Strong chromatic index of sparse graphs with maximum degree 4*

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Abstract

A *strong edge-coloring* of a graph G is a proper edge coloring such that every path of length 3 uses three different colors. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the least possible number of colors in a strong edge coloring of G. Choi, Kim, Kostochka and Raspaud (2018) proved that if $\Delta(G) \geq 9$ and maximum average degree is less than $\frac{8}{3}$, then $\chi'_s(G) \leq 3\Delta(G) - 3$; and if $\Delta(G) \geq 7$, maximum average degree is less than 3 and there is no 3-regular subgraphs, then $\chi'_s(G) \leq 3\Delta(G)$. In this paper, we prove that if G is a graph with $\Delta(G) = 4$ and maximum average degree is less than $\frac{8}{3}$(resp.$\frac{14}{5}$), then $\chi'_s(G) \leq 10$(resp.11).

Keywords: Strong edge-coloring, strong chromatic index, maximum average degree, sparse graphs.

AMS classification: 05C15.

1 Introduction

A proper edge coloring is an assignment of colors to the edges such that adjacent edges receive distinct colors. The chromatic index $\chi'(G)$ is the minimum number of colors in a proper edge coloring of G. We denote the minimum and maximum degrees of vertices in G by $\delta(G)$ and $\Delta(G)$ (for short δ and Δ), respectively.

A *strong edge-colouring* (called also distance 2 edge-coloring) of a graph G is a proper edge coloring of G, such that the edges of any path of length 3 use three different colors. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-colored with k colors. Strong edge-coloring was introduced by Fouquet and Jolivet in [7,8]. Strong edge-coloring can be used to model the conflict-free channel assignment in radio networks [16,17].

In 1985, Erdös and Nešetšil gave the following conjecture, which is still open, and provided an example to show that it would be sharp, if true.

Conjecture 1.1 ([6]) *For every graph G,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & if\ \Delta\ is\ even, \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & if\ \Delta\ is\ odd. \end{cases}$$

The conjecture was verified for graphs having $\Delta \leq 3$ [1,13]. When $\Delta > 3$, the only case on which some progress was made is when $\Delta = 4$ and the best upper bound stated is $\chi'_s(G) \leq 21$ [10]. When Δ is sufficiently large, Molloy and Reed in [15] proved that $\chi'_s(G) \leq 1.998\Delta^2$, using probabilistic techniques. This bound is improved to $1.93\Delta^2$ by Bruhn and Joos [3], and very recently, is further improved to $1.835\Delta^2$ by Bonamy, Perrett, and Postle [2].

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The maximum average degree $mad(G)$ of a graph G is the largest average degree of its subgraphs, that is,

$$mad(G) = max\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\}.$$

Hocquard *et al.* [11,12] studied the strong chromatic index of subcubic graphs in terms of maximum average degree and proved that for any graph G with $\Delta = 3$, if $mad(G) < \frac{7}{3}$ (resp. $\frac{5}{2}$, $\frac{8}{3}$, $\frac{20}{7}$), then $\chi'_s(G) \leq 6$ (resp. 7, 8, 9). Lv *et al.* [14] consider graphs with maximum degree 4 and bounded maximum average degree and proved that

Theorem 1.2 *For every graph G with $\Delta = 4$, if $mad(G) < \frac{61}{18}$ (resp. $\frac{7}{2}$, $\frac{18}{5}$, $\frac{15}{4}$, $\frac{51}{13}$), then $\chi'_s(G) \leq 16$ (resp. 17, 18, 19, 20).*

Recently, Choi, Kim, Kostochka and Raspaud [4] obtained the following results.

Theorem 1.3 ([4]) *(1) For every graph G with maximum degree $\Delta \geq 9$ and $mad(G) < \frac{8}{3}$, $\chi'_s(G) \leq 3\Delta - 3$.*

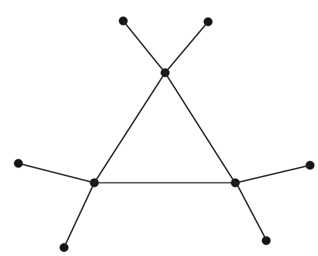
(2) For every graph G with maximum degree $\Delta \geq 7$, $mad(G) \leq 3$ and no 3-regular subgraphs, $\chi'_s(G) \leq 3\Delta$.

Observe that the maximum average degree is more than 3 in Theorem 1.2 and $\Delta \geq 7$ in Theorem 1.3. One naturally find a gap if the maximum average degree decreases to less than 3 in Theorem 1.2 and if Δ decreases to 4 in Theorem 1.3. Motivated by this, we prove the following results in this paper.

Theorem 1.4 *For every graph G with $\Delta = 4$, we have:*
(1) If $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 10$.
(2) If $mad(G) < \frac{14}{5}$, then $\chi'_s(G) \leq 11$.

From Theorem 1.4, one can derive the following result.

Corollary 1.5 *Let G be a planar graph with $\Delta = 4$ and girth g :*
(1) If $g \geq 8$, then $\chi'_s(G) \leq 10$.
(2) If $g \geq 7$, then $\chi'_s(G) \leq 11$.



G

Figure 1: G with $mad(G) = 2$ and $\chi'_s(G) = 9$.

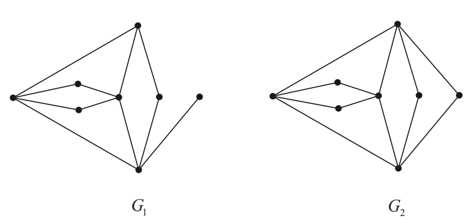


Figure 2: G_1 with $mad(G_1) = \frac{20}{7}$ and $\chi'_s(G_1) = 11$, G_2 with $mad(G_2) = 3$ and $\chi'_s(G_2) = 12$.

It is easy to see that the graph G of Figure 1 is $\chi'_s(G) = 9$ and $mad(G) = 2$, the graph G_1 of Figure 2 is $\chi'_s(G) = 11$ and $mad(G) = \frac{20}{7}$, the graph G_2 of Figure 2 is $\chi'_s(G) = 12$ and $mad(G) = 3$. Therefore, the bounds on the maximum average degree are close to optimal.

We first introduce notations of graphs. Two edges are at distance 1 if they share one of their ends and they are at distance 2 if they are not at distance 1 and there exists an edge adjacent to both of them. Let $d_G(v)$ (or $d(v)$ if it is clear from the context) denote the degree of a vertex v in a graph G. A vertex is a *k-vertex* if it is of degree k. Similarly, a neighbor of a vertex v is a *k-neighbor* of v if it is of degree k. A 3-vertex is a *3_k-vertex* if it is adjacent to exactly k 2-vertices. A 4-vertex is a *4_k-vertex* if it is adjacent to exactly k 2-vertices. We define a *partial coloring* to be a strong edge-coloring except that some edges may be uncolored.

In the proof of the Theorem 1.4, we applied the well-known result of Hall [9] in terms of systems of distinct representatives.

Theorem 1.6 ([9]) *Let $A_1, \ldots, A_n$ be n subsets of a set U. A system of distinct representatives of $\{A_1, \ldots, A_n\}$ exists if and only if for all $k, 1 \le k \le n$ and every choice of subcollection of size k, $\{A_{i_1}, \ldots, A_{i_k}\}$, we have $|A_{i_1} \cup \ldots \cup A_{i_k}| \ge k$.*

2 Proof of Theorem 1.4

Let H be a minimum counterexample to Theorem 1.4 with $|V(H)| + |E(H)|$ minimized. Thus, for some

$$(m, k) \in \{(\frac{8}{3}, 10),\ (\frac{14}{5}, 11)\}$$

we have $mad(H) < m$ and $\chi'_s(H) > k$.

By the minimality of H, $\chi'_s(H - e) \le k$ for each $e \in E(H)$, and we may assume that H is connected.

Let H^* be the graph obtained from H by deleting all vertices of degree 1. Since H^* is the subgraph of H, $mad(H^*) \le mad(H)$. It is sufficient to show that such H^* does not exist. Denote by $N(v)$ and $N_2(uv)$ the neighborhood of the vertex v and the set of edges at distance at most 2 from the edge uv, respectively. Denote by $SC(N_2(uv))$ the set of colors used by edges in $N_2(uv)$. Denote by $L = \{1, 2, \ldots, k\}$ the set of colors and let $L'(e) = L \setminus SC(N_2(e))$. We first establish some properties of H^*.

Lemma 2.1 *If $k \ge 10$, then each of the following holds.*

(1) There is no 1-vertex in H^*.

(2) If $d_{H^*}(v) = 2$, then $d_H(v) = 2$.

(3) If a 3-vertex v is adjacent to two 2-vertices in H^*, then $d_H(v) = d_{H^*}(v) = 3$.

(4) No 3_2-vertex is adjacent to any 3_2-vertex in H^*.

Proof. (1) Suppose that H^* contains a 1-vertex v such that u is its neighbor. Thus, there is at least one 1-vertex v_1 adjacent to v in H. By the minimality of H, $H' = H \setminus \{v_1\}$ has a strong edge coloring with k colors. Observe that $|L'(vv_1)| \ge 4$ since $\Delta = 4$. Thus, we can color vv_1 and obtain the strong edge-coloring of H, a contradiction.

(2) Suppose that $d_H(v) > 2$. Thus, there is at least one 1-vertex v_1 adjacent to v in H. By the minimality of H, $H' = H \setminus \{v_1\}$ has a strong edge coloring c with k colors. Observe that $|L'(vv_1)| \ge 1$. Thus, we can color vv_1, a contradiction.

(3) Suppose that a 3-vertex v is adjacent to two 2-vertices v_1, v_2 in H^* and $d_H(v) > d_{H^*}(v) = 3$. Then v is adjacent to one 1-vertex v' in H. By (2), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(v_2) = d_{H^*}(v_2) = 2$. By the minimality of H, $H' = H \setminus \{v'\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(vv')| \ge 2$. Thus, we can color vv', a contradiction.

(4) Suppose otherwise that a 3_2-vertex v is adjacent to 3_2-vertex u. Let v_1 and v_2 be two 2-neighbors of v, and let u_1 and u_2 be two 2-neighbors of u. By (2) and (3), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(u_2) = d_{H^*}(u_2) = 2$, $d_H(v) = d_{H^*}(v) = 3$, and $d_H(u) = d_{H^*}(u) = 3$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most k colors. Observe that

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$|L'(vv_1)| \geq 3$, $|L'(vv_2)| \geq 3$, and $|L'(vu)| \geq 4$. Thus, we can color vv_1, vv_2, and vu, and obtain a desired strong edge-coloring with k colors, a contradiction. ■

Lemma 2.2 *If $k \geq 10$, then each of the following holds.*

(1) No 2-vertex adjacent to a 2-vertex is adjacent to a 3-vertex in H^*.

(2) No 4-vertex is adjacent to three 2-vertices in H^*, one of which is adjacent to a 2-vertex.

(3) No 3-vertex is adjacent to three 2-vertices in H^*.

Proof. (1) Suppose otherwise that a 2-vertex v is adjacent to a 2-vertex u and a 3-vertex w in H^*. By Lemma 2.1(2), $d_H(v) = d_{H^*}(v) = 2$, and $d_H(u) = d_{H^*}(u) = 2$. If $d_H(w) = d_{H^*}(w) = 3$, then by the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most k colors. It is easy to verify that $|L'(uv)| \geq 4$, $|L'(vw)| \geq 1$. Thus, we can color vw, vu in turn, a contradiction.

