Observer based control for practical stabilization of one-sided Lipschitz nonlinear systems

Nadhem ECHI[a] and Fehmi MABROUK[b]

[a,b]Gafsa University, Faculty of Sciences of Gafsa
Department of Mathematics, Zarroug Gafsa 2112 Tunisia
[a]e-mail: nadhemechi_fsg@yahoo.fr
[b]e-mail: fehmi.mabrouki@gmail.com

Abstract

This paper focuses on the observer design for a class of nonlinear systems involving bounded disturbances. This class of systems is a larger class of nonlinearities than the Lipschitz. We provide sufficient conditions for the observer design based on the one-sided Lipschitz and quadratically inner-bounded ones. In addition, we show the practical stability of the closed-loop system. Furthermore, we establish a separation principle for a class of nonlinear systems with bounded uncertain part. Besides, we give a numerical example to demonstrate the effectiveness and applicability of the proposed controller.

Keywords: One-sided Lipschitz nonlinear systems, observer design, practical stability, output feedback stabilization, separation principle.

1 Introduction

The separation approach using high gain observers for the stabilization design presents a motivation for several works of modern control theory. Many papers and monographs have dealt with the theory of output feedback stabilization satisfying a global Lipschitz condition for the last decades. In this regard, [4] and [5] used a high gain observer to prove the separation principle for nonlinear systems. In [11], Rajamani built a sufficient condition ensuring the stability of the observers for Lipschitz systems.

However, if the Lipschitz constant becomes large, the Lipschitz observer may no longer be valid. In mathematics, to solve this problem, Hairer et al. introduced in [9] a new condition called 'one-sided Lipschitz continuity less restrictive than the classical Lipschitz condition.

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

By comparing the one-sided Lipschitz constant and that of the traditional Lipschitz condition, it is clear that the region of the one-sided Lipschitz constant is much larger, because the Lipschitz constant must be positive while the one-sided Lipschitz constant can be zero or even negative. Therefore, designing observers for one-sided Lipschitz nonlinear systems has been the subject of numerous papers [1, 2, 3, 19, 20, 22] and references therein. Using the Riccati equation method, the observer design problem for one-sided Lipschitz nonlinear systems. The problem of exponential observer design for one-sided Lipschitz nonlinear systems is investigated in [24]. By solving linear matrix inequalities [21], an observer for a class of one-sided Lipschitz nonlinear systems is introduced.

In practice, dynamics, measurement noises, and other disturbances often prevent the origin from being an equilibrium point of the uncertain system. In the existing literature, many researchers have studied the problem of observer design for systems with unknown inputs [6, 16]. But, the greater number of the found results is focused on a special class of nonlinear systems. In recent years, a special attention has been paid to the observer design of nonlinear systems to unknown disturbances, which remains a thorny issue. A linear matrix inequality approach is developed in [12] to propose an observer for a one-sided Lipschitz nonlinear system with unknown inputs. Under Lipschitz constant, Farza et al. have built in [8], an observer for a class of nonlinear systems with unknown input. The adaptive observer design problem is studied in [10] for quasi-one-sided Lipschitz nonlinear systems with uncertain parameters. Sufficient conditions formulated in terms of linear matrix inequalities obtained by [13] to solve the unknown input observer design problem of one-sided Lipschitz nonlinear systems. In [15], Treangle et al. have built a design of state observers for a class of uniformly observable systems written in a triangular form involving bounded disturbances.

In this paper, we focus on a particular class of nonlinear systems in the presence of uncertainties in state equations. Uncertainties will be treated as an unknown function that explicitly depends on time. The nonlinear part satisfies the one-sided Lipschitz condition, whilst the uncertain part is bounded. Our objective is to extend the observer design proposed in [1] and [10] to this class of systems, while providing practical stability. More specifically, referring to Lyapunov function, we investigate the problem of designing an observer to estimate the system states. Uncertain bounded and sufficient conditions are given to ensure the practical stability of the proposed observer. In the presence of uncertainties, the asymptotic estimation error remains in a ball whose radius, particularly depends on the one sided Lipschitz constant of the system nonlinearities as well as on the bound of the uncertainties. Then, we show that practical stability of the closed-loop system with linear state feedback is achieved. Finally, a separation principle is established. In other words, we implement the control law with estimate states given by the practical

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

observer and we prove that the closed-loop system is practically stable.

The paper is organized as follows: In Section 2, basic definitions and the system description are given. The design of the proposed observer by constructing Lyapunov functions is presented in Section 3. Moreover, the required assumptions and the statement of the main results are provided. Section 4 illustrates the validity of our design method in the selected numerical example. Conclusions are drawn in section 5.

