# THE SECOND MINIMAL EXCLUDANT AND MEX SEQUENCES

PRABH SIMRAT KAUR, MEENAKSHI RANA, AND PRAMOD EYYUNNI

Abstract. The minimal excludant of an integer partition, first studied prominently by Andrews and Newman from a combinatorial viewpoint, is the smallest positive integer missing from a partition. Several generalizations of this concept are being explored by mathematicians nowadays. We analogously consider the *second* minimal excludant of a partition and analyze its relationship with the minimal excludant. This leads us to the notion of a *mex sequence* and we derive two neat identities involving the number of partitions whose mex sequence has length at least $r$.

## 1. Introduction

The notion of the *minimal excludant* of a set $S$ of positive integers, namely, the smallest positive integer missing from that set, was introduced by Fraenkel and Peled [11]. They denoted this number by "mex($S$)". In the recent past, Andrews and Newman [2] examined this idea in the context of integer partitions. They defined [2, Equation (1.1)] the minimal excludant of an integer partition to be the least positive integer missing from the partition, and denoted it by $\text{mex}(\pi)$ for a partition $\pi$. For a positive integer $n$, define

$$\sigma\text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi),$$

where $\mathcal{P}(n)$ represents the collection of integer partitions of $n$. By deriving the generating function of $\sigma\text{mex}(n)$, Andrews and Newman [2, Theorem 1.1] proved the following intriguing identity:

$$\sigma\text{mex}(n) = \mathcal{D}_2(n), \tag{1.1}$$

where $\mathcal{D}_2(n)$ is the number of distinct parts partitions of $n$ into two colors. This was earlier also proved by Grabner and Knopfmacher [13], where they undertook an analytic study of the minimal excludant under a different name, viz., the least gap in a partition. A bijective proof of (1.1) was attained by Ballantine and Merca [4]. Andrews and Newman [2], also defined another function:

$$a(n) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \text{mex}(\pi)\ \text{odd}}} 1,$$

2020 *Mathematics Subject Classification.* Primary 11P81, 11P84; Secondary 05A17.

*Keywords and phrases.* Minimal excludant, second minimal excludant, mex sequences, partition identities.



19 Apr 2023 11:57:58 PDT
220812-Eyyunni Version 3 - Submitted to Rocky Mountain J. Math.

i.e., $a(n)$ enumerates the number of partitions of $n$ with an odd minimal excludant. They observed that $\sigma\mathrm{mex}(n)$ and $a(n)$ have the same parity and proved that $a(n)$ is "usually" even by showing that it is odd exactly when $n$ is of the form $j(3j \pm 1)$.

Analogues of the minimal excludant have begun to appear in the literature. For example, Andrews and Newman [2] also studied $\mathrm{moex}(\pi)$, the smallest odd integer missing from a partition. In a subsequent paper [3], they introduced the function $p_{A,a}(n)$, which counts the number of partitions $\pi$ of $n$ satisfying $\mathrm{mex}_{A,a}(\pi) \equiv a \pmod{2A}$, where $\mathrm{mex}_{A,a}(\pi)$ denotes the smallest positive integer congruent to $a$ modulo $A$ that is not a part of $\pi$. Note that $p_{1,1}(n)$ simply equals $a(n)$ and $\mathrm{mex}_{2,1}(\pi)$ is nothing but $\mathrm{moex}(\pi)$. Two of their significant results are partition identities relating instances of $p_{A,a}(n)$ to the partition statistics rank and crank. We have that $p_{1,1}(n)$ equals the number of partitions of $n$ with non-negative crank, and $p_{3,3}(n)$ equals the number of partitions of $n$ with rank $\geq -1$. Sellers and da Silva [14] gave complete parity characterizations of $p_{1,1}(n)$ and $p_{3,3}(n)$, along with some congruences modulo 2 for other partition functions of this type.