If $d_H(w) = 4$, then w is adjacent to one 1-vertex w_1 in H. Let $N(u) = \{u_1, v\}$. By the minimality of H, $H' = H \setminus \{uv\}$ has a strong edge-coloring c with at most k colors. We can switch the colors on vw and ww_1 if necessary such that $c(u_1u) \neq c(vw)$. It is easy to verify that $|L'(uv)| \geq 2$. Thus, we can color uv, a contradiction.

(2) Suppose otherwise that a 4-vertex v is adjacent to three 2-vertices v_1, v_2 and v_3 where v_1 is adjacent to a 2-vertex. Let v_1' be a 2-neighbor of v_1 other than v. By Lemma 2.1(2), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(v_3) = d_{H^*}(v_3) = 2$, and $d_H(v_1') = d_{H^*}(v_1') = 2$. By the minimality of H, $H' = H \setminus \{v_1\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(vv_1)| \geq 1$, $|L'(v_1v_1')| \geq 3$. Thus, we color vv_1, v_1v_1' in turn, a contradiction.

(3) Suppose otherwise that a 3-vertex v is adjacent to three 2-vertices v_1, v_2 and v_3 in H^*. By Lemma 2.1(2)(3), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(v_3) = d_{H^*}(v_3) = 2$, and $d_H(v) = d_{H^*}(v) = 3$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(vv_1)| \geq 4$, $|L'(vv_2)| \geq 4$, and $|L'(vv_3)| \geq 4$. Thus, we can color vv_1, vv_2 and vv_3 in turn, a contradiction. ■

By Lemma 2.2(1) and (2), we classify 2-vertices as follows. A 2-vertex is *very poor* if it is adjacent to a 2-vertex, *poor* if it is adjacent to a 3_2-vertex, and *rich* otherwise.

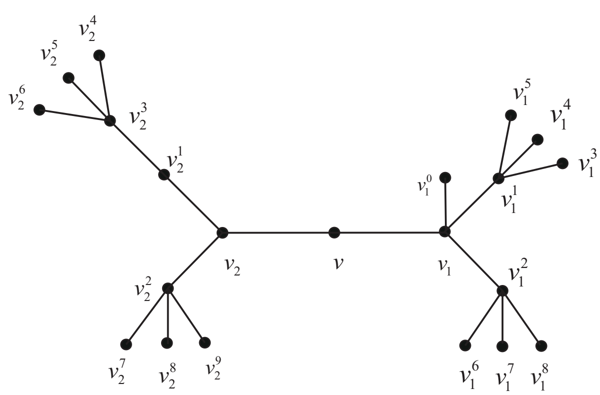

Figure 3: 2-vertex v is adjacent to a 3_1-vertex v_1 and a 3_2-vertex v_2 in H^*, which v_1 is adjacent to one 1-vertex v_1^0 in H.

Lemma 2.3 *If $k \geq 10$, then no 2-vertex is adjacent to a 3_1-vertex and a 3_2-vertex in H^*. Moreover, no 2-vertex is adjacent to two 3_2-vertices in H^*.*

Proof. Suppose otherwise that a 2-vertex v is adjacent to a 3_1-vertex v_1 and a 3_2-vertex v_2 in H^*. Let v_2^1 be a 2-neighbor of v_2 other than v. By Lemma 2.1(2) and (3), $d_H(v) = d_{H^*}(v) = 2$, $d_H(v_2^1) = d_{H^*}(v_2^1) = 2$, and $d_H(v_2) = d_{H^*}(v_2) = 3$.

Assume first that $d_H(v_1) \neq d_{H^*}(v_1)$. The vertex v_1 is adjacent to one 1-vertex v_1^0. We shall use the notations in Figure 3. We claim that v_2^1 is not adjacent to v_1. Suppose otherwise. Then $v_2^3 = v_1$. By the minimality of H, $H' = H \setminus \{v_1^0\}$ has a strong edge-coloring c with at most k colors. Observe that $|L'(v_1v_1^0)| \geq 2$, and we can color $v_1v_1^0$, a contradiction. Similarly, v_2 is not adjacent to v_1.

By the minimality of H, $H' = H \setminus \{v, v_1^0\}$ has a strong edge-coloring with at most k colors. We erase the color of edge $v_2v_2^1$. Observe that $|L'(vv_1)| \geq 1$, $|L'(vv_2)| \geq 3$, $|L'(v_1v_1^0)| \geq 2$, and $|L'(v_2v_2^1)| \geq 2$. We first color edge vv_1. At this time, H has a partial coloring c and uncolored edges are vv_2, $v_1v_1^0$, and $v_2v_2^1$. $|L'(vv_2)| \geq 2$, $|L'(v_1v_1^0)| \geq 1$, and $|L'(v_2v_2^1)| \geq 1$. If $L'(v_1v_1^0) \cap L'(v_2v_2^1) \neq \emptyset$, we color $v_1v_1^0$ and $v_2v_2^1$ with $\alpha \in L'(v_1v_1^0) \cap L'(v_2v_2^1)$, and color vv_2, and obtain a desired strong edge-coloring with k colors, a contradiction. If $L'(v_1v_1^0) \cap L'(v_2v_2^1) = \emptyset$. We claim that $|L'(v_1v_1^0)| = 1$. Suppose otherwise that $|L'(v_1v_1^0)| \geq 2$. We can color $v_2v_2^1$, vv_2 and $v_1v_1^0$ in this order, and obtain a desired strong edge-coloring with k colors, a contradiction. Similarly, $|L'(v_2v_2^1)| = 1$. If we can not assign three distinct colors to three uncolored edges, by Theorem 1.6, $L'(v_2v_2^1) \subseteq L'(vv_2)$, $L'(v_1v_1^0) \subseteq L'(vv_2)$, and $|L'(vv_2)| = 2 (k = 10)$. We assume, without loss of generality, that $L'(v_2v_2^1) = \{1\}$, $L'(v_1v_1^0) = \{2\}$, and $L'(vv_2) = \{1, 2\}$. Since $L'(v_2v_2^1) = \{1\}$ and $L'(vv_2) = \{1, 2\}$, $c(v_2v_2^2)$, $c(v_2^1v_2^3)$, $c(v_2^3v_2^4)$, $c(v_2^3v_2^5)$, $c(v_2^3v_2^6)$, $c(v_2^2v_2^7)$, $c(v_2^2v_2^8)$, $c(v_2^2v_2^9)$ and $c(vv_1)$ are distinct, $2 \notin \{c(v_2^1v_2^3), c(v_2v_2^2), c(v_2^2v_2^7), c(v_2^2v_2^8), c(v_2^2v_2^9), c(vv_1)\}$. Otherwise, $|L'(v_2v_2^1)| \geq 2$, a contradiction. Thus, we may assume, without loss of generality, that $c(v_2^1v_2^3) = 3$, $c(v_2^3v_2^4) = 2$, $c(v_2^3v_2^5) = 4$, $c(v_2^3v_2^6) = 5$, $c(v_2v_2^2) = 6$, $c(v_2^2v_2^7) = 7$, $c(v_2^2v_2^8) = 8$, $c(v_2^2v_2^9) = 9$, and $c(vv_1) = 10$. Since $L'(vv_2) = \{1, 2\}$, $\{c(v_1v_1^1), c(v_1v_1^2)\} = \{4, 5\}$. Since $L'(v_1v_1^0) = \{2\}$, $\{c(v_1^1v_1^3), c(v_1^1v_1^4), c(v_1^1v_1^5), c(v_1^2v_1^6), c(v_1^2v_1^7), c(v_1^2v_1^8)\} = \{1, 3, 6, 7, 8, 9\}$. We recolor vv_1 with 2 and color $v_2v_2^1$ and $v_1v_1^0$ with same color 10, vv_2 with 1. So, we obtain a desired strong edge-coloring with k colors, a contradiction.

Thus, assume that $d_H(v_1) = d_{H^*}(v_1) = 3$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most k colors. We erase the color of edge $v_2v_2^1$. Observe that $|L'(vv_1)| \geq 1$, $|L'(vv_2)| \geq 3$, and $|L'(v_2v_2^1)| \geq 2$. We can color vv_1, $v_2v_2^1$, and vv_2 in turn, a contradiction. ■

Let the initial charge of $x \in V(H^*)$ be $\omega(x) = d(x) - m$. It follows from the hypothesis that $\sum_{x \in V(H^*)} \omega(x) < 0$. Then we define discharging rules to redistribute weights and once the discharging is finished, a new weight function ω^* will be produced. During the discharging process the total sum of weights is kept fixed. Nevertheless, we can show that $\omega^*(x) \geq 0$ for all $x \in V(H^*)$. This leads to the following contradiction:

$$0 \leq \sum_{x \in V(H^*)} \omega^*(x) = \sum_{x \in V(H^*)} \omega(x) < 0.$$

Therefore, such a counterexample cannot exist.

2.1 Case $(\frac{8}{3}, 10)$

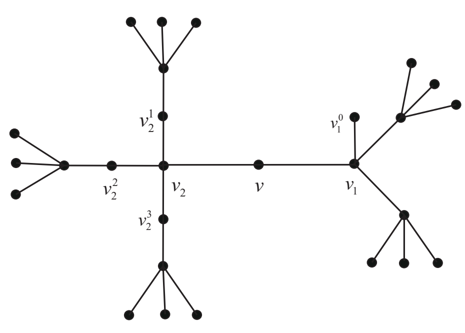

Figure 4: 2-vertex v is adjacent to a 3_1-vertex v_1 and 4_4-vertex v_2 in H^*, which v_1 is adjacent to one 1-vertex v_1^0 in H.

Lemma 2.4 *No 2-vertex is adjacent to a 3_1-vertex and 4_4-vertex in H^*.*

Proof. Suppose otherwise that a 2-vertex v is adjacent to a 3_1-vertex v_1 and a 4_4-vertex v_2 in H^*. Let v_2^1, v_2^2 and v_2^3 be three 2-neighbors of v_2 other than v. By Lemma 2.1(2), $d_H(v) = d_{H^*}(v) = 2$, $d_H(v_2^1) = d_{H^*}(v_2^1) = 2$, $d_H(v_2^2) = d_{H^*}(v_2^2) = 2$, and $d_H(v_2^3) = d_{H^*}(v_2^3) = 2$.

