically isomorphic to the curve singularity $V(y^2 - x^m)$ for some $m \geq 2$, which is a consequence of Weierstrass preparation theorem [5, Theorem 1.8]. As we do not know that the linear type property is an analytic property, we give a direct ring theoretic proof by using the standard calculations in plane curve theory.

Theorem 1.5. *Any reduced plane curve with singular points of multiplicity 2 is of Jacobian linear type.*

Proof. Let $f \in R = k[x, y]$ be a reduced polynomial defining the reduced plane curve $X = V(f) \subseteq \mathbb{A}_k^2$. By Proposition 1.4, it is enough to show that X is locally Eulerian. Let $p \in X$ be a singular point of multiplicity 2. By a linear change of coordinates, we may assume that $p = (0, 0)$. Denote by $\mathfrak{m} = (x, y)$ the maximal ideal corresponding to the point p. We can write $f = x^2 + \lambda xy + \alpha y^2 + F(x, y)$, where $F(x, y) := f_3 + \ldots + f_d$ has the initial degree at least 3. Replacing x by $x - (\lambda/2)y$ and rewriting f we obtain

$$f = x^2 + \alpha y^2 + F(x, y), \quad \alpha \in k. \tag{3}$$

Assume that $\alpha \neq 0$. By scaling y, we may assume that $\alpha = 1$. Then $f = x^2 + y^2 + F(x, y)$. Set $f_x := \partial f/\partial x$ and $f_y := \partial f/\partial y$. We have

$$f_x = 2x + F_x(x, y) \quad , \quad f_y = 2y + F_y(x, y).$$

By the condition on the initial degree of $F(x, y)$, we can write $F_x = xg_1(x, y) + yg_2(y)$ and $F_y = yh_1(x, y) + xh_2(x)$. One has

$$f_x = x(2 + g_1(x, y)) + yg_2(y) \quad , \quad f_y = y(2 + h_1(x, y)) + xh_2(x)$$

Since the elements $2 + g_1(x, y)$ and $2 + h_1(x, y)$ are units locally at maximal ideal $\mathfrak{m}$, it follows that $(f_x, f_y)_{\mathfrak{m}} = (x, y)_{\mathfrak{m}}$, which proves that f is locally Eulerian in this case.

6 Feb 2023 20:03:49 PST
220821-Nasrollah Version 4 - Submitted to Rocky Mountain J. Math.

Now assume that $\alpha = 0$ in (3) and $k \geq 3$ is the smallest positive integer such that y^k belongs to the support of F. We write $f_k = y^k + x\varphi(x,y)$, where $\varphi \in \mathfrak{m}^{k-1}$. Then $f = x^2+y^k+x\varphi(x,y)+h(x,y)$, where $h(x,y)$ has the initial degree at least $k+1$. Replacing x by $x-(1/2)\varphi$ and rewriting f, we obtain

$$f = x^2 + y^k + h(x,y).$$

The gradient ideal of f is generated by

$$f_x = 2x + h_x \quad , \quad f_y = ky^{k-1} + h_y.$$

We write $h_x = xg_1(x,y) + y^{k-1}g_2(y)$, $h_y = xh_1(x,y) + y^{k-1}h_2(y)$. We obtain

$$J(f) = (x(2+g_1(x,y)) + y^{k-1}g_2(y)\ ,\ y^{k-1}(k+h_2(y)) + xh_1(x,y)).$$

Since the elements $2+g_1(x,y)$ and $k+h_2(y)$ are units locally at $\mathfrak{m}$, it follows that $f \in J(f)_{\mathfrak{m}} = (x, y^{k-1})$ which proves that f is locally Eulerian.