Chern [10] studied the complementary problem for the largest integer less than the largest part that goes missing in a partition, calling it the maximal excludant (denoted by $\mathrm{maex}(\pi)$ for a partition $\pi$) and investigated the corresponding function $\sigma\mathrm{maex}(n) := \sum_{\pi \in \mathcal{P}(n)} \mathrm{maex}(\pi)$. In particular, he derived the generating function of $\sigma\mathrm{maex}(n)$ and showed the asymptotic relation $\sigma\mathrm{maex}(n) \sim \sigma L(n)$, as $n \to \infty$, where $\sigma L(n)$ stands for the sum of largest parts in all partitions of $n$. In [8] the first and the third authors, along with Bhoria and Maji, looked at restricted versions of $\sigma\mathrm{mex}$ type functions. Instead of taking the sum over the set of all partitions of $n$, they only considered those in $\mathcal{D}(n)$, the set of distinct parts partitions of $n$. More precisely, they defined $\sigma_d\mathrm{mex}(n) := \sum_{\pi \in \mathcal{D}(n)} \mathrm{mex}(\pi)$ and allied functions. They showed that this function is related to Ramanujan's function $\sigma(q) = \displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}$ via its generating function:

$$\sum_{n=0}^{\infty} \sigma_d\mathrm{mex}(n) q^n = (-q;q)_\infty \sigma(q).$$

Here, and in the sequel, the '$q$'- products are as defined below. For complex numbers $a$ and $q$, we have

$$(a;q)_n := \prod_{j=0}^{n-1}(1 - aq^j), \quad \text{for } n \geq 1, \quad \text{and} \quad (a;q)_0 := 1,$$
$$(a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j), \quad \text{for } |q| < 1.$$

A natural continuation to the study of minimal excludants is the second minimal excludant which we define as follows:

**Definition 1** (Second minimal excludant). *The second smallest integer missing from an integer partition $\pi$ is known as the second minimal excludant, denoted by* $\mathrm{mex}_2(\pi)$.

19 Apr 2023 11:57:58 PDT
220812-Eyyunni Version 3 - Submitted to Rocky Mountain J. Math.

For instance, $\mathrm{mex}_2(1+1)=3,\ \mathrm{mex}_2(9+7+7+5+5+5+3+3+2)=4$. We analogously define the $\sigma_2\mathrm{mex}(n)$ function as

$$\sigma_2\mathrm{mex}(n) := \sum_{\pi\in\mathcal{P}(n)} \mathrm{mex}_2(\pi). \tag{1.2}$$

We derive the generating function for $\sigma_2\mathrm{mex}(n)$ and study partitions with a fixed difference between the minimal excludant and the second minimal excludant. For this, we define $\Delta_t(n)$, the number of partitions $\pi$ of $n$ with $\mathrm{mex}_2(\pi)-\mathrm{mex}(\pi)=t$. We derive its generating function and as special cases, obtain interesting identities connecting $\Delta_t(n)$ to $\sigma\mathrm{mex}(n)$ and certain restricted partition functions.

Starting with the observation that $\Delta_1(n)$ enumerates the partitions of $n$ in which the minimal excludant and the second minimal excludant are consecutive integers, we are naturally led to examine the longest sequence of missing integers in a partition, starting from the minimal excludant.

**Definition 2** (Mex sequence)**.** *The mex sequence of a partition is the longest sequence of consecutive missing integers in the partition, starting from its minimal excludant.*

For example, the mex sequences of some partitions of 6 are tabulated below:

| Partition | Minimal excludant | Mex sequence | Length of mex sequence |
|---|---|---|---|
| $6$ | 1 | $(1,2,3,4,5)$ | 5 |
| $5+1$ | 2 | $(2,3,4)$ | 3 |
| $4+2$ | 1 | $(1)$ | 1 |
| $4+1+1$ | 2 | $(2,3)$ | 2 |
| $3+3$ | 1 | $(1,2)$ | 2 |
| $3+1+1+1$ | 2 | $(2)$ | 1 |
| $2+2+2$ | 1 | $(1)$ | 1 |

A remark is in place here. The mex sequence of a partition can be *infinitely long*. For instance, the partitions of 6 not alluded to in the table above, namely, $3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$ all have such mex sequences.

**Definition 3** (The function $p_r^{\mathrm{mex}}(n)$)**.** *The function $p_r^{\mathrm{mex}}(n)$ enumerates the number of partitions of $n$ whose corresponding mex sequences have length at least $r$.*

The generating function for $p_r^{\mathrm{mex}}(n)$ and its consequences form an important component of the present work. Interestingly, depending on the parity of $r$, this gives us two elegant partition identities for $p_r^{\mathrm{mex}}(n)$. We now state our main results in the next section.