Assume first that $d_H(v_1) \neq d_{H^*}(v_1)$. Then v_1 is adjacent to one 1-vertex v_1^0 in H (see Figure 4). We claim that v_2^1 is not adjacent to v_1. Suppose otherwise. By the minimality of H, $H' = H\backslash\{v_1^0\}$ has a strong edge-coloring c with at most k colors. Observe that $|L'(v_1v_1^0)| \geq 2$, and we can color $v_1v_1^0$, and obtain a desired strong edge-coloring with k colors, a contradiction. By the minimality of H, $H' = H \setminus \{v, v_1^0, v_2\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(vv_1)| \geq 2$, $|L'(vv_2)| \geq 5$, $|L'(v_1v_1^0)| \geq 2$, $|L'(v_2v_2^1)| \geq 4$, $|L'(v_2v_2^2)| \geq 4$, and $|L'(v_2v_2^3)| \geq 4$. Note that v_2 is a 4_4-vertex and v_1 is not a 2-vertex, thus v_2 is not adjacent to v_1. Recall that v_2^1 is not adjacent to v_1. Therefore, $v_1v_1^0$ and $v_2v_2^1$ have distance greater than 2. If $L'(v_1v_1^0) \cap L'(v_2v_2^1) \neq \emptyset$, we color $v_1v_1^0$ and $v_2v_2^1$ with $\alpha \in L'(v_1v_1^0) \cap L'(v_2v_2^1)$, and color vv_1, $v_2v_2^2$, $v_2v_2^3$, and vv_2 in turn, and obtain a desired strong edge-coloring with k colors, a contradiction. If $L'(v_1v_1^0) \cap L'(v_2v_2^1) = \emptyset$, then let $T = \{vv_1, vv_2, v_2v_2^1, v_2v_2^2, v_2v_2^3, v_1v_1^0\}$. For any $S \subseteq T$, we have $|\bigcup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign six distinct colors to six uncolored edges, and we obtain a desired strong edge-coloring with k colors, a contradiction.

Suppose that $d_H(v_1) = d_{H^*}(v_1) = 3$. By the minimality of H, $H' = H \setminus \{v, v_2\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(vv_1)| \geq 2$, $|L'(vv_2)| \geq 5$, $|L'(v_2v_2^1)| \geq 4$, $|L'(v_2v_2^2)| \geq 4$, and $|L'(v_2v_2^3)| \geq 4$. We can color vv_1, $v_2v_2^1$, $v_2v_2^2$, $v_2v_2^3$, and vv_2 in turn, a contradiction. ■

Lemma 2.5 *No 4-vertex is adjacent to three poor 2-vertices in H^*.*

Proof. Suppose otherwise that H^* contain a 4-vertex v adjacent to three poor 2-vertices u, w and t. Let u_0 be 3_2-neighbor of u, let w_0 be 3_2-neighbor of w, and let t_0 be 3_2-neighbor of t. Let u_1 be 2-neighbor of u_0 other than u, let w_1 be 2-neighbor of w_0 other than w, and let t_1 be 2-neighbor of t_0 other than t. By Lemma 2.1(2) and (3), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(t) = d_{H^*}(t) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 2$, $d_H(t_1) = d_{H^*}(t_1) = 2$, $d_H(u_0) = d_{H^*}(u_0) = 3$, $d_H(w_0) = d_{H^*}(w_0) = 3$, and $d_H(t_0) = d_{H^*}(t_0) = 3$. We shall use the notations in Figure 5. We claim that $u_0 \neq t_0$. Suppose otherwise. By the minimality of H, $H' = H \setminus \{u, t\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(uv)| \geq 3$, $|L'(vt)| \geq 3$, $|L'(uu_0)| \geq 4$, and $|L'(tt_0)| \geq 4$. Thus, we can color uv, vt, uu_0 and tt_0 in this order, and obtain a desired strong edge-coloring with k colors, a contradiction. Similarly, $u_0 \neq w_0$, $w_0 \neq t_0$. Lemma 2.1(4), u_0 is not adjacent to t_0, u_0 is not adjacent to w_0, w_0 is not adjacent to t_0.

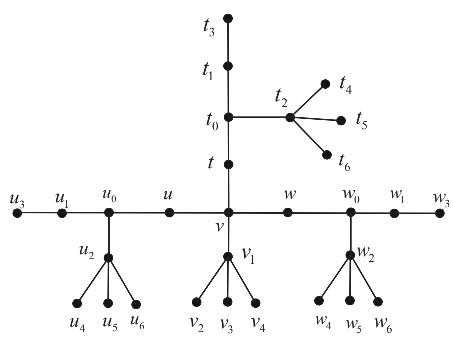

Figure 5: 4-vertex v is adjacent to three poor 2-vertices u, w and t in H^*.

By the minimality of H, $H' = H \setminus \{u, w, t\}$ has a strong edge-coloring with at most k colors. Observe that $|L'(uu_0)| \geq 3$, $|L'(ww_0)| \geq 3$, $|L'(tt_0)| \geq 3$, $|L'(uv)| \geq 4$, $|L'(vw)| \geq 4$, and $|L'(vt)| \geq 4$.
Claim 1. $L'(uu_0) \cap L'(tt_0) = \emptyset$; $L'(uu_0) \cap L'(ww_0) = \emptyset$; $L'(ww_0) \cap L'(tt_0) = \emptyset$.
Proof of Claim 1. Recall that $u_0 \neq t_0$ and u_0 is not adjacent to t_0, thus uu_0 and tt_0 have distance greater than 2. Suppose otherwise that $L'(uu_0) \cap L'(tt_0) \neq \emptyset$. We first color uu_0 and tt_0 with same color, and color ww_0. In this case, H has a partial coloring c and uncolored edges are uv, vw and vt, where $|L'(uv)| \geq 2$, $|L'(vw)| \geq 2$, and $|L'(vt)| \geq 2$. If we can not assign three distinct colors to three uncolored edges, by Theorem 1.6, $L'(uv) = L'(vw) = L'(vt)$ and $|L'(uv)| = 2$. We assume that without loss of generality, that $L'(uv) = L'(vw) = L'(vt) = \{1, 2\}$. Since $L'(uv) = \{1, 2\}$ and $c(uu_0) = c(tt_0)$, $c(uu_0)$, $c(u_0u_1)$, $c(u_0u_2)$, $c(vv_1)$, $c(v_1v_2)$, $c(v_1v_3)$, $c(v_1v_4)$, and $c(ww_0)$ are distinct. Thus, we may assume,

without loss of generality, that $c(uu_0) = c(tt_0) = 3$, $c(u_0u_1) = 4$, $c(u_0u_2) = 5$, $c(vv_1) = 6$, $c(v_1v_2) = 7$, $c(v_1v_3) = 8$, $c(v_1v_4) = 9$, $c(ww_0) = 10$. Since $L'(wv) = \{1, 2\}$, $\{c(w_0w_1), c(w_0w_2)\} = \{4, 5\}$. Since $L'(vt) = \{1, 2\}$, $\{c(t_0t_1), c(t_0t_2)\} = \{4, 5\}$.

We claim that $\{c(w_1w_3), c(w_2w_4), c(w_2w_5), c(w_2w_6)\} = \{3, 7, 8, 9\}$. Suppose otherwise. We assume, without loss of generality, that $3 \notin \{c(w_1w_3), c(w_2w_4), c(w_2w_5), c(w_2w_6)\}$. In this case, we recolor ww_0 with 3 and color vw with 10, uv with 1, vt with 2, and we obtain a desired strong edge-coloring with k colors, a contradiction.

Now, we erase the color of edge uu_0, tt_0. In this case, $|L'(uu_0)| \geq 3$, $|L'(tt_0)| \geq 3$. Recall that $3 \in L'(uu_0) \cap L'(tt_0)$. We claim that $L'(uu_0) \cap L'(tt_0) = \{3\}$. Suppose otherwise that there exist $\alpha \in L'(uu_0) \cap L'(tt_0) \setminus \{3\}$. If $\alpha \notin \{1, 2\}$, we color uu_0 and tt_0 with α, color uv with 3, vt with 1, vw with 2. So we obtain a desired strong edge-coloring with k colors, a contradiction. If $\alpha \in \{1, 2\}$, by symmetry, assume that $\alpha = 1$. In this case, we color uu_0 and tt_0 with 1, recolor ww_0 with 1, color uv with 3, vw with 10, vt with 2. Thus, we obtain a desired strong edge-coloring with k colors, a contradiction.

We claim that $|\{1, 2\} \cap L'(uu_0)| \leq 1$ and $|\{1, 2\} \cap L'(tt_0)| \leq 1$. Suppose otherwise that $\{1, 2\} \subset L'(uu_0)$. Since $L'(uu_0) \cap L'(tt_0) = \{3\}$, $|L'(uu_0)| \geq 3$ and $|L'(tt_0)| \geq 3$, $|L'(tt_0) \setminus L'(uu_0)| \geq 2$. Thus, we can choose $\beta \in L'(tt_0)$ such that $\beta \notin \{1, 2, 3, 10\}$. Thus, we color uu_0 with 1, recolor ww_0 with 1, color tt_0 with β, uv with 3, vt with 2, vw with 10, and so we obtain a desired strong edge-coloring with k colors, a contradiction. The proof is similar for the case that $\{1, 2\} \subset L'(tt_0)$.

Thus, we can get $\gamma_1 \in L'(uu_0)$, $\gamma_2 \in L'(tt_0)$, and $\gamma_1 \notin \{1, 2, 3\}$, $\gamma_2 \notin \{1, 2, 3\}$. We can color uu_0 with γ_1, tt_0 with γ_2, uv with 3, vt with 1, wv with 2, and we obtain a desired strong edge-coloring with k colors, a contradiction.

We can similarly prove that $L'(uu_0) \cap L'(ww_0) = \emptyset$ and $L'(ww_0) \cap L'(tt_0) = \emptyset$. This proves our claim.

Let $T = \{uu_0, ww_0, tt_0, uv, vt, wv\}$. By Claim 1, for any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign six distinct colors to six uncolored edges and we obtain a desired strong edge-coloring with k colors, a contradiction. ■

The discharging rules are defined as follows:

(R1) 4-vertex sends $\frac{2}{3}$ to the adjacent very poor 2-vertex.

(R2) 4-vertex sends $\frac{1}{2}$ to the adjacent poor 2-vertex.

(R3) 4-vertex sends $\frac{1}{3}$ to the adjacent rich 2-vertex.

(R4) 3_1-vertex sends $\frac{1}{3}$ to the adjacent 2-vertex.

(R5) 3_2-vertex sends $\frac{1}{6}$ to the adjacent poor 2-vertex.

Now we consider the new charge $\omega^*(v)$ for each vertex $v \in H^*$.

Let $v \in V(H^*)$ be a k-vertex. By Lemma 2.1, $k \geq 2$.

(1) $k = 2$. If v is a very poor 2-vertex, then v is adjacent to one 4-vertex by Lemma 2.2(1). By (R1), $\omega^*(v) = 2 - \frac{8}{3} + \frac{2}{3} = 0$. If v is a poor 2-vertex, then v is adjacent to one 4-vertex by Lemma 2.3. By (R2) and (R5), $\omega^*(v) = 2 - \frac{8}{3} + \frac{1}{2} + \frac{1}{6} = 0$. If v is a rich 2-vertex, then v is adjacent to two 3_1-vertices or one 3_1-vertex and one 4-vertex, or two 4-vertices. By (R3) and (R4), $\omega^*(v) = 2 - \frac{8}{3} + \frac{1}{3} + \frac{1}{3} = 0$.

(2) $k = 3$. By Lemma 2.2(3), v is adjacent to at most two 2-vertices. If v is not adjacent to 2-vertex, then $\omega^*(v) = 3 - \frac{8}{3} = \frac{1}{3} > 0$. If v is a 3_1-vertex, by (R4), $\omega^*(v) = 3 - \frac{8}{3} - \frac{1}{3} = 0$. If v is a 3_2-vertex , by (R5), $\omega^*(v) = 3 - \frac{8}{3} - 2 \times \frac{1}{6} = 0$.