To complete the proof, we need to consider the case that $\alpha = 0$ and y^k for $k \geq 3$ does not belong to the support of F in (3). Then $f(x,y) = x^2+xh(x,y)$, where $h(x,y)$ has the initial degree at least 2. Replacing x by $x-(1/2)h(x,y)$, we obtain $f(x,y) = x^2-(1/2)h^2(x,y)$, where $h^2(x,y)$ has the initial degree at least 4. Since f is reduced, it follows that y^k belongs to the support of $h^2(x,y)$ for some $k \geq 4$. Then we apply the same argument as above to prove that f is locally Eulerian in this case. $\square$

We now discuss the Jacobian linear type plane curve with singular points of multiplicity 3. We will need the following lemma for the characterization.

Lemma 1.6. *Let $X = V(f)$ be an affine plane curve defined by the polynomial $f(x,y) = x^3+xy^{k+3}+h(x,y)$ with $k \geq 0$, where $h(x,y)$ has the initial degree $k+5$ and y^{k+5} belongs to the support of $h(x,y)$. Then X is of Jacobian linear type if and only if $k = 0, 1$.*

Proof. The curve X has a singular point at the origin of $\mathbb{A}^2_k$. The gradient ideal of f is generated by

$$f_x = 3x^2 + y^{k+3} + h_x \quad , \quad f_y = (k+3)xy^{k+2} + h_y.$$

By the condition on the initial degree of $h(x,y)$, we write

$$\begin{aligned} h_x &= y^{k+3}g_1(x,y) + y^{k+2}x^2g_2(x) + y^{k+1}x^3g_3(x) + \ldots + yx^{k+3}g_{k+3}(x) + x^{k+4}g_{k+4}(x), \\ h_y &= y^{k+4}u_0(y) + xy^{k+2}u_1(x,y) + x^3y^{k+1}u_3(x) + x^4y^ku_4(x) + \ldots + x^{k+4}u_{k+4}(x). \end{aligned}$$

Thus

$$\begin{aligned} f_x &= y^{k+3}(1+g_1) + x^2(3 + y^{k+2}g_2 + y^{k+1}xg_3 + \cdots + x^{k+2}g_{k+4}), \\ f_y &= xy^{k+2}(k+3+u_1) + y^{k+4}u_0 + x^{k+4}u_{k+4} + x^3y(y^ku_3 + \cdots + x^ku_{k+3}). \end{aligned}$$

The elements $1+g_1$ and $k+3+u_1$ are units locally at $\mathfrak{m} = (x,y)$ and we denote by a and b the inverses of these elements, respectively. We obtain

$$\begin{aligned} af_x &= y^{k+3} + ax^2(3 + y^{k+2}g_2 + y^{k+1}xg_3 + \cdots + x^{k+2}g_{k+4}), \\ bf_y &= xy^{k+2} + by^{k+4}u_0 + bx^{k+4}u_{k+4} + bx^3y(y^ku_3 + \cdots + x^ku_{k+3}). \end{aligned}$$

Assume that $k = 0$. The gradient ideal locally at $\mathfrak{m}$ is generated by

$$f_x = y^3 + ax^2(3 + y^2g_2 + xyg_3 + x^2g_4),\ f_y = xy^2 + by^4u_0 + bx^4u_4 + bx^3yu_3.$$

We claim that $(x^3, x^2y^2, xy^3, y^5)_{\mathfrak{m}} \subseteq J(f)_{\mathfrak{m}}$ in the local ring $R_{\mathfrak{m}}$. Using the division algorithm with respect to the negative degree reverse lexicographical local ordering on $R_{\mathfrak{m}}$ with $x > y$, we obtain the following relation in $R_{\mathfrak{m}}$:

$$\begin{aligned} (3a + P(x,y))x^3 &= H_1 f_x + H_2 f_y, \\ (1 + Q(x,y))x^2y^2 &= b^2u_0^2y^3H_3 + (x - bu_0y^2)H_4, \\ (1 + 3ab^2u_0^2y)y^5 &= (y^3 + 3ax^2)y^2 + 3a(bu_0y^2 - x)H_4, \end{aligned}$$

where $P(x,y), Q(x,y) \in R_{\mathfrak{m}}$ are polynomials with the initial degree at least one and

$$\begin{aligned} H_1 &= x + bu_0(y^2 + bu_4x^3 + 3abu_0xy - ag_2x^2y), \\ H_2 &= y + 3abu_0x, \\ H_3 &= y^3 + 3ax^2 + ax^2y^2g_2, \\ H_4 &= xy^2 + by^4u_0. \end{aligned}$$