2 System description and preliminary

Consider the following system:

$$\begin{cases} \dot{x}(t) = f(x(t), u), \\ x(t_0) = x_0. \end{cases} \tag{1}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input.

Definition 1 *A solution of* (1) *is said to be globally uniformly bounded if for every* $\alpha > 0$ *there exists* $c = c(\alpha)$ *such that, for all* $t_0 \geq 0$,

$$\|x_0\| \leq \alpha \;\Rightarrow\; \|x(t)\| \leq c(\alpha),\; \forall\, t \geq 0.$$

We will recall the definition of uniform stability and uniform attractivity of (1) towards

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leq r;\; r \geq 0\}$$

Definition 2

- *B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $t_0 \geq 0$,*
$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \;\forall t \geq t_0$$
- *B_r is globally uniformly stable if it is uniformly stable and the solutions of system* (1) *are globally uniformly bounded.*

Definition 3 *B_r is globally uniformly attractive if for all $\varepsilon > r$ and $c > 0$, there exists $T(\varepsilon, c) > 0$ such that for all $t_0 \geq 0$,*

$$\|x(t)\| < \varepsilon \;\forall t \geq t_0 + T(c), \|x_0\| < c.$$

Definition 4 *System* (1) *is globally uniformly practically asymptotically stable if there exists $r \geq 0$ such that B_r is globally uniformly stable and globally uniformly attractive.*

In this paper, we consider a class of uncertain nonlinear system described by:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + f(t, x(t)) + B\varepsilon(t), \\ y(t) &= Cx(t), \end{cases} \tag{2}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $\varepsilon : \mathbb{R}_+ \to \mathbb{R}^p$ is an unknown disturbance, $f(t, x(t))$ represents the nonlinear dynamics associated with the state vector with $f(t, 0) = 0$, and A, B and C correspond to the linear constant matrices of a system of appropriate dimensions.

Notation 1 *Throughout the paper, A^T means the transpose of A. $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and minimal eigenvalue of a matrix A respectively. $P > 0$ means that the matrix P is symmetric positive definite matrix. I is an appropriately dimensioned identity matrix. $\langle ., . \rangle$ is the inner product in $\mathbb{R}^n$, i.e., given $x, y \in \mathbb{R}^n$, then $\langle x, y \rangle = x^T y$ and $\| \; \|$ being the Euclidean-norm in $\mathbb{R}^n$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$.*

Definition 5 *[1] Let $\mathcal{D}$ a region including the origin. The nonlinear function $f(t, x)$ is said to be one-sided Lipschitz if there exists $\rho \in \mathbb{R}$ such that $\forall \; x_1, x_2 \in \mathcal{D}$*

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq \rho \|x_1 - x_2\|^2, \tag{3}$$

where $\rho \in \mathbb{R}$ is called the one-sided Lipschitz constant.

Definition 6 *[1] The nonlinear function $f(t, x)$ is called quadratically inner-bounded in the region $\widetilde{\mathcal{D}}$, if $\forall \; x_1, x_2 \in \widetilde{\mathcal{D}}$ there exist $\beta, \; \gamma \in \mathbb{R}$ such that*

$$[f(t, x_1) - f(t, x_2)]^T [f(t, x_1) - f(t, x_2)] \leq \beta \|x_1 - x_2\|^2 + \gamma \langle x_1 - x_2, f(t, x_1) - f(t, x_2) \rangle. \tag{4}$$

Remark 1 *In [1] and [19] the authors consider a class of nonlinear systems satisfying a one-sided Lipschitz condition. However, the system* (2) *generalizes the systems introduced by [1] and [19] for the case of an nonlinear system without disturbance.*

Remark 2 *[8] and [15] consider a specific class of nonlinear systems. The system have been considered in which the nonlinear parts satisfy the Lipschitz condition while the uncertain part is bounded. The current work is the extension of that one proposed [8] and [15] which the nonlinear parts satisfy the one-sided Lipschitz constants. It easy to see that the one-sided Lipschitz constants are significantly smaller than the classical Lipschitz constants.*

3 Separation principle

The aim of this paper is to give a separation principle in the practical sense of the uncertain nonlinear dynamical systems described by (2). The following assumption is introduced to design the proposed observer:

Assumption 1. The unknown disturbance ε is an essentially bounded function, i.e.

$$\exists \delta_\varepsilon > 0 \text{ such that } \|\varepsilon\| \triangleq \operatorname{ess}\sup_{t\geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon. \tag{5}$$

Assumption 2. The pairs (A, C) is observable.