(3) $k = 4$. If v is a 4_4-vertex, then v is not adjacent to a very poor 2-vertex or a poor 2-vertex by Lemma 2.2 (2) and 2.4. By (R3), $\omega^*(v) = 4 - \frac{8}{3} - 4 \times \frac{1}{3} = 0$. If v is a 4_3-vertex, then v is not adjacent to a very poor 2-vertex by Lemma 2.2(2). By Lemma 2.5 v is not adjacent to three poor 2-vertices. By (R2) and (R3), $\omega^*(v) \geq 4 - \frac{8}{3} - 2 \times \frac{1}{2} - \frac{1}{3} = 0$. If v is a 4_2-vertex, by (R1), (R2) and (R3), $\omega^*(v) \geq 4 - \frac{8}{3} - 2 \times \frac{2}{3} = 0$. If v is a 4_1-vertex, by (R1), (R2) and (R3), $\omega^*(v) \geq 4 - \frac{8}{3} - \frac{2}{3} = \frac{2}{3} > 0$. If v is a 4_0-vertex, $\omega^*(v) \geq 4 - \frac{8}{3} = \frac{4}{3} > 0$.

2.2 Case $(\frac{14}{5}, 11)$

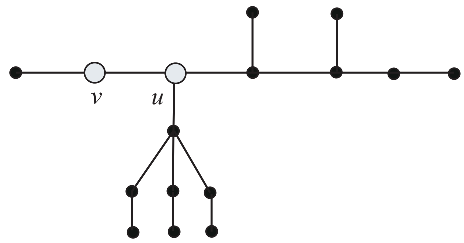

Figure 6: special 3_1-vertex u and semi-rich 2-vertex v

In this section, we give the definition of a special vertex as follows. A 3_1-vertex is a *special* 3_1*-vertex* if it is adjacent to one 4_3-vertex and one 3_0-vertex adjacent to two 3_1-vertices. By Lemma 2.3, no 2-vertex is adjacent to one 3_1-vertex and one 3_2-vertex. A rich 2-vertex is a *semi-rich* 2*-vertex* if it is adjacent to a special 3_1-vertex and a *super-rich* 2*-vertex* otherwise (see Figure 6).

Lemma 2.6 *(1) If a 3-vertex v is adjacent to a 2-vertex in H^*, then $d_H(v) = d_{H^*}(v) = 3$.*

(2) No 3_2-vertex v is adjacent to any 3-vertex in H^*.

(3) No 3_2-vertex v is adjacent to a 4-vertex with at least two 2-neighbors in H^*.

Proof. (1) Suppose otherwise that a 3-vertex v adjacent to a 2-vertex v_1 in H^* and $d_H(v) > d_{H^*}(v) = 3$. Then v is adjacent to one 1-vertex v' in H. By Lemma 2.1(2), $d_H(v_1) = d_{H^*}(v_1) = 2$. By the minimality of H, $H' = H \setminus \{v'\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vv')| \geq 1$. Thus, we can color vv' and obtain a desired strong edge-coloring with eleven colors, a contradiction.

(2) Suppose otherwise that a 3_2-vertex v is adjacent to a 3-vertex v_1 in H^*. Let v_2 and v_3 be two 2-neighbors of v in H^* other than v_1. By Lemma 2.1(2) and (1) of this lemma, $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(v_3) = d_{H^*}(v_3) = 2$, and $d_H(v) = d_{H^*}(v) = 3$. If $d_H(v_1) > d_{H^*}(v_1) = 3$, v_1 is adjacent to one 1-vertex v_1' in H. By the minimality of H, $H' = H \setminus \{v_1', v\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(v_1v_1')| \geq 3$, $|L'(vv_1)| \geq 1$, $|L'(vv_2)| \geq 4$, and $|L'(vv_3)| \geq 4$. Thus, we can color vv_1, v_1v_1', vv_2 and vv_3 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. If $d_H(v_1) = d_{H^*}(v_1) = 3$, by the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vv_1)| \geq 1$, $|L'(vv_2)| \geq 4$, and $|L'(vv_3)| \geq 4$. Thus, we can color vv_1, vv_2, and vv_3 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

(3) Suppose otherwise that a 3_2-vertex v is adjacent to a 4-vertex v_1 with at least two 2-neighbors in H^*. Let v_2, v_3 be two 2-neighbors of v in H^*, let v_1^1, v_1^2 be two 2-neighbors of v_1 in H^*. By Lemma 2.1(2), $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(v_3) = d_{H^*}(v_3) = 2$, $d_H(v_1^1) = d_{H^*}(v_1^1) = 2$, and $d_H(v_1^2) = d_{H^*}(v_1^2) = 2$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vv_1)| \geq 1$, $|L'(vv_2)| \geq 3$, $|L'(vv_3)| \geq 3$. Thus, we can color vv_1, vv_2, and vv_3 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. ■

Lemma 2.7 *(1) No 2-vertex v is adjacent to two 3-vertices u and w in H^* such that one of u and w is adjacent to a 3-vertex.*

(2) No 2-vertex v is adjacent to two 3-vertices u and w in H^* such that one of u and w is adjacent to a 4_3-vertex.

Proof. (1) Suppose otherwise that a 2-vertex v is adjacent to two 3-vertices u and w which is adjacent to a 3-vertex s in H^*. By Lemma 2.1 (2) and 2.6(1), $d_H(v) = d_{H^*}(v) = 2$, $d_H(u) = d_{H^*}(u) = 3$, and $d_H(w) = d_{H^*}(w) = 3$. We claim that $d_H(s) > d_{H^*}(s) = 3$. Suppose otherwise that $d_H(s) = d_{H^*}(s) = 3$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 1$, $|L'(vw)| \geq 2$. Thus, we can color vu and vw in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

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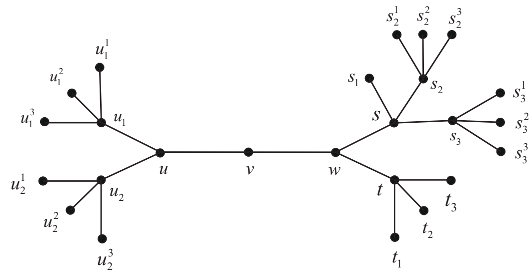

Figure 7: 2-vertex v adjacent to two 3-vertices u and w with w adjacent to a 3-vertex s in H^*.

Therefore, s is adjacent to one 1-vertex s_1 in H. We shall use the notations in Figure 7. Recall that $d_H(u) = d_{H^*}(u) = 3$ and $d_H(s) > d_{H^*}(s) = 3$, then $u \neq s$. We claim that u is not adjacent to s. Suppose otherwise that u is adjacent to s. By the minimality of H, $H' = H \setminus \{s_1\}$ has a strong edge-coloring c with at most eleven colors. Observe that $|L'(ss_1)| \geq 1$. Thus, we can color ss_1, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Therefore, vu and ss_1 have distance greater than 2. By the minimality of H, $H' = H \setminus \{v, s_1\}$ has a strong edge-coloring c with at most eleven colors. Observe that $|L'(vu)| \geq 1$, $|L'(vw)| \geq 2$, and $|L'(ss_1)| \geq 1$. If $L'(vu) \cap L'(ss_1) \neq \emptyset$, we color vu and ss_1 with the same color and then color vw, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

Thus, assume that $L'(vu) \cap L'(ss_1) = \emptyset$. We claim that $|L'(ss_1)| = 1$. Suppose otherwise. We can color uv, vw and ss_1 in this order. Similarly, we can prove that $|L'(vu)| = 1$ and $|L'(vw)| = 2$. We claim that $L'(vu) \cup L'(ss_1) = L'(vw)$. Suppose otherwise. By Theorem 1.6, we can assign three distinct colors to uncolored edge uv, ss_1 and vw. Thus, we assume, without loss of generality, that $L'(vu) = \{1\}$, $L'(ss_1) = \{2\}$, and $L'(vw) = \{1, 2\}$. Since $L'(vu) = \{1\}$, $c(uu_1)$, $c(uu_2)$, $c(u_1u_1^1)$, $c(u_1u_1^2)$, $c(u_1u_1^3)$, $c(u_2u_2^1)$, $c(u_2u_2^2)$, $c(u_2u_2^3)$, $c(ws)$ and $c(wt)$ are distinct. Since $L'(vw) = \{1, 2\}$, $2 \notin \{c(uu_1), c(uu_2), c(ws), c(wt)\}$. We may assume, without loss of generality, that $c(uu_1) = 3$, $c(uu_2) = 4$, $c(u_1u_1^1) = 2$, $c(u_1u_1^2) = 7$, $c(u_1u_1^3) = 8$, $c(u_2u_2^1) = 9$, $c(u_2u_2^2) = 10$, $c(u_2u_2^3) = 11$, $c(ws) = 5$ and $c(wt) = 6$. Since $L'(ss_1) = \{2\}$ and $L'(vw) = \{1, 2\}$, $2 \notin \{c(tt_1), c(tt_2), c(tt_3), c(ss_2), c(ss_3), c(s_2s_2^1), c(s_2s_2^2), c(s_2s_2^3), c(s_3s_3^1), c(s_3s_3^2), c(s_3s_3^3)\}$. Thus, we can recolor ws with 2, color ss_1 with 5, uv with 5, vw with 1, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

(2) Suppose otherwise that a 2-vertex v is adjacent to two 3-vertices u and w such that u is adjacent to a 4_3-vertex s. Let s_1, s_2 and s_3 be three 2-neighbors of s. By Lemma 2.1(2) and 2.6(1), $d_H(v) = d_{H^*}(v) = 2$, $d_H(s_1) = d_{H^*}(s_1) = 2$, $d_H(s_2) = d_{H^*}(s_2) = 2$, $d_H(s_3) = d_{H^*}(s_3) = 2$, $d_H(u) = d_{H^*}(u) = 3$, and $d_H(w) = d_{H^*}(w) = 3$. We claim that s_1 is not adjacent to w. Suppose otherwise. By the minimality of H, $H' = H \setminus \{s\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(us)| \geq 2$, $|L'(ss_1)| \geq 4$, $|L'(ss_2)| \geq 3$, and $|L'(ss_3)| \geq 3$, and color us, ss_2, ss_3, and ss_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

By the minimality of H, $H' = H \setminus \{v, s\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 5$, $|L'(vw)| \geq 2$, $|L'(us)| \geq 4$, $|L'(ss_1)| \geq 4$, $|L'(ss_2)| \geq 4$, and $|L'(ss_3)| \geq 4$. If $L'(vw) \cap L'(ss_1) \neq \emptyset$, we color edges vw and ss_1 with same color, and color ss_2, ss_3, us, and uv in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. If $L'(vw) \cap L'(ss_1) = \emptyset$, let $T = \{uv, vw, us, ss_1, ss_2, ss_3\}$, for any $S \subseteq T$, we have $|\bigcup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign six distinct colors to six uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction. ■

Lemma 2.8 *(1) No 3_1-vertex v is adjacent to one 3_1-vertex u and one 3-vertex w in H^*.*

(2) No 3_1-vertex v is adjacent to one 3_1-vertex u and one 4_3-vertex w in H^*.

(3) No 3_1-vertex v is adjacent to two 4_3-vertices w and t in H^*.

(4) No 3-vertex v is adjacent to three 3_1-vertices u, w and t in H^*.

Proof. (1) Suppose otherwise that a 3_1-vertex v is adjacent to one 3_1-vertex u and one 3-vertex w in H^*. Let v_1 be 2-neighbor of v, u_1 be 2-neighbor of u. By Lemmas 2.1(2) and 2.6(1), $d_H(v_1) = d_{H^*}(v_1) = 2$,

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$d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(v) = d_{H^*}(v) = 3$, and $d_H(u) = d_{H^*}(u) = 3$.