Since the elements $3a + P(x,y)$, $1 + Q(x,y)$, $1 + 3ab^2u_0^2y$ are units locally at $\mathfrak{m}$, it follows that $(x^3, x^2y^2, y^5)_{\mathfrak{m}} \in J(f)_{\mathfrak{m}}$. The relation $xy^3 = by^5u_0 - yH_4$ implies that $xy^3 \in J(f)_{\mathfrak{m}}$ which proves the claim. By the above claim and the assumption on the initial degree of $h(x,y) = \sum_{i=0}^{r} A_{i,r-i}x^iy^{r-i}$ with $r \geq 5$, we conclude that all terms of $f(x,y)$ belong to the ideal $(x^3, x^2y^2, xy^3, y^5)_{\mathfrak{m}} \subseteq J(f)_{\mathfrak{m}}$ which proves that X is locally Eulerian.

Now assume that $k = 1$. The gradient ideal locally at $\mathfrak{m}$ is generated by

$$\begin{aligned} f_x &= y^4 + ax^2(3 + y^3g_2 + xy^2g_3 + x^2yg_4 + x^3g_5), \\ f_y &= xy^3 + by^5u_0 + bx^5u_5 + bx^3y(yu_3 + xu_4). \end{aligned}$$

We claim that

$$(x^4,\ x^3y,\ x^2y^3,\ xy^5,\ y^7)_{\mathfrak{m}} \subseteq J(f)_{\mathfrak{m}}.$$

To prove the claim, we apply the division algorithm with respect to the negative degree reverse lexicographical local ordering on $R_{\mathfrak{m}}$ with $x > y$ in $R_{\mathfrak{m}}$. We put

$$\omega_1 = a\lambda + P(x,y), \quad \omega_2 = a\lambda + Q(x,y), \quad \omega_3 = ab^2u_0^2g_2y^3 + \lambda - 2,$$

where $\lambda = 3ab^2u_0^2 + 3$ and $P(x,y), Q(x,y) \in R_{\mathfrak{m}}$ are polynomials with the initial degree at least one. We have

$$\begin{aligned} \omega_1x^4 &= H_1f_x + H_2f_y \\ \omega_2x^3y &= M_1H_3 + M_2H_4 \\ \omega_3x^2y^3 &= b^2u_0^2y^3M_3 + (x - bu_0y^2)M_4 \end{aligned}$$

where

$$\begin{array}{ll} M_1 = y^4 + ax^2(3 + y^3g_2 + xy^2g_3) \quad , & M_2 = xy^3 + by^5u_0 + bx^3y^2u_3 \\ M_3 = y^4 + 3ax^2 + ax^2y^3g_2 \quad , & M_4 = xy^3 + by^5u_0, \end{array}$$

and

$$\begin{aligned} H_1 &= -(\lambda - 2)x^2 + \lambda b^4u_0^2u_3^2x^4y^2 + b^3u_0^3(ag_3 + bu_4)x^2y^4 - b^2u_0^2((\lambda - 1)bu_3 + ag_2)x^2y^3 \\ &+ b^4u_0^3u_3(\lambda bu_3 + ag_2)x^3y^4 + b^2u_0^2y(y^3 + bu_5x^4 + bu_4x^3y - b^2u_0u_3xy^4), \\ H_2 &= xy - bu_0y^3 + b^2u_0u_3(\lambda - 2)x^2y^2 - b^2u_0^2(ag_3 + bu_4)x^2y^3 + b^3u_0^2u_3xy^4, \\ &+ \lambda b^4u_0^2u_3^2x^3y^3 \\ H_3 &= (\lambda - 2)xy + bu_0y^3 + abu_0(3b^3u_0^2u_3 - g_2)x^2y^2, \\ H_4 &= y^2 + 3abu_0x(1 + b^2u_0u_3xy). \end{aligned}$$