## 2. Main Results

Let $\sigma_2\mathrm{mex}(n)$ be the function defined in (1.2). For instance, $\sigma_2\mathrm{mex}(4)=2+4+3+4+3=16$, the listed summands being the respective second minimal excludants of the partitions $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$, of the integer 4. We then have the following result:

19 Apr 2023 11:57:58 PDT
220812-Eyyunni Version 3 - Submitted to Rocky Mountain J. Math.

**Theorem 2.1.** *The generating function of $\sigma_2\mathrm{mex}(n)$ is given by*

$$\sum_{n=0}^{\infty}\sigma_2\mathrm{mex}(n)q^n = \frac{1}{(q;q)_\infty}\left\{\frac{1}{(1-q)} - \sum_{s=0}^{\infty}(s-1)q^{\binom{s+1}{2}}\right\}. \tag{2.1}$$

As outlined in Section 1, for a positive integer $t$ we define:

$$\Delta_t(n) = \text{number of partitions } \pi \text{ of } n \text{ satisfying } \mathrm{mex}_2(\pi) - \mathrm{mex}(\pi) = t. \tag{2.2}$$

**Example 1.** *See that $\Delta_1(5) = 5$, as the relevant partitions of $5$ are $5,\ 4+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$, whereas $\Delta_2(5) = 1$, by taking into consideration the partition $3+1+1$.*

Let $\psi(q)$ be one of Ramanujan's theta functions defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}. \tag{2.3}$$

The generating function for $\Delta_t(n)$ has a nice representation in terms of a "tail" of $\psi(q)$.

**Theorem 2.2.** *Let $t$ be a positive integer. Then the following is the generating function for $\Delta_t(n)$:*

$$\sum_{n=0}^{\infty}\Delta_t(n)q^n = \frac{q^{t-1}}{(q^2;q)_\infty}\sum_{r=0}^{\infty} q^{\binom{r+t}{2}}. \tag{2.4}$$

**Corollary 2.3.** *For $t = 1, 2$, (2.4) gives us the partition identities listed below.*

$$\Delta_1(n) = \sigma\mathrm{mex}(n) - \sigma\mathrm{mex}(n-1), \tag{2.5}$$

$$\Delta_2(n) = \sigma\mathrm{mex}(n-1) - \sigma\mathrm{mex}(n-2) - p(n|\textit{exactly one } 1). \tag{2.6}$$

Here, $p(n|\text{condition})$ means the number of partitions of $n$ satisfying the condition appearing after the $|$ symbol. For instance, $p(n|\textit{exactly one } r)$ counts the number of partitions of $n$ in which the integer $r$ appears exactly once. We discuss the main results pertaining to mex sequences in the upcoming subsection.

2.1. **Mex sequences.** We let $\mathcal{M}_r(n)$ denote the set of partitions of $n$ whose mex sequence has length *at least* $r$. We also put

$$|\mathcal{M}_r(n)| := p_r^{\mathrm{mex}}(n),$$

agreeing with the definition of $p_r^{\mathrm{mex}}(n)$ as stated in Definition 3.

Firstly, observe that $p_1^{\mathrm{mex}}(n)$ is simply the partition function $p(n)$, since the mex sequence of every partition has length at least 1. From the definition of $\Delta_1(n)$ in (2.2), one can also deduce that $p_2^{\mathrm{mex}}(n) = \Delta_1(n)$. In fact, this observation was the motivation for defining the mex sequence in Section 1. (see the paragraph just before Definition 2)

**Example 2.** *Consider the set* $\mathcal{M}_3(6)$*, which consists of the partitions* $6,\ 5+1,\ 3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$*, and so* $p_3^{\mathrm{mex}}(6)=6$*. Similarly, the partitions belonging to the set* $\mathcal{M}_4(6)$ *are* $6,\ 3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$*, which gives us* $p_4^{\mathrm{mex}}(6)=5$*. Note that* $\mathcal{M}_4(6)\subset\mathcal{M}_3(6)$ *and hence,* $p_4^{\mathrm{mex}}(6)\le p_3^{\mathrm{mex}}(6)$*.*