Assume first $d_H(w) = d_{H^*}(w) = 3$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. And we erase the color of edge uu_1. Observe that $|L'(vu)| \geq 3$, $|L'(vw)| \geq 1$, $|L'(vv_1)| \geq 4$, and $|L'(uu_1)| \geq 3$. We can color vw, vu, uu_1, and vv_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

Thus, assume that $d_H(w) > d_{H^*}(w) = 3$. Let w_1 be the 1-neighbor of w. By the minimality of H, $H' = H \setminus \{v, w_1\}$ has a strong edge-coloring with at most eleven colors. We erase the color of edge uu_1. We claim that u_1 is not adjacent to w. Suppose otherwise that u_1 is adjacent to w. In this case, $|L'(vu)| \geq 4$, $|L'(vw)| \geq 4$, $|L'(vv_1)| \geq 4$, $|L'(uu_1)| \geq 5$, and $|L'(ww_1)| \geq 6$. Thus, we can color vu, vv_1, vw, uu_1 and ww_1 in turn and obtain a desired strong edge-coloring with eleven colors, a contradiction. Similarly, we can prove that u is not adjacent to w. We now go back to H. Observe that $|L'(vu)| \geq 3$, $|L'(vw)| \geq 1$, $|L'(vv_1)| \geq 4$, $|L'(uu_1)| \geq 3$, and $|L'(ww_1)| \geq 3$. We now color vw and available colors for vu, vv_1, uu_1, and ww_1 are changed as follows: $|L'(vu)| \geq 2$, $|L'(vv_1)| \geq 3$, $|L'(uu_1)| \geq 2$, and $|L'(ww_1)| \geq 2$. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, we color edges uu_1 and ww_1 with the same color, and color vu and vv_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. If $L'(uu_1) \cap L'(ww_1) = \emptyset$, let $T = \{uu_1, ww_1, vu, vv_1\}$. For any $S \subseteq T$, we have $| \cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign four distinct colors to four uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction.

(2) Suppose otherwise that a 3_1-vertex v is adjacent to one 3_1-vertex u and one 4_3-vertex w. Let v_1 be 2-neighbor of v, u_1 be 2-neighbor of u. Let w_1, w_2, w_3 be three 2-neighbors of w. By Lemmas 2.1(2) and 2.6(1), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 2$, $d_H(w_2) = d_{H^*}(w_2) = 2$, $d_H(w_3) = d_{H^*}(w_3) = 2$, $d_H(v) = d_{H^*}(v) = 3$, and $d_H(u) = d_{H^*}(u) = 3$. We claim that u_1 is not adjacent to w. Suppose otherwise that $u_1 = w_1$ by symmetry. By the minimality of H, $H' = H \setminus \{v, w\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 5$, $|L'(vw)| \geq 6$, $|L'(vv_1)| \geq 5$, $|L'(ww_1)| \geq 7$, $|L'(ww_2)| \geq 5$, and $|L'(ww_3)| \geq 5$, we color vu, ww_2, ww_3, vv_1, vw, and ww_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction.

We now go back to H. By the minimality of H, $H' = H \setminus \{v, w\}$ has a strong edge-coloring with at most eleven colors. We now erase the color of edge uu_1. Observe that $|L'(vu)| \geq 5$, $|L'(vw)| \geq 6$, $|L'(vv_1)| \geq 6$, $|L'(uu_1)| \geq 3$, $|L'(ww_1)| \geq 5$, $|L'(ww_2)| \geq 5$, and $|L'(ww_3)| \geq 5$. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, we color edges uu_1 and ww_1 with same color, and color vu, ww_2, ww_3, vw and vv_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. If $L'(uu_1) \cap L'(ww_1) = \emptyset$, let $T = \{uu_1, vv_1, vu, vw, ww_1, ww_2, ww_3\}$. For any $S \subseteq T$, we have $|\bigcup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign seven distinct colors to seven uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction.

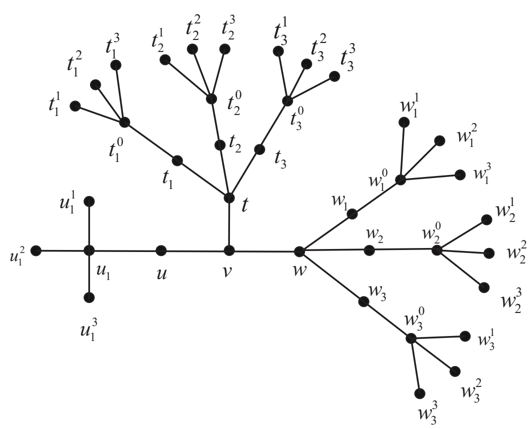

Figure 8: 3_1-vertex v is adjacent to two 4_3-vertices w and t in H^*.

(3) Suppose otherwise that a 3_1-vertex v adjacent to two 4_3-vertices w and t. Let u be 2-neighbor of v, let w_1, w_2, and w_3 be 2-neighbors of w, and let t_1, t_2, and t_3 be 2-neighbors of t. By Lemmas 2.1(2) and

2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 2$, $d_H(w_2) = d_{H^*}(w_2) = 2$, $d_H(w_3) = d_{H^*}(w_3) = 2$, $d_H(t_1) = d_{H^*}(t_1) = 2$, $d_H(t_2) = d_{H^*}(t_2) = 2$, $d_H(t_3) = d_{H^*}(t_3) = 2$, and $d_H(v) = d_{H^*}(v) = 3$. We shall use the notations in Figure 8. By the minimality of H, $H' = H \setminus \{v, w, t\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 7$, $|L'(vw)| \geq 7$, $|L'(vt)| \geq 7$, $|L'(ww_1)| \geq 5$, $|L'(ww_2)| \geq 5$, $|L'(ww_3)| \geq 5$, $|L'(tt_1)| \geq 5$, $|L'(tt_2)| \geq 5$, and $|L'(tt_3)| \geq 5$.
Claim 2. *$L'(ww_i) \cap L'(tt_j) = \emptyset$, for all $i, j \in \{1, 2, 3\}$.*
Proof of Claim 2. We only prove that $L'(ww_1) \cap L'(tt_1) = \emptyset$. The proofs are similar for other cases. Suppose otherwise that $L'(ww_1) \cap L'(tt_1) \neq \emptyset$. We claim that $w_1 \neq t_1$. Suppose otherwise that $w_1 = t_1$. In this case, $|L'(vu)| \geq 7$, $|L'(vw)| \geq 8$, $|L'(vt)| \geq 8$, $|L'(ww_1)| \geq 9$, $|L'(ww_2)| \geq 6$, $|L'(ww_3)| \geq 6$, $|L'(tt_1)| \geq 9$, $|L'(tt_2)| \geq 6$, and $|L'(tt_3)| \geq 6$, we color ww_2, ww_3, tt_2, tt_3, vu, vw, vt, ww_1 and tt_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. We claim that w_1 is not adjacent to t_1. Suppose otherwise that w_1 is adjacent to t_1. In this case, we erase the color of edge w_1t_1. Now, we have $|L'(vu)| \geq 7$, $|L'(vw)| \geq 8$, $|L'(vt)| \geq 8$, $|L'(ww_1)| \geq 9$, $|L'(ww_2)| \geq 6$, $|L'(ww_3)| \geq 6$, $|L'(tt_1)| \geq 9$, $|L'(tt_2)| \geq 6$, $|L'(tt_3)| \geq 6$, and $|L'(w_1t_1)| = 11$, we color ww_2, ww_3, tt_2, tt_3, vu, vw, vt, ww_1, tt_1 and w_1t_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Therefore, ww_1 and tt_1 have distance greater than 2. We first color ww_1 and tt_1 with same color, and color ww_2, ww_3, tt_2, tt_3. Now, we have a partial coloring c and uncolored edges are vu, vw and vt, $|L'(vu)| \geq 2$, $|L'(vw)| \geq 2$, $|L'(vt)| \geq 2$. If we cannot assign three distinct colors to these three uncolored edges. By Theorem 1.6, $L'(vu) = L'(vw) = L'(vt)$ and $|L'(vw)| = 2$. We assume, without loss of generality, that $L'(vu) = L'(vw) = L'(vt) = \{1, 2\}$. Since $L'(vu) = \{1, 2\}$ and $c(ww_1) = c(tt_1)$, $c(uu_1)$, $c(u_1u_1^1)$, $c(u_1u_1^2)$, $c(u_1u_1^3)$, $c(tt_2)$, $c(tt_3)$, $c(ww_2)$, $c(ww_3)$, and $c(ww_1)$ are distinct. Thus, we may assume, without loss of generality, that $c(ww_1) = c(tt_1) = 3$, $c(uu_1) = 4$, $c(u_1u_1^1) = 5$, $c(u_1u_1^2) = 6$, $c(u_1u_1^3) = 7$, $c(tt_2) = 8$, $c(tt_3) = 9$, $c(ww_2) = 10$, and $c(ww_3) = 11$. Since $L'(vw) = L'(vt) = \{1, 2\}$, $\{c(t_1t_1^0), c(t_2t_2^0), c(t_3t_3^0)\} = \{5, 6, 7\}$, $\{c(w_1w_1^0), c(w_2w_2^0), c(w_3w_3^0)\} = \{5, 6, 7\}$. We claim that $\{c(t_2^0t_2^1), c(t_2^0t_2^2), c(t_2^0t_2^3)\} = \{4, 10, 11\}$. Suppose otherwise that $4 \notin \{c(t_2^0t_2^1), c(t_2^0t_2^2), c(t_2^0t_2^3)\}$. We recolor tt_2 with 4 and color vt with 8, vu with 1, vw with 2. So, we obtain a desired strong edge-coloring with eleven colors. This contradiction proves that $4 \in \{c(t_2^0t_2^1), c(t_2^0t_2^2), c(t_2^0t_2^3)\}$. Similarly, we can prove that $10, 11 \in \{c(t_2^0t_2^1), c(t_2^0t_2^2), c(t_2^0t_2^3)\}$. Similarly, $\{c(w_2^0w_2^1), c(w_2^0w_2^2), c(w_2^0w_2^3)\} = \{4, 8, 9\}$. Now, we recolor tt_2 and ww_2 with the same color 1, and color vt with 8, vw with 10, vu with 2, and obtain a desired strong edge-coloring with eleven colors, a contradiction. This proves our claim.

Let $T = \{uv, vt, vw, tt_1, tt_2, tt_3, ww_1, ww_2, ww_3\}$. For any $S \subseteq T$, by Claim 2, $|\cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign nine distinct colors to nine uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction.

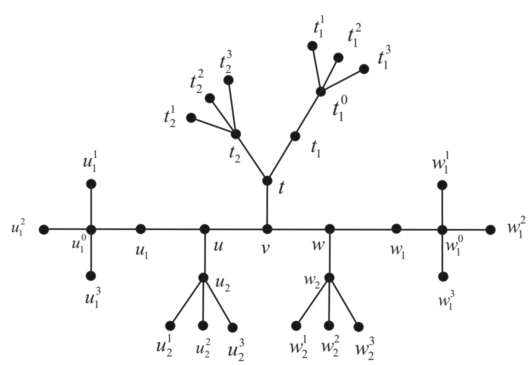

Figure 9: 3-vertex v is adjacent to three 3_1-vertices u, w and t in H^*.