Assumption 3. The pairs (A, B) is stabilizable.

Remark 3 *Since (A, C) is observable, then there exists a gain matrix L such that for all positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P which satisfies solution of the Lyapunov equation*

$$(A - LC)^T P + P(A - LC) = -Q. \tag{6}$$

3.1 Observer design

In this paragraph, we study the designing of an observer in order to have the states of the uncertain nonlinear system. In practice, we cannot do the direct measurement of all states of the system. In what follows, we try to build an observer in order to ensure the practical stability of the system. We propose the following state observer:

$$\begin{cases} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + f(t, \hat{x}(t)) + L(y(t) - C\hat{x}(t)), \\ \hat{y}(t) &= C\hat{x}(t), \end{cases} \tag{7}$$

where $L = [l_1, \dots, l_n]^T$ a gain matrix such that $A_L := A - LC$ is Hurwitz.

Theorem 1 *Consider a nonlinear system* (2) *satisfying Assumption 1, Assumption 2 and the conditions* (3) *and* (4) *with constants ρ, β and γ, along with the observer* (7)*. In addition, suppose that there exist two matrices P and Q which verify the Lyapunov equation* (6)*, such that*

$$\{(\beta + 1) + \rho(\gamma + 2)\} \lambda_{\max}(P) + 1 < \lambda_{\min}(P) + \lambda_{\min}(Q), \tag{8}$$

then the system (7) *is globally practically asymptotically observer for system* (2)*.*

Proof. Let $e = \hat{x} - x$ denote the estimation error. The dynamics of the observer error is expressed as follows:

$$\dot{e}(t) = (A - LC)e(t) + f(t, \hat{x}(t)) - f(t, x(t)) - B\varepsilon(t). \tag{9}$$

Let us consider the following Lyapunov function candidate:

$$V(e) = e^T Pe. \tag{10}$$

The derivative of V along the trajectories of (7) is given by

$$\begin{aligned}\dot{V}(e) &= e^T(A_L^T P + PA_L)e + 2e^T P(f(t,\hat{x}(t) - f(t,x(t)) - 2e^T PB\varepsilon(t) \\ &= -e^T Qe + 2e^T P(f(t,\hat{x}(t) - f(t,x(t)) - 2e^T PB\varepsilon(t).\end{aligned}$$

We denote $f(t,\hat{x}(t)) := \hat{f}$ and $f(t,x(t)) := f$. On one hand, we have

$$2e^T P(\hat{f} - f) = \left[e + (\hat{f} - f)\right]^T P\left[e + (\hat{f} - f)\right] - e^T Pe - (\hat{f} - f)^T P(\hat{f} - f). \tag{11}$$

Using the quadratic boundedness property, we get

$$\begin{aligned}[e + (\hat{f} - f)]^T[e + (\hat{f} - f)] &= e^T e + 2e^T(\hat{f} - f) + (\hat{f} - f)^T(\hat{f} - f) \\ &\leq e^T e + 2e^T(\hat{f} - f) + \beta e^T e + \gamma\langle\hat{f} - f, \hat{x} - x\rangle \\ &\leq (\beta + 1)e^T e + (\gamma + 2)e^T(\hat{f} - f).\end{aligned} \tag{12}$$

Since P satisfies the following inequality,

$$\lambda_{\min}(P)\|\hat{f} - f\|^2 \leq (\hat{f} - f)^T P(\hat{f} - f) \leq \lambda_{\max}(P)\|\hat{f} - f\|^2. \tag{13}$$

So,

$$\left[e + (\hat{f} - f)\right]^T P\left[e + (\hat{f} - f)\right] \leq \lambda_{\max}(P)\left[e + (\hat{f} - f)\right]^T\left[e + (\hat{f} - f)\right]. \tag{14}$$

Combining (12), (13), (14) and using the one-sided Lipschitz inequality (3), one gets

$$\begin{aligned}2e^T P(\hat{f} - f) &\leq \lambda_{\max}(P)\left[(\beta + 1)e^T e + (\gamma + 2)e^T(\hat{f} - f)\right] - e^T Pe - \lambda_{\min}(P)\|\hat{f} - f\|^2 \\ &\leq \Big\{\lambda_{\max}(P)\left[(\beta + 1) + \rho(\gamma + 2)\right] - \lambda_{\min}(P)\Big\}e^T e.\end{aligned}$$