In fact, from the definition of the sets $\mathcal{M}_r(n)$, it follows that $\mathcal{M}_{r+1}(n)\subset\mathcal{M}_r(n)$, and consequently

$$p_{r+1}^{\mathrm{mex}}(n)\le p_r^{\mathrm{mex}}(n)\ \forall\ r\ge 1. \tag{2.7}$$

At the other end of the spectrum, it is interesting to see what happens for "large" values of $r$. The next result speaks to this:

**Proposition 2.4.** *Let* $r,n$ *be positive integers and suppose* $q(n)$ *denotes the number of distinct parts partitions of* $n$*. Then*

$$p_r^{\mathrm{mex}}(n)=q(n)\quad\Longleftrightarrow\quad r\ge n.$$

We now turn to the generating function for $p_r^{\mathrm{mex}}(n)$, and see that it has a succinct representation in terms of $q$-products.

**Theorem 2.5.** *If* $r$ *is a positive integer, then the generating function of* $p_r^{\mathrm{mex}}(n)$ *is given by*

$$\sum_{n=0}^{\infty}p_r^{\mathrm{mex}}(n)q^n=\frac{1}{(q;q^2)_\infty(q^{r+1};q^2)_\infty}. \tag{2.8}$$

Indeed, with $r=1$ this gives the generating function for $p_1^{\mathrm{mex}}(n)$ to be $1/(q;q)_\infty$, the generating function for $p(n)$, as it should be. Also, as $r\to\infty$ the right hand side of (2.8) tends to $1/(q;q^2)_\infty=(-q;q)_\infty$, the generating function of distinct parts partitions, which is in agreement with Proposition 2.4. Interpreting Theorem 2.5 combinatorially leads to two identities, based on the parity of $r$.

**Corollary 2.6.** *Define* $p_e^{>r}(n)$ *to be the number of partitions of* $n$ *in which no even integer less than* $r$ *is allowed to be a part and* $p_{o,2}^{>r}(n)$ *is the number of partitions of* $n$ *into odd parts where parts greater than* $r$ *come in two colors. Then,*

- *For odd integers* $r$*, we have* $p_r^{\mathrm{mex}}(n)=p_e^{>r}(n)$ *and*
- *For even integers* $r$*, we have* $p_r^{\mathrm{mex}}(n)=p_{o,2}^{>r}(n)$*.*

**Example 3.** *We have* $p_2^{\mathrm{mex}}(5)=5$*, because the relevant partitions are* $5,\ 4+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$*. The partitions of* $5$ *into odd parts where parts greater than* $2$ *come in two colors, say* $r$ *and* $b$*, are* $5_r,\ 5_b,\ 3_r+1+1,\ 3_b+1+1,\ 1+1+1+1+1$*, also five in number. Next,* $5,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$ *are the four partitions of* $5$ *with mex sequence of length at least* $3$*. We also see that there are four partitions of* $5$ *with no '2's, namely,* $5,\ 4+1,\ 3+1+1,\ 1+1+1+1+1$*.*

**Remark 1.** *Observe that we can also calculate the number of partitions of* $n$ *whose mex sequence has length* $r$*. Such partitions are precisely the members of the set* $\mathcal{M}_r(n) \setminus \mathcal{M}_{r+1}(n)$ *and thus total* $p_r^{\mathrm{mex}}(n) - p_{r+1}^{\mathrm{mex}}(n)$ *in number. From the example above, we see that there is* $p_2^{\mathrm{mex}}(5) - p_3^{\mathrm{mex}}(5) = 1$ *partition of* $5$ *with mex sequence of length two, namely,* $4+1$*.*

Finally, as a consequence of Theorem 2.5 for $r = 2$, we obtain a $q$-product representation for $\psi(q)$, which is usually derived using Jacobi's triple product identity [7, Equation (1.3.14), p. 11].

**Corollary 2.7.** *Let* $\psi(q)$ *be the theta function of Ramanujan defined in* (2.3)*. Then,*

$$\psi(q) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

We now collect some auxiliary results in the upcoming section.

## 3. Preliminaries

We shall require the following result which is due to Andrews and Newman [2, Theorem 1.1].