(4) Suppose otherwise that a 3-vertex v is adjacent to three 3_1-vertices u, w and t. Let u_1 be 2-neighbor of u, w_1 be 2-neighbor of w, t_1 be 2-neighbor of t. By Lemmas 2.1(2) and 2.6(1), $d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 2$, $d_H(t_1) = d_{H^*}(t_1) = 2$, $d_H(u) = d_{H^*}(u) = 3$, $d_H(w) = d_{H^*}(w) = 3$, and $d_H(t) = d_{H^*}(t) = 3$. We shall use the notations in Figure 9. We claim that $d_H(v) = d_{H^*}(v) = 3$. Suppose otherwise that v is adjacent to one 1-vertex v_1 in H. By the minimality of H, $H' = H \setminus \{v_1\}$

has a strong edge-coloring with at most eleven colors. Observe that $|L'(vv_1)| \geq 2$. We can color vv_1 and obtain a desired strong edge-coloring with eleven colors, a contradiction.

By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. We now erase the color of edges uu_1, ww_1 and tt_1. Observe that $|L'(vu)| \geq 4$, $|L'(vw)| \geq 4$, $|L'(vt)| \geq 4$, $|L'(uu_1)| \geq 3$, $|L'(ww_1)| \geq 3$, and $|L'(tt_1)| \geq 3$.

Claim 3. $L'(uu_1) \cap L'(tt_1) = \emptyset$, $L'(uu_1) \cap L'(ww_1) = \emptyset$, and $L'(ww_1) \cap L'(tt_1) = \emptyset$.

Proof of Claim 3. We only prove that $L'(uu_1) \cap L'(tt_1) = \emptyset$. The proofs for other cases are similar. Suppose otherwise that $L'(uu_1) \cap L'(tt_1) \neq \emptyset$. We claim that $u_1 \neq t_1$. Suppose otherwise that $u_1 = t_1$. In this case, we have $|L'(vu)| \geq 5$, $|L'(vw)| \geq 4$, $|L'(vt)| \geq 5$, $|L'(uu_1)| \geq 6$, $|L'(ww_1)| \geq 3$, and $|L'(tt_1)| \geq 6$. We can color ww_1, vw, vu, vt, uu_1, and tt_1 in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Recall Lemma 2.2(1), no 2-vertex adjacent to a 2-vertex is adjacent to a 3-vertex in H^*, then u_1 is not adjacent to t_1. We claim that u is not adjacent to t. Suppose otherwise that u is adjacent to t. In this case, we have $|L'(vu)| \geq 8$, $|L'(vw)| \geq 5$, $|L'(vt)| \geq 8$, $|L'(uu_1)| \geq 6$, $|L'(ww_1)| \geq 3$, and $|L'(tt_1)| \geq 6$. We can color ww_1, vw, tt_1, uu_1, vu, and vt in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Recall that u and t are 3_1-vertices, $d_H(u_1) = d_{H^*}(u_1) = 2$, and $d_H(t_1) = d_{H^*}(t_1) = 2$, then $t_1 \neq u_2$ and $t_2 \neq u_1$. Therefore, uu_1 and tt_1 have distance greater than 2. We first color uu_1 and tt_1 with the same color and then color ww_1. We now have a partial coloring c and uncolored edges are vu, vw and vt, where $|L'(vu)| \geq 2$, $|L'(vw)| \geq 2$, and $|L'(vt)| \geq 2$. If we cannot assign three distinct colors to these three uncolored edges, then by Theorem 1.6, $L'(vu) = L'(vw) = L'(vt)$ and $|L'(vw)| = 2$. We assume, without loss of generality, that $L'(vu) = L'(vw) = L'(vt) = \{1, 2\}$. Since $L'(vu) = \{1, 2\}$ and $c(uu_1) = c(tt_1)$, $c(u_1u_1^0)$, $c(uu_2)$, $c(u_2u_2^1)$, $c(u_2u_2^2)$, $c(u_2u_2^3)$, $c(tt_2)$, $c(ww_1)$, and $c(ww_2)$ are distinct. Thus, we may assume, without loss of generality, that $c(uu_1) = c(tt_1) = 3$, $c(uu_2) = 4$, $c(u_1u_1^0) = 5$, $c(u_2u_2^1) = 6$, $c(u_2u_2^2) = 7$, $c(u_2u_2^3) = 8$, $c(tt_2) = 9$, $c(ww_1) = 10$, and $c(ww_2) = 11$. Since $L'(vt) = \{1, 2\}$, $\{c(t_1t_1^0), c(t_2t_2^1), c(t_2t_2^2), c(t_2t_2^3)\} = \{5, 6, 7, 8\}$. Since $L'(vw) = \{1, 2\}$, $\{c(w_1w_1^0), c(w_2w_2^1), c(w_2w_2^2), c(w_2w_2^3)\} = \{5, 6, 7, 8\}$. We claim that $\{c(w_1^0w_1^1), c(w_1^0w_1^2), c(w_1^0w_1^3)\} = \{3, 4, 9\}$. Suppose otherwise. We assume that $3 \notin \{c(w_1^0w_1^1), c(w_1^0w_1^2), c(w_1^0w_1^3)\}$. We recolor ww_1 with 3 and color uv with 1, vt with 2, vw with 10. So we obtain a desired strong edge-coloring with eleven colors, a contradiction. Similarly, we can prove that $4, 9 \in \{c(w_1^0w_1^1), c(w_1^0w_1^2), c(w_1^0w_1^3)\}$. Now we erase the color of edge uu_1, tt_1. In this time, $|L'(uu_1)| \geq 3$, $|L'(tt_1)| \geq 3$. Recall that $3 \in L'(uu_1) \cap L'(tt_1)$. We claim that $L'(uu_1) \cap L'(tt_1) = \{3\}$. Suppose otherwise that there exist $\alpha \in L'(uu_1) \cap L'(tt_1) \setminus \{3\}$. If $\alpha \notin \{1, 2\}$, we color uu_1 and tt_1 with the same color α, color uv with 3, vt with 1, vw with 2, and we obtain a desired strong edge-coloring with eleven colors, a contradiction. If $\alpha \in \{1, 2\}$, we assume, without loss of generality, that $\alpha = 1$. We color both uu_1 and tt_1 with 1, recolor ww_1 with 1, color uv with 3, vw with 10, vt with 2, a contradiction.

We claim that $\{1, 2\} \not\subseteq L'(uu_1)$ and $\{1, 2\} \not\subseteq L'(tt_1)$. Suppose otherwise that $\{1, 2\} \subset L'(uu_1)$. Since $L'(uu_1) \cap L'(tt_1) = \{3\}$ and $|L'(uu_1)| \geq 3$, $|L'(tt_1)| \geq 3$ and $|L'(tt_1) \setminus L'(uu_1)| \geq 2$. We can choose $\beta \in L'(tt_1)$ and $\beta \notin \{1, 2, 3, 10\}$. In this case, we color uu_1 with 1, recolor ww_1 with 1, color tt_1 with β, uv with 3, vt with 2, vw with 10, a contradiction. The proof for the case that $\{1, 2\} \subset L'(tt_1)$ is similar.

Thus, we can get $\gamma_1 \in L'(uu_1)$, $\gamma_2 \in L'(tt_1)$ and $\gamma_1 \notin \{1, 2, 3\}$, $\gamma_2 \notin \{1, 2, 3\}$. We can color uu_1 with γ_1, tt_1 with γ_2, uv with 3, vt with 1, wv with 2, a contradiction. This proves our claim.

Let $T = \{uv, vt, vw, uu_1, ww_1, tt_1\}$. For any $S \subseteq T$, by Claim 3, $|\bigcup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign six distinct colors to six uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction. ■

Lemma 2.9 *(1) No 4-vertex is adjacent to two very poor 2-vertices in H^*.*

(2) No 4-vertex is adjacent to four 2-vertices in H^*.

(3) No 4-vertex is adjacent to two poor 2-vertices in H^*.

(4) No 4-vertex is adjacent to a very poor 2-vertex and a poor 2-vertex in H^*.

(5) No 4-vertex is adjacent to a very poor 2-vertex, one rich 2-vertex and one 3-vertex with at least one 2-neighbor in H^*.

(6) No 4-vertex is adjacent to a very poor 2-vertex, three 3-vertices with at least one 2-neighbor in H^*.

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(7) No 4-vertex is adjacent to a poor 2-vertex and two 2-vertices in H^*.

(8) No 4-vertex is adjacent to a poor 2-vertex, one rich 2-vertex and one 3-vertex with at least one 2-neighbor in H^*.

Proof. (1) Suppose otherwise that H^* contain a 4-vertex v adjacent to two very poor 2-vertices u and w. Let u_1 be the 2-neighbor of u, w_1 be the 2-neighbor of w in H^*. By Lemma 2.1(2), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, and $d_H(w_1) = d_{H^*}(w_1) = 2$. By the minimality of H, $H' = H \setminus \{u, w\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(uv)| \geq 2$, $|L'(vw)| \geq 2$, $|L'(uu_1)| \geq 5$, and $|L'(ww_1)| \geq 5$. Thus, we can color uv, vw, uu_1, and ww_1 in turn, a contradiction.

(2) Suppose otherwise that H^* contain a 4-vertex v adjacent to four 2-vertices v_1, v_2, v_3 and v_4. By Lemma 2.1(2), $d_H(v_1) = d_{H^*}(v_1) = 2$, $d_H(v_2) = d_{H^*}(v_2) = 2$, $d_H(v_3) = d_{H^*}(v_3) = 2$, and $d_H(v_4) = d_{H^*}(v_4) = 2$. By the minimality of H, $H' = H \setminus \{v\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vv_1)| \geq 4$, $|L'(vv_2)| \geq 4$, $|L'(vv_3)| \geq 4$, and $|L'(vv_4)| \geq 4$. Thus, we can color vv_1, vv_2, vv_3, and vv_4 in turn, a contradiction.

(3) Suppose otherwise that H^* contain a 4-vertex v adjacent to two poor 2-vertices u and w. Let u_1 be 3_2-neighbor of u in H^*, w_1 be 3_2-neighbor of w in H^*. Let u_1^1 be 2-neighbor of u_1 other than u, let w_1^1 be 2-neighbor of w_1 other than w. By Lemma 2.1(2) and 2.6(2), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(u_1^1) = d_{H^*}(u_1^1) = 2$, $d_H(w_1^1) = d_{H^*}(w_1^1) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 3$, and $d_H(w_1) = d_{H^*}(w_1) = 3$. We claim that $w_1 \neq u_1$. Suppose otherwise that $w_1 = u_1$. By the minimality of H, $H' = H \setminus \{u, w\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 2$, $|L'(vw)| \geq 2$, $|L'(uu_1)| \geq 5$, and $|L'(ww_1)| \geq 5$. Thus, we can color vu, vw, uu_1, and ww_1, a contradiction. We also claim that u_1 is not adjacent to w_1. Suppose otherwise that u_1 is adjacent to w_1. By the minimality of H, $H' = H \setminus \{u, w\}$ has a strong edge-coloring with at most eleven colors. Now, we erase the color of edge u_1w_1. It is easy to verify that $|L'(vu)| \geq 2$, $|L'(vw)| \geq 2$, $|L'(uu_1)| \geq 6$, $|L'(ww_1)| \geq 6$, and $|L'(u_1w_1)| \geq 7$. Thus, we can color vu, vw, uu_1, ww_1, and u_1w_1 in turn, a contradiction.