So using Assumption 1, we deduce that

$$\begin{aligned}\dot{V}(e) &\leq -e^T Qe + \{\lambda_{\max}(P)\left[(\beta + 1) + \rho(\gamma + 2)\right] - \lambda_{\min}(P)\}e^T e - 2e^T PB\varepsilon(t) \\ &\leq -\{\lambda_{\min}(Q) + \lambda_{\min}(P) - \alpha\lambda_{\max}(P)\}\|e\|^2 + 2\|P\|\|B\|\|\varepsilon(t)\|\|e\| \\ &\leq -\{\lambda_{\min}(Q) + \lambda_{\min}(P) - \alpha\lambda_{\max}(P)\}\|e\|^2 + 2\delta_\varepsilon\|P\|\|B\|\|e\|,\end{aligned} \tag{15}$$

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

where

$$\alpha = (\beta + 1) + \rho(\gamma + 2). \tag{16}$$

Let $\mu = \delta_\varepsilon \|P\| \|B\|$. Using the fact that

$$2\mu\|e\| \leq \mu^2 + \|e\|^2. \tag{17}$$

By Equation (15) and (17), we obtain,

$$\dot{V}(e) \quad \leq \quad -\left\{\lambda_{\min}(Q) + \lambda_{\min}(P) - \alpha\lambda_{\max}(P) - 1\right\} \|e\|^2 + \mu^2. \tag{18}$$

So the error dynamic (9) is globally practically asymptotically stable. ∎

Remark 4 *[12] considers a class of nonlinear singular system without disturbances and the nonlinear term satisfies one-sided Lipschitz condition. The observer proposed in [12] has a Markovian jump systems. Different form [12], one-sided Lipschitz nonlinear systems with disturbances are studied in this paper, which is more general. Moreover, the conditions proposed in this paper which are easier to realize in practice.*

Remark 5 *An algorithm which contains two steps given in [2] and [18] to solve the design problem of the observer. On the other hand, a sufficient hypothesis* (8) *is given to guarantee a practical stability of the observer proposed in Theorem 1 which is the best feasibility to implement than those of the methods given in [2] and [18].*

3.2 Global stabilization by state feedback

In this subsection, we establish a condition for the asymptotical state feedback stabilization of the nonlinear system (2). The state feedback controller is given by

$$u = Kx, \tag{19}$$

where $K = [k_1, \ldots, k_n]$ such that $A_K := A + BK$ is Hurwitz. Let S be the symmetric positive definite solution of the Lyapunov equation

$$A_K^T S + S A_K = -Q_1, \tag{20}$$

for any given positive definite symmetric matrix Q_1.

Theorem 2 *Consider a nonlinear system* (2) *satisfying Assumption 1, Assumption 3 and the conditions* (3) *and* (4) *with constants ρ, β and γ. In addition, there exist two matrices S and Q_1 which verify the Lyapunov equation* (20)*, such that*

$$\left\{(\beta + 1) + \rho(\gamma + 2)\right\} \lambda_{\max}(S) + 1 < \lambda_{\min}(S) + \lambda_{\min}(Q_1). \tag{21}$$

Then the origin of the closed loop system (2)*-*(19) *is globally practically asymptotically stable.*

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

Proof. The closed loop system is given by

$$\dot{x}(t) = (A + BK)x + f(t, x(t)) + B\varepsilon(t). \tag{22}$$

Let us choose a Lyapunov function candidate as follows

$$W(x) = x^T Sx. \tag{23}$$

The derivative of W along the trajectories of (22) is given by

$$\begin{aligned} \dot{W}(x) &= x^T(A_K^T S + SA_K)x + 2x^T Sf(t, x(t) + 2x^T SB\varepsilon(t) \\ &= -x^T Q_1 x + 2x^T Sf(t, x(t)) + 2x^T SB\varepsilon(t). \end{aligned} \tag{24}$$

We denote $f(t, x(t)) := f$. Since $f(t, 0) = 0$, we have

$$2x^T Sf = [x + f]^T S [x + f] - x^T Sx - f^T Sf. \tag{25}$$

Using the quadratic boundedness, we get

$$\begin{aligned} [x + f]^T[x + f] &= x^T x + 2x^T f + f^T f \\ &\le x^T x + 2x^T f + \beta x^T x + \gamma\langle x, f(t, x(t))\rangle \\ &\le (\beta + 1)x^T x + (\gamma + 2)x^T f. \end{aligned}$$