Note that the element λ is invertible in $R_{\mathfrak{m}}$, otherwise u_0 has degree at most zero and $-3u_0y^2 = (1+g_1(x,y))(4+u_1(x,y))^2$, which is impossible as g_1, u_1 have the initial degree at least 1. Since the elements a and b are the inverses of $1+g_1$ and $4+u_1$, respectively, it follows that the elements ω_1, ω_2, ω_3 are units locally at $\mathfrak{m}$. Hence x^4, x^3y, x^2y^3 locally belong to $J(f)_{\mathfrak{m}}$. Using the polynomials y^3M_3 and y^2M_4, we conclude that y^7, $xy^5 \in J(f)_{\mathfrak{m}}$ which completes the proof of the claim. By the claim and the assumption on the initial degree of $h(x,y)$, it follows that all terms of $f(x,y)$ except the terms x^3, xy^4, y^6 belong to $J(f)_{\mathfrak{m}}$. Set $F = x^3 + xy^4 + cy^6$ with $c \in k$. We write $g_1 = xG_1(x,y) + yG_2(y)$ and $u_1 = xU_1(x,y) + yU_2(y)$. We obtain

$$\begin{aligned} f_x &= x^4(yg_4 + xg_5) + x^3y^2g_3 + x^2y^3g_2 + F_1(x,y), \\ f_y &= x^4(yu_4 + xu_5) + x^3y^2u_3 + x^2y^3U_1 + F_2(x,y), \end{aligned}$$

where

$$\begin{aligned} F_1(x,y) &= 3x^2 + y^4(1 + xG_1 + yG_2), \\ F_2(x,y) &= xy^3(4 + yU_2) + y^5u_0 = xy^3(4 + yU_2) + y^5(6c + yT(y)). \end{aligned}$$

Clearly, F_1, F_2 locally belong to $J(f)_{\mathfrak{m}}$. One has

$$3(16 + 4U_2y + (3cU_2 - 2T(y))G_1y^3)F = xT_1F_1 - yT_2F_2 + y^7D(x,y),$$

where $T_1 = 16 + 4U_2y + (3cU_2 - 2T)G_1y^3$, $T_2 = 2G_1x + 2G_2y - 3cG_1y^2 - 4$ and $D(x,y)$ is a polynomial in $R_{\mathfrak{m}}$. Thus $F = x^3 + xy^4 + cy^6 \in J(f)_{\mathfrak{m}}$, which proves that f is locally Eulerian and hence X is of gradient linear type.

For $k \geq 2$, the same argument as the case $k = 1$, shows that y^{k+6} belongs locally to the gradient ideal $J(f)_{\mathfrak{m}}$ and the polynomials f_x, f_y, y^{k+6} are a standard basis of $J(f)_{\mathfrak{m}}$ with respect to the negative degree reverse lexicographical local ordering on $R_{\mathfrak{m}}$ with $x > y$. Thus

$$\mu_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/J(f)_{\mathfrak{m}} = \dim_k R_{\mathfrak{m}}/(x^2, xy^{k+1}, y^{k+6}) = 2(k+5) - 2 = 2k + 8.$$

It is easy to check that the polynomials f_x, f_y, y^{k+5} are a standard basis for $I(f)_{\mathfrak{m}}$. One has

$$\tau_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/I(f)_{\mathfrak{m}} = \dim_k R_{\mathfrak{m}}/(x^2, xy^{k+1}, y^{k+5}) = 2(k+5) - 1 = 2k + 7.$$

Thus $\mu_{\mathfrak{m}}(f) \neq \tau_{\mathfrak{m}}(f)$ for $k \geq 2$, which completes the proof. $\square$

Theorem 1.7. *Let X be a reduced plane curve with a singular points of multiplicity 3 at the origin of $\mathbb{A}^2_k$. The following are equivalent.*