**Theorem 3.1.** *We have the ensuing identity for the generating function of* $\sigma\mathrm{mex}(n)$*:*

$$\sum_{n=0}^{\infty} \sigma\mathrm{mex}(n) q^n = \frac{\psi(q)}{(q; q)_\infty}, \tag{3.1}$$

*where* $\psi(q)$ *is as defined in* (2.3)*.*

The famous $q$-binomial theorem is given by [1, Equation (2.2.1), p. 17]

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}. \quad (|z| < 1) \tag{3.2}$$

A version of Heine's transformation for $q$-hypergeometric series is given below [12, Equation (III.1), p. 359]:

$${}_2\phi_1 \left[ \begin{matrix} a, & b \\ & c \end{matrix} ; q, z \right] = \frac{(b; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} \frac{c}{b}, & z \\ & az \end{matrix} ; q, b \right], \tag{3.3}$$

where the $q$-hypergeometric series ${}_{r+1}\phi_r$ is defined to be

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_{r+1}; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_r; q)_n} z^n. \tag{3.4}$$

## 4. Proofs of the main results

*Proof of Theorem* 2.1. If $A_{r,s}(n)$ denotes the number of partitions of $n$ with minimal excludant $r$ and second minimal excludant $s$, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} A_{r,s}(n)q^n &= \frac{q^1}{1-q^1}\cdots\frac{q^{r-1}}{1-q^{r-1}}\cdot\frac{q^{r+1}}{1-q^{r+1}}\cdots\frac{q^{s-1}}{1-q^{s-1}}\cdot\frac{1}{1-q^{s+1}}\cdot\frac{1}{1-q^{s+2}}\cdots\text{to }\infty\\ &= \frac{q^{\binom{s}{2}-r}(1-q^r)(1-q^s)}{(q;q)_\infty}. \end{aligned} \tag{4.1}$$

We introduce two parameters $z$ and $w$ and let the exponents of $z$ and $w$ keep track of the minimal excludant and the second minimal excludant of a partition respectively. Note that the second minimal excludant $s$ of a partition is *at least two* and the minimal excludant then ranges between 1 and $s-1$. We thus have the three parameter generating function for the number of partitons with a specified minimal excludant and second minimal excludant as follows:

$$\sum_{n=0}^{\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} A_{r,s}(n)z^r w^s q^n = \sum_{s=2}^{\infty}\sum_{r=1}^{s-1} z^r w^s \sum_{n=0}^{\infty} A_{r,s}(n)q^n \tag{4.2}$$

$$= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} z^r w^s q^{\binom{s}{2}-r}(1-q^r)(1-q^s), \tag{4.3}$$

by (4.1).

If we put $z=1$ in the left side of (4.2), we get

$$\sum_{n=0}^{\infty}\sum_{s=2}^{\infty}\left(\sum_{r=1}^{s-1} A_{r,s}(n)\right) w^s q^n = \sum_{n=0}^{\infty}\sum_{s=2}^{\infty} p^{\mathrm{mex}_2}(s,n) w^s q^n, \tag{4.4}$$

where $p^{\mathrm{mex}_2}(s,n)$ is the number of partitions of $n$ with second minimal excludant $s$. This is because in $\sum_{r=1}^{s-1} A_{r,s}(n)$, we are summing over all possible values of $r$ for a given $s$. Now, put

19 Apr 2023 11:57:58 PDT
220812-Eyyunni Version 3 - Submitted to Rocky Mountain J. Math.

$z=1$ in the right hand side of (4.3) obtaining

$$\begin{aligned}
&\frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} q^{\binom{s}{2}-r}(1-q^r)(1-q^s)w^s \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\sum_{r=1}^{s-1}\frac{1-q^r}{q^r} \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\{q^{-1}+q^{-2}+\cdots+q^{-(s-1)}-(s-1)\} \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\left\{\frac{1-q^{s-1}-q^{s-1}(1-q)(s-1)}{q^{s-1}(1-q)}\right\} \\
&= \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}.
\end{aligned}$$

But by (4.4), this gives us

$$\sum_{n=0}^{\infty}\sum_{s=2}^{\infty} p^{\mathrm{mex}_2}(s,n)w^sq^n = \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{4.5}$$

Note that differentiating the left hand side of (4.5) with respect to $w$ and putting $w=1$, we arrive at

$$\sum_{n=0}^{\infty}\left(\sum_{s=2}^{\infty} sp^{\mathrm{mex}_2}(s,n)\right)q^n = \sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n, \tag{4.6}$$

since each partition of $n$ contributes $s$, its second minimal excludant, to the sum $\sum_{s=2}^{\infty} sp^{\mathrm{mex}_2}(s,n)$.