Since S satisfies the following inequality,

$$\lambda_{\min}(S)\|f\|^2 \le f^T Sf \le \lambda_{\max}(S)\|f\|^2. \tag{26}$$

So,

$$[x + f]^T S [x + f] \le \lambda_{\max}(S) [x + f]^T [x + f]. \tag{27}$$

From (27) and sided Lipschitz, we get

$$\begin{aligned} 2x^T Sf &\le \lambda_{\max}(S) \left[(\beta + 1)x^T x + (\gamma + 2)x^T f\right] - x^T Sx - \lambda_{\min}(S)\|f\|^2 \\ &\le \Big\{\lambda_{\max}(S) [(\beta + 1) + \rho(\gamma + 2)] - \lambda_{\min}(S)\Big\}x^T x. \end{aligned}$$

Since Q_1 is symmetric positive definite then for all $x \in \mathbb{R}^n$,

$$\lambda_{\min}(Q_1)\|x\|^2 \le x^T Q_1 x \le \lambda_{\max}(Q_1)\|x\|^2.$$

This implies that

$$\begin{aligned}\dot{W}(x) &\leq -x^T Q_1 x + \{\lambda_{\max}(S)\,[(\beta+1)+\rho(\gamma+2)] - \lambda_{\min}(S)\}\, x^T x + 2x^T SB\varepsilon(t)\\ &\leq -\lambda_{\min}(Q_1)\|x\|^2 + \{\lambda_{\max}(S)\,[(\beta+1)+\rho(\gamma+2)] - \lambda_{\min}(S)\}\, x^T x + 2x^T SB\varepsilon(t).\end{aligned}$$

So using Assumption 1, and Equation (16) we deduce that

$$\begin{aligned}\dot{W}(x) &\leq -\{\lambda_{\min}(Q_1) + \lambda_{\min}(S) - \alpha\lambda_{\max}(S)\}\,\|x\|^2 + 2\|S\|\|B\|\|\varepsilon(t)\|\|x\|\\ &\leq -\{\lambda_{\min}(Q_1) + \lambda_{\min}(S) - \alpha\lambda_{\max}(S)\}\,\|x\|^2 + 2\delta_\varepsilon\|S\|\|B\|\|x\|. \qquad (28)\end{aligned}$$

Let $\mu_1 = \delta_\varepsilon\|S\|\|B\|$. Using the fact that

$$2\mu_1\|x\| \leq \mu_1^2 + \|x\|^2. \qquad (29)$$

By Equation (28) and (17), we obtain,

$$\dot{W}(x) \leq -\{\lambda_{\min}(Q_1) + \lambda_{\min}(S) - \alpha\lambda_{\max}(S) - 1\}\,\|x\|^2 + \mu_1^2. \qquad (30)$$

We deduce that the origin of the closed loop system (22) is practically asymptotically stable. ■

3.3 Observer-based control stabilization

Now, a separation principle is established, that is, we implement the control law with estimate states given by the practical observer and we prove that the closed loop system is practical stable. The design of the observer-based controller is established in the subsection. We implement the control law with estimate states. The observer based controller is given by:

$$u = K\hat{x}, \qquad (31)$$

where $\hat{x}$ is provided by the observer (7).

Remark 6 *The separation principle is defined as follows: if we know that a state feedback control law of the $u(x)$ stabilizes the system then $u(\hat{x})$ also stabilizes. The aim is to check the asymptotic stability in a practical sense the presence of the uncertainty of the system introduced by $f(t,\hat{x}(t))$ in (7) to reach this purpose, we propose the controller (31). This principle of separation is theoretically supported by recent results on cascaded nonlinear systems and standard Lyapunov theory.*

Theorem 3 *Suppose that conditions (3) and (4) and Assumption 1 to Assumption 3 are satisfied. Moreover the conditions (8) and (21) hold. Then the origin of the closed loop system (2)-(31) is globally practically asymptotically stable.*

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

Proof. The closed loop system in the (e, x) coordinates can be represented by:

$$\begin{aligned} \dot{x} &= A_K x + BKe + f(t, x(t)) + B\varepsilon(t), \\ \dot{e} &= A_L e + f(t, \hat{x}) - f(t, x) - B\varepsilon(t). \end{aligned} \tag{32}$$

Let

$$U(e, x) = \theta V(e) + W(x), \tag{33}$$

where V and W are given by (10) and (23) respectively and $\theta > 0$ is to be determined. From (24), we have

$$\dot{W}(x) = -x^T Q_1 x + 2x^T S f(t, x(t)) + 2x^T eSBK + 2x^T SB\varepsilon(t).$$