(a) *X is not of Jacobian linear type.*
(b) *For $k \geq 6$, $X = V(x^3 + xy^{k-1} + h(x,y))$ where $h(x,y)$ has the initial degree $k+1$ such that y^{k+1} belongs to the support of $h(x,y)$.*

Proof. Let $f \in R = k[x,y]$ be a reduced polynomial defining the plane curve $X = V(f)$. Let $p \in X$ be a singular point of multiplicity 3. By a linear change of coordinates, we may assume that $p = (0,0)$. We can write $f = x^3 + \alpha x^2y + \beta xy^2 + \gamma y^3 + F(x,y)$, where $F(x,y)$ has the initial degree at least 4 and α, $\beta \in k$. Replacing x by $x - (1/3)\alpha y$ and rewriting f, we may assume that

$$f = x^3 + \alpha xy^2 + \beta y^3 + F(x,y).$$

The gradient ideal of f is generated by

$$f_x = 3x^2 + \alpha y^2 + F_x(x,y) \quad , \quad f_y = 2\alpha xy + 3\beta y^2 + F_y(x,y).$$

6 Feb 2023 20:03:49 PST
220821-Nasrollah Version 4 - Submitted to Rocky Mountain J. Math.

Set $F(x,y) = f_4 + f_5 + \ldots + f_d$, where f_i is a homogeneous polynomial of degree i. We can write

$$F_x(x,y) = f_4^{(x)} + \ldots + f_d^{(x)} \quad , \quad F_y(x,y) = f_4^{(y)} + \ldots + f_d^{(y)},$$

where $f_i^{(x)} = (f_i)_x$ and $f_i^{(y)} = (f_i)_y$. Consider the negative degree reverse lexicographical local ordering with $x > y$ on $R_{\mathfrak{m}}$. Assume that $\alpha \neq 0$. The S-polynomial of f_x and f_y is

$$S(f_x, f_y) = 2\alpha^2 y^3 - 9\beta xy^2 + 2\alpha y F_x - 3xF_y,$$

which upon division by f_x, f_y is reducible to

$$2\alpha^2 y^3 + a_1 y^4 + \ldots + a_{d-3} y^{d-1},$$

where a_i belong to k. Hence $y^3 \in J(f)_{\mathfrak{m}}$. If we add the monomial y^3 to the generators of $J(f)$, then we can conclude that the set $\{3x^2 + \alpha y^2,\, 2\alpha xy + 3\beta y^2,\, y^3\}$ is an standard basis for $J(f)_{\mathfrak{m}}$ as the S-polynomials of the pairs $\{3x^2 + \alpha y^2,\, 2\alpha xy + 3\beta y^2\}$, $\{3x^2 + \alpha y^2,\, y^3\}$ and $\{2\alpha xy + 3\beta y^2,\, y^3\}$ reduces to zero. Since an standard basis is a specific generating set of an ideal, it follows that $J(f)_{\mathfrak{m}} = (3x^2 + \alpha y^2,\, 2\alpha xy + 3\beta y^2,\, y^3)_{\mathfrak{m}}$. One has

$$x^3 = 1/3x(3x^2 + \alpha y^2) - 1/6y(2\alpha xy + 3\beta y^2) + 1/2\beta(y^3).$$

If $\beta = 0$, then clearly $xy^2 \in J(f)_{\mathfrak{m}}$. Otherwise we obtain the relation

$$xy^2 = (1/2\alpha)y(2\alpha xy + 3\beta y^2) - ((2/3)(\alpha/\beta)(y^3)$$

Therefore, $f \in (x^3, xy^2, y^3)_{\mathfrak{m}} \subseteq J(f)_{\mathfrak{m}}$, which proves that f is locally Eulerian and hence X is of Jacobian linear type.