By the minimality of H, $H' = H \setminus \{u, w\}$ has a strong edge-coloring with at most eleven colors. We erase the color of edge $u_1u_1^1$. Observe that $|L'(vu)| \geq 2$, $|L'(vw)| \geq 1$, $|L'(uu_1)| \geq 4$, $|L'(ww_1)| \geq 3$, and $|L'(u_1u_1^1)| \geq 3$. Since $u_1u_1^1$ and w_1w are at distance 3 and u_1u and w_1w are at distance 3, we can color vw, vu, ww_1, $u_1u_1^1$, and uu_1 in turn, a contradiction.

(4) Suppose otherwise that H^* contain a 4-vertex v adjacent to one very poor 2-vertex u and one poor 2-vertex w. Let u_1 be 2-neighbors of u in H^*, w_1 be 3_2-neighbors of w in H^*. Let w_1^1 be a 2-neighbor of w_1 other than w. By Lemma 2.1(2) and 2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(w_1^1) = d_{H^*}(w_1^1) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, and $d_H(w_1) = d_{H^*}(w_1) = 3$. By the minimality of H, $H' = H \setminus \{u, w\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 2$, $|L'(uu_1)| \geq 5$, $|L'(vw)| \geq 1$, and $|L'(ww_1)| \geq 3$. Thus, we can color vw, vu, ww_1, and uu_1 in order, a contradiction.

(5) Suppose otherwise that H^* contain a 4-vertex v adjacent to one very poor 2-vertex u, one rich 2-vertex w and one 3-vertex s with at least one 2-neighbor. Let u_1 be 2-neighbors of u in H^*. By Lemma 2.1(2) and 2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, and $d_H(s) = d_{H^*}(s) = 3$. By the minimality of H, $H' = H \setminus \{u\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 1$, $|L'(uu_1)| \geq 4$. Thus, we can color uv and uu_1 in order, a contradiction.

(6) Suppose otherwise that H^* contain a 4-vertex v adjacent to one very poor 2-vertex u and three 3-vertices w, s, t with at least one 2-neighbor. Let u_1 be 2-neighbor of u. By Lemma 2.1(2) and 2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 2$, $d_H(w) = d_{H^*}(w) = 3$, $d_H(s) = d_{H^*}(s) = 3$, and $d_H(t) = d_{H^*}(t) = 3$. By the minimality of H, $H' = H \setminus \{u\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(vu)| \geq 1$, $|L'(uu_1)| \geq 4$. Thus, we can color vu and uu_1 in order, a contradiction.

(7) Suppose otherwise that H^* contain a 4-vertex v adjacent to one poor 2-vertex u and two 2-vertices w and t. Let u_1 be 3_2-neighbor of u, let u_1^1 be 2-neighbor of u other than u in H^*. By Lemma 2.1(2) and 2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(t) = d_{H^*}(t) = 2$, $d_H(u_1^1) = d_{H^*}(u_1^1) = 2$, and $d_H(u_1) = d_{H^*}(u_1) = 3$. By the minimality of H, $H' = H \setminus \{u\}$ has a strong edge-coloring with at

most eleven colors. Observe that $|L'(vu)| \geq 1$, $|L'(uu_1)| \geq 2$. Thus, we can color vu and uu_1 in order, a contradiction.

(8) Suppose otherwise that H^* contain 4-vertex v adjacent to a poor 2-vertex u, one rich 2-vertex w and one 3-vertex s with at least one 2-neighbor. Let u_1 be 3_2-neighbor of u, let u_1^1 be 2-neighbor of u_1 other than u in H^*. By Lemma 2.1(2) and 2.6(1), $d_H(u) = d_{H^*}(u) = 2$, $d_H(w) = d_{H^*}(w) = 2$, $d_H(u_1^1) = d_{H^*}(u_1^1) = 2$, $d_H(u_1) = d_{H^*}(u_1) = 3$, and $d_H(s) = d_{H^*}(s) = 3$. By the minimality of H, $H' = H \setminus \{u\}$ has a strong edge-coloring with at most eleven colors. We now erase the color of edge $u_1u_1^1$. Observe that $|L'(vu)| \geq 1$, $|L'(uu_1)| \geq 3$, and $|L'(u_1u_1^1)| \geq 3$. Thus, we can color vu, uu_1, and $u_1u_1^1$ in order, a contradiction. ■

Lemma 2.10 *No 4-vertex is adjacent to one semi-rich 2-vertex and two 2-vertices in H^*. Moreover, no 4-vertex adjacent to one semi-rich 2-vertex, one 2-vertex and and one 3-vertex with at least one 2-neighbor in H^*.*

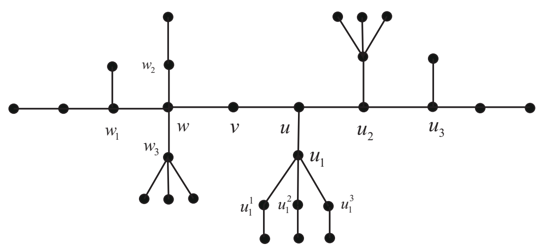

Figure 10: 4-vertex w is adjacent to one semi-rich 2-vertex v, one 2-vertex w_2 and one 3-vertex w_1 with at least one 2-neighbor.

Proof. We only prove the latter case. The proof is similar for the former case. Suppose otherwise that a 4-vertex w is adjacent to a semi-rich 2-vertex v, one 2-vertex w_2 and one 3-vertex w_1 with at least one 2-neighbor (see Figure 10). Let u be special 3_1-neighbor of v. Let u_1 be 4_3-neighbor of u, u_2 be 3-neighbor of u where u_2 is adjacent to other 3_1-vertex u_3. Let u_1^1, u_1^2, u_1^3 be three 2-neighbors of u_1. By Lemma 2.1(2) and 2.6(1), $d_H(v) = d_{H^*}(v) = 2$, $d_H(w_2) = d_{H^*}(w_2) = 2$, $d_H(u_1^1) = d_{H^*}(u_1^1) = 2$, $d_H(u_1^2) = d_{H^*}(u_1^2) = 2$, $d_H(u_1^3) = d_{H^*}(u_1^3) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 3$, and $d_H(u_3) = d_{H^*}(u_3) = 3$.

We claim that $d_H(u_2) = d_{H^*}(u_2) = 3$. Suppose otherwise that u_2 is adjacent to one 1-vertex u_2^1 in H. By the minimality of H, $H' = H \setminus \{u_2^1\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(u_2u_2^1)| \geq 1$. Thus, we can color $u_2u_2^1$, a contradiction.

We claim that u_1^1 is not adjacent to w. Suppose otherwise. Let $u_1^1 = w_2$. By the minimality of H, $H' = H \setminus \{v, u, u_1, u_1^1\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(wv)| \geq 4$, $|L'(uv)| \geq 7$, $|L'(uu_1)| \geq 7$, $|L'(uu_2)| \geq 4$, $|L'(u_1u_1^1)| \geq 7$, $|L'(u_1u_1^2)| \geq 6$, $|L'(u_1u_1^3)| \geq 6$, and $|L'(u_1^1w)| \geq 4$. We claim that w is not adjacent to u_2. Suppose otherwise that w is adjacent to u_2. In this case, we have $|L'(wv)| \geq 6$, $|L'(uv)| \geq 8$, $|L'(uu_1)| \geq 7$, $|L'(uu_2)| \geq 6$, $|L'(u_1u_1^1)| \geq 7$, $|L'(u_1u_1^2)| \geq 6$, $|L'(u_1u_1^3)| \geq 6$, and $|L'(u_1^1w)| \geq 3$. We can color u_1^1w, vw, uu_2, $u_1u_1^2$, $u_1u_1^3$, uu_1, $u_1u_1^1$ and uv in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Therefore, uu_2 and u_1^1w have distance greater than 2. If $L'(uu_2) \cap L'(u_1^1w) \neq \emptyset$, we color edges uu_2 and u_1^1w with same color, and color wv, $u_1u_1^2$, $u_1u_1^3$, $u_1u_1^1$, uu_1, and uv in order, a contradiction. If $L'(uu_2) \cap L'(u_1^1w) = \emptyset$, let $T = \{uu_2, u_1^1w, wv, u_1u_1^2, u_1u_1^3, u_1u_1^1, uu_1, uv\}$. For any $S \subseteq T$, we have $|\bigcup_{e\in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign eight distinct colors to eight uncolored edges and we obtain a desired strong edge-coloring with eleven colors, a contradiction.

By the minimality of H, $H' = H \setminus \{v, u, u_1\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(wv)| \geq 2$, $|L'(uv)| \geq 6$, $|L'(uu_1)| \geq 6$, $|L'(uu_2)| \geq 4$, $|L'(u_1u_1^1)| \geq 5$, $|L'(u_1u_1^2)| \geq 5$, and $|L'(u_1u_1^3)| \geq 5$. If $L'(wv) \cap L'(u_1u_1^1) \neq \emptyset$, we color edges wv and $u_1u_1^1$ with same color, and color uu_2, $u_1u_1^2$, $u_1u_1^3$, uu_1, and uv in order, a contradiction. If $L'(wv) \cap L'(u_1u_1^1) = \emptyset$, let $T = \{uu_2, wv, u_1u_1^2, u_1u_1^3, u_1u_1^1, uu_1, uv\}$. For any $S \subseteq T$, we have $|\bigcup_{e\in S} L'(e)| \geq |S|$. By Theorem 1.6, we can assign seven distinct colors to seven uncolored edges, a contradiction. ■

Lemma 2.11 *No 4-vertex is adjacent to one semi-rich 2-vertex and one very poor 2-vertex in H^*.*

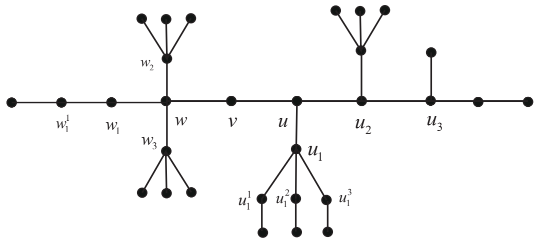

Figure 11: 4-vertex w is adjacent to one semi-rich 2-vertex v and one very poor 2-vertex w_1 in H^*.

Proof. Suppose otherwise that a 4-vertex w is adjacent to a semi-rich 2-vertex v, one very poor 2-vertex w_1(see Figure 11). Let u be special 3_1-neighbor of v. Let w_1^1 be 2-neighbor of w_1. Let u_1 be 4_3-neighbor of u, u_2 be 3-neighbor of u where u_2 is adjacent to other 3_1-vertex u_3. Let u_1^1, u_1^2, u_1^3 be three 2-neighbors of u_1. By Lemma 2.1(2) and 2.6(2), $d_H(v) = d_{H^*}(v) = 2$, $d_H(w_1) = d_{H^*}(w_1) = 2$, $d_H(w_1^1) = d_{H^*}(w_1^1) = 2$, $d_H(u_1^1) = d_{H^*}(u_1^1) = 2$, $d_H(u_1^2) = d_{H^*}(u_1^2) = 2$, $d_H(u_1^3) = d_{H^*}(u_1^3) = 2$, $d_H(u) = d_{H^*}(u) = 3$, and $d_H(u_3) = d_{H^*}(u_3) = 3$.