Now differentiate the right hand side of (4.5) with respect to $w$ and put $w=1$ to get:

$$\frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{4.7}$$

From (4.5), (4.6) and (4.7), we obtain

$$\sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n = \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{4.8}$$

We start by re-indexing the sum on the right hand side of (4.8) as shown:

$$\begin{aligned}
&\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^{s+1}-(s+1)q^s\} \\
&= \sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^s(q-1)-q^s\} \\
&= \sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}} - (1-q)\sum_{s=1}^{\infty} s(s+1)(1-q^{s+1})q^{\binom{s+1}{2}} - \sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s+1}{2}}.
\end{aligned} \tag{4.9}$$

We shall consider the three terms in the right hand side above one by one. First, start with

$$\begin{aligned}\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}} &= \sum_{s=1}^{\infty}(s+1)q^{\binom{s}{2}} - q\sum_{s=1}^{\infty}(s+1)q^{\binom{s+1}{2}} \\ &= 2+\sum_{s=2}^{\infty}(s+1)q^{\binom{s}{2}} - q\sum_{s=2}^{\infty}sq^{\binom{s}{2}} \\ &= 2+\sum_{s=2}^{\infty}q^{\binom{s}{2}} + (1-q)\sum_{s=2}^{\infty}sq^{\binom{s}{2}} = \sum_{s=0}^{\infty}q^{\binom{s}{2}} + (1-q)\sum_{s=2}^{\infty}sq^{\binom{s}{2}}.\end{aligned} \tag{4.10}$$

Next, look at the second term in (4.9):

$$\begin{aligned}(1-q)\sum_{s=1}^{\infty}s(s+1)(1-q^{s+1})q^{\binom{s+1}{2}} &= (1-q)\sum_{s=1}^{\infty}s(s+1)\left(q^{\binom{s+1}{2}}-q^{\binom{s+2}{2}}\right) \\ &= (1-q)\sum_{s=1}^{\infty}s(s+1)q^{\binom{s+1}{2}} - (1-q)\sum_{s=2}^{\infty}s(s-1)q^{\binom{s+1}{2}} \\ &= 2q(1-q)+2(1-q)\sum_{s=2}^{\infty}sq^{\binom{s+1}{2}} \\ &= 2(1-q)\sum_{s=1}^{\infty}sq^{\binom{s+1}{2}} = 2(1-q)\sum_{s=2}^{\infty}(s-1)q^{\binom{s}{2}}.\end{aligned} \tag{4.11}$$

And the last term in (4.9) is

$$\begin{aligned}\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s+1}{2}} &= \sum_{s=2}^{\infty}s(1-q^{s})q^{\binom{s}{2}} \\ &= \sum_{s=2}^{\infty}s\left(q^{\binom{s}{2}}-q^{\binom{s+1}{2}}\right) = \sum_{s=2}^{\infty}sq^{\binom{s}{2}} - \sum_{s=3}^{\infty}(s-1)q^{\binom{s}{2}} \\ &= 2q+\sum_{s=3}^{\infty}q^{\binom{s}{2}} = q+\sum_{s=2}^{\infty}q^{\binom{s}{2}}.\end{aligned} \tag{4.12}$$

Putting (4.10), (4.11), and (4.12) into (4.9), we obtain

$$\begin{aligned}&\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^{s+1}-(s+1)q^{s}\} \\ &= \sum_{s=0}^{\infty}q^{\binom{s}{2}} + (1-q)\sum_{s=2}^{\infty}sq^{\binom{s}{2}} - 2(1-q)\sum_{s=2}^{\infty}(s-1)q^{\binom{s}{2}} - q - \sum_{s=2}^{\infty}q^{\binom{s}{2}} \\ &= 2-q-(1-q)\sum_{s=2}^{\infty}(s-2)q^{\binom{s}{2}} = 2-q-(1-q)\sum_{s=0}^{\infty}sq^{\binom{s+2}{2}}.\end{aligned} \tag{4.13}$$