Assume that $\alpha = 0$ and $\beta \neq 0$. We may assume that $\beta = 1$ by scaling y. Then $f_x = 3x^2 + F_x(x,y)$ and $f_y = 3y^2 + F_y(x,y)$. By the condition on the initial degree of $F(x,y)$, we can write $F_x = x^2 g_1(x,y) + y^2 g_2(x,y)$ and $F_y = y^2 h_1(x,y) + x^2 h_2(x,y)$. One has

$$f_x = x^2(3 + g_1(x,y)) + y^2 g_2(x,y) \quad , \quad f_y = y^2(3 + h_1(x,y)) + x^2 h_2(x,y)$$

Since the elements $3 + g_1(x,y)$ and $3 + h_1(x,y)$ are units locally at maximal ideal $\mathfrak{m}$, it follows that $J(f)_{\mathfrak{m}} = (x^2, y^2)_{\mathfrak{m}}$, which proves that f is locally Eulerian in this case.

Now assume that $\alpha = \beta = 0$. Let f_k be the smallest nonzero homogeneous component of $F(x,y) = f_4 + \cdots + f_d$. We write $f_k = ay^k + bxy^{k-1} + x^2\varphi(x,y)$, where $a, b \in k$ and $\varphi(x,y) \in \mathfrak{m}^{k-2}$. Then

$$f(x,y) = x^3 + ay^k + bxy^{k-1} + x^2\varphi(x,y) + H(x,y), \tag{4}$$

where $H(x,y)$ has the initial degree at least $k+1$.

Assume that $a \neq 0$. We may assume that $a = 1$ by scaling y. Then by replacing first y by $y - (b/4)x$ and second x by $x - (1/3)\varphi(x,y)$ and rewriting f, we obtain

$$f(x,y) = x^3 + y^k + h(x,y),$$

where $h(x,y)$ has the initial degree at least $k+1$. The gradient ideal of f is generated by

$$f_x = 3x^2 + h_x \quad , \quad f_y = ky^{k-1} + h_y.$$

We can write

$$h_x = x^2 g_1(x,y) + y^{k-1} g_2(x,y) \quad , \quad h_y = y^{k-1} h_1(x,y) + x^2 h_2(x,y),$$

where g_1 and h_2 has the initial degree at least $k-2$ and g_2, h_1 has the initial degree at least one. We obtain

$$f_x = x^2(3 + g_1(x,y)) + y^{k-1} g_2(x,y) \quad , \quad f_y = y^{k-1}(k + h_1(x,y)) + x^2 h_2(x,y).$$

Since $3 + g_1(x,y), k + h_1(x,y)$ are units locally at $\mathfrak{m}$, it follows that

$$J(f)_{\mathfrak{m}} = (x^2 + \lambda_1 y^{k-1} g_2(x,y)\ ,\ y^{k-1} + \lambda_2 x^2 h_2(x,y)),$$

where λ_1, λ_2 are the inverses of $3 + g_1(x,y)$ and $k + h_1(x,y)$ in $R_{\mathfrak{m}}$, respectively. Hence

$$\omega x^2 = f_x - (\lambda_1 g_2) f_y \quad , \quad \omega y^{k-1} = f_y - (\lambda_2 h_2) f_x, \tag{5}$$

where $\omega = 1 - \lambda_1\lambda_2 g_2 h_2$. Since $\omega \notin \mathfrak{m}$, the relations (5) show that $(x^2,\, y^{k-1}) \in J(f)$ locally at $\mathfrak{m}$ and hence $J(f)_{\mathfrak{m}} = (x^2, y^{k-1})_{\mathfrak{m}}$, which proves that f is locally Eulerian.

Now assume that $a = 0$ and $b \neq 0$ in (4). We may assume that $b = 1$ by scaling y. Replacing x by $x - (1/3)\varphi(x,y)$ and rewriting f we obtain

$$f(x,y) = x^3 + xy^{k-1} + h(x,y),$$

where $h(x,y)$ has the initial degree at least $k+1$. Then Lemma 1.6 completes the proof. □

It is natural to pose the problem of *characterizing the Jacobian linear type plane curve with points of multiplicity* ≥ 4. We can find plane curves of higher multiplicity such that the Jacobian ideal is not of linear type.