Using Inequality (18) and (30), we get

$$\begin{aligned} \dot{U}(e, x) \leq\; & -\theta\{\lambda_{min}(Q) + \lambda_{min}(P) - \alpha\lambda_{max}(P) - 1\}\|e\|^2 + \theta\mu^2 \\ & -\{\lambda_{min}(Q_1) + \lambda_{min}(S) - \alpha\lambda_{max}(S) - 1\}||x||^2 + \mu_1^2 \\ & +2||S||||K||||B||||e||||x||. \end{aligned}$$

Since for all $\theta_1 > 0$, we have

$$2||x||||e|| \leq \theta_1||x||^2 + \frac{1}{\theta_1}||e||^2,$$

we deduce that

$$\begin{aligned} \dot{U}(e, x) - \theta\mu^2 - \mu_1^2 \leq\; & -\theta\nu(Q, P)\|e\|^2 - \nu(Q_1, S)||x||^2 \\ & +\theta_1||S||||K||||B||||x||^2 + \frac{1}{\theta_1}||S||||K||||B||||e||^2, \end{aligned}$$

where

$$\begin{cases} \nu(Q, P) = \lambda_{min}(Q) + \lambda_{min}(P) - \alpha\lambda_{max}(P) - 1, \\ \nu(Q_1, S) = \lambda_{min}(Q_1) + \lambda_{min}(S) - \alpha\lambda_{max}(S) - 1. \end{cases}$$

Now, select $\theta_1 = \frac{\nu(Q_1,S)}{2||S||||K||||B||}$; we obtain

$$\begin{aligned} \dot{U}(e, x) \leq\; & -\left\{\theta\nu(Q, P) + \frac{2||S||^2||K||^2||B||^2}{\nu(Q_1, S)}\right\}\|e\|^2 \\ & -\frac{\nu(Q_1, S)}{2}||x||^2 + \theta\mu^2 + \mu_1^2. \end{aligned}$$

Finally we select θ such that

$$\theta\nu(Q, P) + \frac{2||S||^2||K||^2||B||^2}{\nu(Q_1, S)} > 0,$$

to deduct that the origin of the system (32) is globally practically asymptotically stable.

■

Remark 7 *Observer-based output feedback stabilization problem of nonlinear systems by means of one-sided Lipschitz condition, is studied for instance in [17]. By solving linear matrix inequalities, an output feedback controller is envisaged to make the closed loop system globally asymptotically stable. In this paper, we suppose that there exists a linear feedback that asserts global asymptotic stability of the linear part. Then, we establish global asymptotic stability of the nonlinear system under the same controller, for a class of one-sided Lipschitz nonlinear systems with external disturbances.*

Remark 8 *In [14], by constructing the Lyapunov function, some conditions were established in terms of linear matrix inequalities, which can guarantee the global asymptotic stability. Compare with [14], the conditions in Theorem 1 and 3 are expected to be less conservatism to ensure the global asymptotical stability.*

Remark 9 *Asymptotically stabilization of nonlinear uncertain systems is investigated in [7]. Feedback controllers are synthesized under sufficient conditions expressed in terms of linear matrix inequalities. It is obvious that, in this paper, the computational load is reduced compared to the previous study.*

4 Numerical example

The above equation describes the movement of an object in Cartesian coordinates (as seen in [1], [2], and [22]). Moreover, in order to improve the example of [1] and [22], we assume that there is an unknown exogenous perturbation $\varepsilon(t)$.

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2) + \varepsilon(t), \\ y &= x_1. \end{aligned} \tag{34}$$

System (34) can be written as the form of (2) with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In this example, the unknown exogenous disturbance has been chosen equal to $\varepsilon(t) = 3.75 + 0.25 \sin 10t$ for all $t \geq 0$. The equation (34) describes the motion of a moving object, see [1]. Since there exist the disturbances, the schemes in [1] and [2] are not applicable. Additionally, according to [1], the system (34) verifies that the condition (3) and (12) condition are held with the one-sided Lipschitz constant and the quadratically inner-bounded property $\rho = 0$, $\gamma < 0$, and $\beta + \frac{\gamma^2}{4} > 0$ respectively. We can solve Theorem 1 and 2 by choosing the one-sided Lipschitz constant and quadratic inner-boundedness