We claim that $d_H(u_2) = d_{H^*}(u_2) = 3$. Suppose otherwise that u_2 is adjacent to one 1-vertex u_2^1 in H. By the minimality of H, $H' = H \setminus \{u_2^1\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(u_2u_2^1)| \geq 1$. Thus, we can color $u_2u_2^1$, a contradiction.

By the minimality of H, $H' = H \setminus \{v, u, w_1\}$ has a strong edge-coloring with at most eleven colors. Observe that $|L'(w_1w_1^1)| \geq 5$, $|L'(ww_1)| \geq 2$, $|L'(wv)| \geq 3$, $|L'(vu)| \geq 4$, $|L'(uu_1)| \geq 3$, and $|L'(uu_2)| \geq 1$. We claim that $w \neq u_1$. Suppose otherwise that $w = u_1$. In this case, we have $|L'(w_1w_1^1)| \geq 6$, $|L'(ww_1)| \geq 6$, $|L'(wv)| \geq 7$, $|L'(vu)| \geq 8$, $|L'(uu_1)| \geq 5$, and $|L'(uu_2)| \geq 3$. We can color uu_2, $w_1w_1^1$, uu_1, ww_1, vu, and wv in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Recall that u_2 is a 3_0-vertex, then $w \neq u_2$. Therefore, w is not adjacent to u. We claim that w is not adjacent to u_2. Suppose otherwise that w is adjacent to u_2. In this case, we have $|L'(w_1w_1^1)| \geq 5$, $|L'(ww_1)| \geq 4$, $|L'(wv)| \geq 5$, $|L'(vu)| \geq 5$, $|L'(uu_1)| \geq 3$, and $|L'(uu_2)| \geq 3$. Note that $|N_2(w_1w_1^1)| = 8 < 11$. We can color uu_2, uu_1, ww_1, wv, vu, and $w_1w_1^1$ in this order, and obtain a desired strong edge-coloring with eleven colors, a contradiction. Therefore, uu_2 and ww_1 have distance greater than 2. If $L'(uu_2) \cap L'(ww_1) \neq \emptyset$, we color edges uu_2 and ww_1 with the same color, and color uu_1, wv, vu, and $w_1w_1^1$ in order, a contradiction. Thus, $L'(uu_2) \cap L'(ww_1) = \emptyset$. Note that u_1 is a 4_3-vertex, then w is not adjacent to u_1. Recall that w is not adjacent to u. Therefore, uu_1 and ww_1 have distance greater than 2. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, we color edges uu_1 and ww_1 with same color $\alpha \in L'(uu_1) \cap L'(ww_1)$. Obviously, $\alpha \notin L'(uu_2)$. Therefore, we color uu_2, wv, vu, and $w_1w_1^1$ in order, a contradiction. If $L'(uu_1) \cap L'(ww_1) = \emptyset$, let $T = \{ww_1, wv, vu, uu_1, uu_2\}$. For any $S \subseteq T$, $| \cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.6, we can first assign five distinct colors to this five uncolored edges, and last color the edge $w_1w_1^1$ since $|N_2(w_1w_1^1)| = 8 < 11$, a contradiction. ■

The discharging rules are defined as follows:

(R1) Every 4-vertex sends $\frac{4}{5}$ to each very poor 2-vertex.

(R2) Every 4-vertex sends $\frac{3}{5}$ to each poor 2-vertex.

(R3) Every 4-vertex sends $\frac{3}{5}$ to each semi-rich 2-vertex, $\frac{2}{5}$ to each super-rich 2-vertex.

(R4) Every 4-vertex which is not a 4_3-vertex sends $\frac{1}{5}$ to the 3_1-vertex adjacent to a 3_1-vertex or a 4_3-vertex; every 4-vertex which is not a 4_3-vertex sends $\frac{1}{10}$ to the 3_1-vertex not adjacent to a 3_1-vertex nor a 4_3-vertex.

(R5) Every 4-vertex sends $\frac{1}{5}$ to each 3_2-vertex.

(R6) Every 3_0-vertex adjacent to one 3_1-vertex sends $\frac{1}{5}$ to the 3_1-vertex; every 3_0-vertex adjacent to two 3_1-vertices sends $\frac{1}{10}$ to each 3_1-vertex.

(R7) Every special 3_1-vertex $\frac{1}{5}$ to the semi-rich 2-vertex. Every non-special 3_1-vertex sends $\frac{2}{5}$ to the 2-vertex.

(R8) Every 3_2-vertex sends $\frac{1}{5}$ to each 2-vertex.

Now we consider the new charge $\omega^*(v)$ for each vertex $v \in H^*$. Let $v \in V(H^*)$ be a k-vertex. By Lemma 2.1(1), $k \geq 2$.

(1) $k = 2$. If v is a very poor 2-vertex, then v is adjacent to one 4-vertex by Lemma 2.2(1). By (R1), $\omega^*(v) = 2 - \frac{14}{5} + \frac{4}{5} = 0$. If v is a poor 2-vertex, then v is adjacent to one 4-vertex by Lemma 2.3. By (R2) and (R8), $\omega^*(v) = 2 - \frac{14}{5} + \frac{3}{5} + \frac{1}{5} = 0$. Thus, assume that v is a rich 2-vertex. If v is adjacent to two 3-vertices u and w, then u and w are 3_1-vertices by Lemma 2.3. By Lemma 2.7(1), each of u and w is not a special 3_1-vertex. By (R7), $\omega^*(v) = 2 - \frac{14}{5} + 2 \times \frac{2}{5} = 0$.

Let v be adjacent to one 3-vertex u and one 4-vertex w. If v is a semi-rich 2-vertex, then u is a special 3_1-vertex, Thus, $\omega^*(v) = 2 - \frac{14}{5} + \frac{3}{5} + \frac{1}{5} = 0$ by (R3) and (R7). If v is a super-rich 2-vertex, then u is a 3_1-vertex but not special one or a 4-vertex. Thus, $\omega^*(v) = 2 - \frac{14}{5} + 2 \times \frac{2}{5} = 0$ by (R3) and (R7).

If v is adjacent to two 4-vertices u and w, then $\omega^*(v) = 2 - \frac{14}{5} + 2 \times \frac{2}{5} = 0$ by (R3).

(2) $k = 3$. By Lemma 2.2(3), v is adjacent to at most two 2-vertices.

If v is a 3_2-vertex, then v is adjacent to one 4-vertex by Lemma 2.6(2). By (R5) and (R8), $\omega^*(v) = 3 - \frac{14}{5} + \frac{1}{5} - 2 \times \frac{1}{5} = 0$.

Let v be a 3_1-vertex. If v is adjacent to two 3-vertices u and w, then each of u and w is a 3_0-vertex by Lemma 2.8(1). By Lemma 2.8(4), u and w are adjacent to at most two 3_1-vertices. By (R6) and (R7), $\omega^*(v) \geq 3 - \frac{14}{5} + 2 \times \frac{1}{10} - \frac{2}{5} = 0$.

Assume next that v is adjacent to one 3-vertex u and one 4-vertex w. If u is a 3_1-vertex, then w is not a 4_3-vertex by Lemma 2.8(2). By (R4) and (R7), $\omega^*(v) = 3 - \frac{14}{5} + \frac{1}{5} - \frac{2}{5} = 0$. If u is a 3_0-vertex and adjacent to the other 3_1-vertex, and w is a 4_3-vertex, then v is a special 3_1-vertex. By (R7), $\omega^*(v) = 3 - \frac{14}{5} - \frac{1}{5} = 0$. Thus, assume that w is a 4_3-vertex and u is adjacent to only one 3_1-vertex v. By (R6) and (R7), $\omega^*(v) = 3 - \frac{14}{5} + \frac{1}{5} - \frac{2}{5} = 0$; If w is a 4-vertex with at least two 2-neighbors, then by Lemma 2.8(4), u is adjacent to at most two 3_1-vertices. By (R4) and (R6), $\omega^*(v) \geq 3 - \frac{14}{5} + 2 \times \frac{1}{10} - \frac{2}{5} = 0$.

Finally, assume that v is adjacent to two 4-vertices u and w. By Lemma 2.8(3), one of u and w is not 4_3-vertex. By (R4) and (R7), $\omega^*(v) = 3 - \frac{14}{5} + \frac{1}{5} - \frac{2}{5} = 0$.

If v is a 3_0-vertex, then by Lemma 2.8(4), v is adjacent to at most two 3_1-vertex. By (R6), $\omega^*(v) \geq 3 - \frac{14}{5} - \frac{1}{10} \times 2 = 0$.

(3) $k = 4$. By Lemma 2.9(2), v is adjacent to at most three 2-vertices.

Let v be a 4_3-vertex. By Lemmas 2.2(2), 2.9(7) and 2.10, v is not adjacent to a very poor 2-vertex nor a poor 2-vertex nor a semi-rich 2-vertex. By (R4), 4_3-vertex sends nothing to adjacent 3_1-vertex. By Lemma 2.6(3), v is not adjacent to any 3_2-vertex. Thus, $\omega^*(v) = 4 - \frac{14}{5} - 3 \times \frac{2}{5} = 0$ by (R3).

Let v be a 4_2-vertex. Let u and w be two 2-neighbors of v. By Lemma 2.9(1), (3) and (4), one, say w, of u and w is a rich 2-vertex. If u is a very poor 2-vertex, by Lemma 2.11, w is a super-rich 2-vertex. By Lemma 2.9(5), v is not adjacent to a 3-vertex with at least one 2-neighbor. By (R1) and (R3), $\omega^*(v) \geq 4 - \frac{14}{5} - \frac{4}{5} - \frac{2}{5} = 0$. If u is a poor 2-vertex, by Lemma 2.9(8), v is not adjacent to a 3-vertex with at least one 2-neighbor. By (R2) and (R3), $\omega^*(v) \geq 4 - \frac{14}{5} - 2 \times \frac{3}{5} = 0$. Thus, assume that u is a rich 2-vertex. If one of u and w is a semi-rich 2-vertex, by Lemma 2.10, v is not adjacent to a 3-vertex with at least one 2-neighbor. By (R3), $\omega^*(v) \geq 4 - \frac{14}{5} - 2 \times \frac{3}{5} = 0$. Thus, assume that both u and w are super-rich 2-vertices. By (R3), (R4) and (R5), $\omega^*(v) \geq 4 - \frac{14}{5} - 2 \times \frac{2}{5} - 2 \times \frac{1}{5} = 0$.

Let v be a 4_1-vertex and u be a 2-neighbor of v. If u is a very poor 2-vertex, then v is not adjacent to three 3-vertices with at least one 2-neighbor by Lemma 2.9(6). By (R1), (R4) and (R5), $\omega^*(v) \geq 4 - \frac{14}{5} - \frac{4}{5} - 2 \times \frac{1}{5} = 0$. If u is not a very poor 2-vertex, then $\omega^*(v) \geq 4 - \frac{14}{5} - \frac{3}{5} - 3 \times \frac{1}{5} = 0$ by (R2), (R3), (R4) and (R5).

Let v be a 4_0-vertex. By (R4) and (R5), $\omega^*(v) \geq 4 - \frac{14}{5} - 4 \times \frac{1}{5} = \frac{2}{5} > 0$.

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