Example 1.8. Let $X = V(f)$ be a plane curve defined by the polynomial $f = y^4 + x^5 + x^3y^2 + x^2y^3 \subseteq R = k[x,y]$. The curve X has a singular point of multiplicity 4 at the origin. Consider the negative degree reverse lexicographical local ordering on $R_{\mathfrak{m}}$ with $x > y$. The partial derivatives $f_x,\ f_y$ are a standard basis for $J(f)_{\mathfrak{m}}$ as the leading term are relatively prime. Then the leading term ideal $\mathrm{LT}(J(f)_{\mathfrak{m}}) = (x^4, y^3)_{\mathfrak{m}}$. We have

$$\mu_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/J(f)_{\mathfrak{m}} = \dim_k R_{\mathfrak{m}}/\mathrm{LT}(J(f)_{\mathfrak{m}}) = 12.$$

A direct calculation in $R_{\mathfrak{m}}$ implies that $I(f) = (f_x, f_y, x^3y^2)$. Hence

$$\tau_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/I(f)_{\mathfrak{m}} = \dim_k R_{\mathfrak{m}}/\mathrm{LT}(I(f)_{\mathfrak{m}}) = \dim_k R_{\mathfrak{m}}/(x^4, y^3, x^3y^2) = 11,$$

which shows that X is not locally Eulerian. A computation in [3] yields that the defining ideal of the Rees algebra of $I(f)$ contains a quadratic polynomial in T_i's.

Example 1.9. Consider the reduced plane curve $X = V(f)$ defined by the polynomials $f = y^{d-1} + x^d + x^2y^{d-2} + xy^{d-1}$ with $d \geq 5$. The curve X has a singular point of multiplicity $d-1$ at the origin. We show that X is of Jacobian linear type if and only if $d = 5$. We consider the negative degree reverse lexicographical local ordering on $R_{\mathfrak{m}}$ with $x > y$. Since the leading monomial of the partial derivatives f_x and f_y are relatively prime, it follows that $f_x,\ f_y$ are a standard basis for $J(f)_{\mathfrak{m}}$. One has

$$\mu_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/\mathrm{LT}(J(f)_{\mathfrak{m}}) = \dim_k R_{\mathfrak{m}}/(x^{d-1}, y^{d-2}) = (d-1)(d-2).$$

It is easy to check that

$$f_x,\ f_y,\ (d-2)^2x^4y^{d-3} + ((d-2)^2 + (d-1)^2 - 1)x^3y^{d-2} + (d-1)^2x^2y^{d-1}$$

are a standard basis for the Jacobian ideal $I(f)$. We get

$$\tau_{\mathfrak{m}}(f) = \dim_k R_{\mathfrak{m}}/\mathrm{LT}(I(f)_{\mathfrak{m}}) = \dim_k R_{\mathfrak{m}}/(x^{d-1}, y^{d-2}, x^4y^{d-3}) = (d-1)(d-2) - (d-5).$$

Therefore, for $d = 5$, we get $\mu_{\mathfrak{m}}(f) = \tau_{\mathfrak{m}}(f) = 12$, which shows that in this case X is locally Eulerian.

Example 1.10. Let $X = V(f)$ be a plane curve with a singular point of multiplicity 4 defined by the polynomial $f = xy(x^2-y^2)+x^5+y^5+x^4y+x^3y^2$. Then X has a singular point at the origin. A computation in the local ring $k[x,y]_{(x,y)}$ shows that X is locally Eulerian and hence X is of Jacobian linear type.

For plane curves of multiplicity 4, we derive the following:

Conjecture 1.11. *Let $X = V(f)$ be a plane curve with singular points of multiplicity 4 defined by the polynomial $f = f_4(x,y) + F(x,y)$, where f_4 is a homogeneous polynomial of degree 4 and F has the initial degree at least 5. Then X is of Jacobian linear type if and only if $V(f_4(x,y))$ has no singular point in $\mathbb{A}^2_k \setminus \{(0,0)\}$.*

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