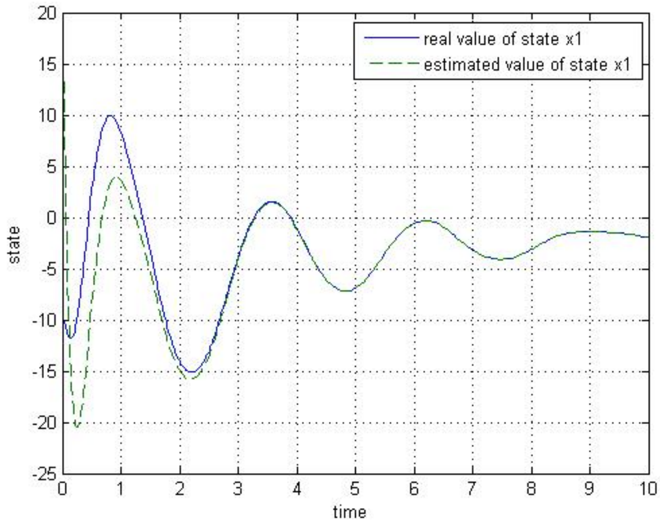

Figure 1: Trajectories of x_1 and its estimate $\hat{x}_1$

constants $\rho = 0$ and $\beta < 1$. Now, select $K = \begin{bmatrix} -12 & 20 \end{bmatrix}$ and $L = \begin{bmatrix} -58 & -16 \end{bmatrix}^T$, *with* A_K and A_L are Hurwitz. We also choose the matrices

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}.$$

The solutions of the Lyapunov equations (6) and (20) are given by

$$P = \begin{bmatrix} 0.0178 & 0.0480 \\ 0.0480 & 3.1836 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.0178 & 0.0480 \\ 0.0480 & 3.1836 \end{bmatrix}.$$

Figs 1 and 2 demonstrate the simulation results of states $x_1(t)$, $x_2(t)$ with disturbances. Consequently, the estimated magnitudes converge practically to the real one. Therefore, the states of this system are well estimated by the design method in this paper.

5 Conclusion

In this article, a separation principle for the perturbed systems is studied. The nonlinearities of this class of systems satisfy the one-sided Lipschitz condition while the uncertain term is bounded. Since the origin is not supposed to be an equilibrium point, A practical observer and state feedback are provided to show that the observer-based controller ensures the globally practically observer-based asymptotic stability of the closed-loop system based on a Lyapunov function. Finally, a simulation study has been undertaken to

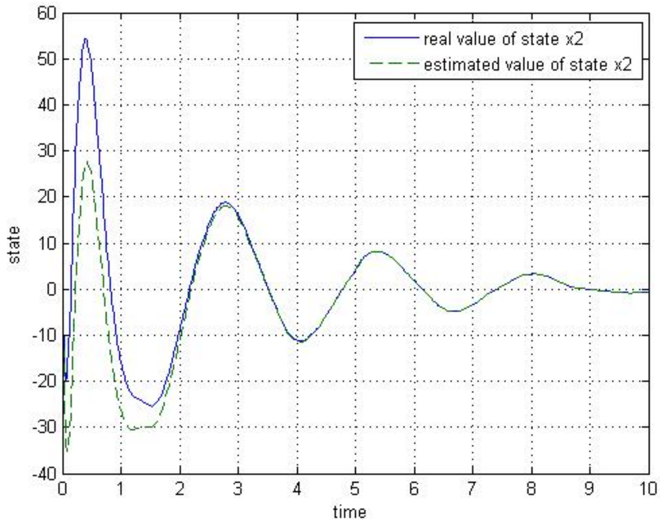

Figure 2: Trajectories of x_2 and its estimate $\hat{x}_2$

illustrate the theory. As a perspective, a new observer design for the same nonlinear systems with external disturbances (2) taking into account the reduction of quasi-one-sided Lipschitz condition given by [10], which is less conservative than the one-sided Lipschitz condition.

References

[1] Abbaszadeh, M., Marquez, H., J., *Nonlinear observer design for onesided Lipschitz systems.* Proc. American Control Conf., Baltimore, USA, 5284-5289 (2010).

[2] Ahmed, S., Rehan M., Hong K., S., *Observer-based robust control of one-sided Lipschitz nonlinear systems.* ISA Transactions., 65,. 230-240 (2016).

[3] Akrouti, I., Echi, N., Global practical exponential stabilization for one-sided lipschitz systems with time delay. Journal of Systems Science and Complexity, 1-17 https://doi.org/10.1007/s11424-022-1061-4 (2022).

[4] Atassi, A. N., Khalil, H. K., *A separation principle for the stabilization of a class of nonlinear systems.* IEEE Transactions on Automatic Control. 44, 1672-1687 (1999).

[5] Atassi, A. N., Khalil, H. K., *Separation results for the stabilization of nonlinear systems using different high-gain observer designs.* Systems and Control Letters. 39, 183-191 (2000).

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

[6] Chen, J., Patton, R.J., Zhang, H.Y., *Design of unknown input observers and robust fault detection filters.* Int. J. Control. 63, 85-105 (1996).

[7] El-Haiek, B., EL AISS, H., Hmamed, A., El-Hajjaji, A., *Robust observer design of one-sided Lipschitz nonlinear systems* Annual American Control Conference, June 27-29 USA, 5250-5255 (2018).

[8] Farza, M., Hernndez-Gonzlez, O., Mnard, T., Targui, B., M'Saad, M., Astorga-Zaragoza, C.-M. *Cascade observer design for a class of uncertain nonlinear systems with delayed outputs.* Automatica 89, 125-134 (2018).

[9] Hairer, E., Norsett, S. P., Wanner, G., *Solving Ordinary Differntial Equations II: Stiff and DAE problems.* Springer-Verlag, (1993).

[10] Huang, J., Yu, L., Shi, M. *Adaptive Observer Design for Quasi-one-sided Lipschitz Nonlinear Systems.*Lecture Notes in Electrical Engineering 459, (2017).

[11] Rajamani, R. *Observers for Lipschitz nonlinear systems.* IEEE Trans Automat Control,43(3) 397-401, (1998).

[12] Tian, J., Ma, S., *Unknown input observer design for one-sided lipschitz nonlinear continuous-time singular markovian jump systems.* World Congress on Intelligent Control and Automation, Guilin, China (2016).

[13] Tian, J., Ma, S., Zhang, C., *Unknown input reduced-order observer design for one-sided lipschitz nonlinear descriptor markovian jump systems.* Asian Journal of Control, (21) 3, 1-13, (2019).

[14] Saad W., Sellami A., and Garcia G., *Robust stabilization of one-sided Lipschitz nonlinear systems via adaptive sliding mode control.* Journal of Vibration and Control, (26) 7-8, 399-412, (2019).

[15] Trangle, C, Farza, M., M'Saad, M., *Observer design for a class of disturbed nonlinear systems with time-varying delayed outputs using mixed time-continuous and sampled measurements.* 107, 231-240 (2019).

[16] Wang, S., Davison, E., Dorato, P., *Observing the states of systems with unmeasurable disturbance.* IEEE Trans. Autom. Control. 20, 716-717 (1975).

[17] Wu R., Zhang W., Song F., Wu Z. and Guo W., *Observer-based stabilization of one-sided Lipschitz systems with application to flexible link manipulator.* Advances in Mechanical Engineering. 7(12) 1-8 (2015).

[18] Zemouche, A., Rajamani, R., Kheloufi, H., Bedouhene, F., *Robust observer-based stabilization of Lipschitz nonlinear uncertain systems via LMIs discussions and new design procedure.* Int. J. Robust Nonlinear Control., 27 (11) 1915-1939 (2016).

18 Nov 2022 12:45:12 PST
220816-Echi Version 2 - Submitted to Rocky Mountain J. Math.

[19] Zhao, Y., Tao, J., N. Z. Shi, N. Z. *A note on observer design for one-sided Lipschitz nonlinear systems.* Syst. Control Lett., (59), No. 1, 66-71 (2010).

[20] Zhang W., Su H., Zhu F., Azar G. *Unknown input observer design for one-sided Lipschitz nonlinear systems.* Nonlinear Dynam., 79 (2) 1469-1479, (2015).

[21] Zhang, W., Liang, Y., Su H.,S., Han Z., Z., *LMI-based Observer Design for One-sided Lipschitz Nonlinear Systems.* Proc. 30th Chinese Control Conference, Yantai, China, 256260 (2011).

[22] Zhang, W., Su H., Liang Y., Han Z. *Nonlinear observer design for one-sided Lipschitz systems: a linear matrix inequality approach'*, IET Control Theory Appl., 6 (9) 1297-1303, (2012).

[23] Zhang, W. Su, H., Wang,H., Han, Z. *Full-order and reduced-order observers for one-sided Lipschitz nonlinear systems using Riccati equations.* Commun. Nonlinear Sci. Numer. Simulat. 17(12), 4968-4977 (2012).

[24] Zhang, W., Su, H., Zhu, F., and Bhattacharyya, S., P., *Improved exponential observer design for one-sided Lipschitz nonlinear systems.* Int. J. Robust Nonlinear Control 26, 3958-3973 (2016).