Substituting (4.13) back into (4.8), we finally arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n &= \frac{1}{(1-q)(q;q)_\infty}\left\{2-q-(1-q)\sum_{s=0}^{\infty}sq^{\binom{s+2}{2}}\right\}\\ &= \frac{1}{(1-q)(q;q)_\infty}\left\{1-(1-q)\sum_{s=0}^{\infty}(s-1)q^{\binom{s+1}{2}}\right\}\\ &= \frac{1}{(q;q)_\infty}\left\{\frac{1}{(1-q)}-\sum_{s=0}^{\infty}(s-1)q^{\binom{s+1}{2}}\right\},\end{aligned}$$

which is precisely (2.1). □

*Proof of Theorem 2.2.* As we have seen in the proof of Theorem 2.1, the generating function for the number of partitions with minimal excludant $r$ and second minimal excludant $s$ is

$$\frac{q^{\binom{s}{2}-r}(1-q^r)(1-q^s)}{(q;q)_\infty} \quad (\text{for } 1 \le r \le s-1).$$

We are interested in the generating function of $\Delta_t(n)$, the number of partitions $\pi$ of $n$ with $\mathrm{mex}_2(\pi)-\mathrm{mex}(\pi)=t$. Suppose that the minimal excludant equals $r$ for some positive integer $r$. Then the generating function for partitions with minimal excludant $r$ and second minimal excludant $r+t$ is given by

$$\frac{q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t})}{(q;q)_\infty}. \tag{4.14}$$

For keeping track of all partitions with $\mathrm{mex}_2(\pi)-\mathrm{mex}(\pi)=t$, we need to sum expressions of the form in (4.14) as $r$ runs over the positive integers. Therefore,

$$\sum_{n=0}^{\infty}\Delta_t(n)q^n = \frac{1}{(q;q)_\infty}\sum_{r=1}^{\infty}q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t}). \tag{4.15}$$

Starting with the right side of (4.15), we have

$$\begin{aligned}\sum_{r=1}^{\infty}q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t}) &= \sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}-r-1}(1-q^{r+1})(1-q^{r+1+t})\\ &= \sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}-r-1}-q^t\sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}}-\sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}}+\sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}+r+t+1}\\ &= q^{t-1}\sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}-(r+t)}-q^t\left(-q^{\binom{t}{2}}+\sum_{r=0}^{\infty}q^{\binom{r+t}{2}}\right)-\sum_{r=0}^{\infty}q^{\binom{r+1+t}{2}}+\sum_{r=0}^{\infty}q^{\binom{r+2+t}{2}}\\ &= q^{t-1}\sum_{r=0}^{\infty}q^{\binom{r+t}{2}}+q^{\binom{t+1}{2}}-q^t\sum_{r=0}^{\infty}q^{\binom{r+t}{2}}-q^{\binom{t+1}{2}}\\ &= (q^{t-1}-q^t)\sum_{r=0}^{\infty}q^{\binom{r+t}{2}}.\end{aligned} \tag{4.16}$$

Hence, putting the information from (4.16) in (4.15), we finally get

$$\sum_{n=0}^{\infty} \Delta_t(n)q^n = \frac{q^{t-1}}{(q^2;q)_\infty} \sum_{r=0}^{\infty} q^{\binom{r+t}{2}}.$$

□

*Proof of Corollary 2.3.* Put $t = 1$ in Theorem 2.2 to see that

$$\sum_{n=0}^{\infty} \Delta_1(n)q^n = \frac{1}{(q^2;q)_\infty} \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{\psi(q)}{(q^2;q)_\infty}, \tag{4.17}$$

by (2.3). Using (3.1), we can write the rightmost expression in (4.17) in terms of the generating function of $\sigma\text{mex}(n)$, which gives us

$$\sum_{n=0}^{\infty} \Delta_1(n)q^n = (1-q)\sum_{n=0}^{\infty} \sigma\text{mex}(n)q^n = \sum_{n=0}^{\infty} (\sigma\text{mex}(n) - \sigma\text{mex}(n-1))\, q^n.$$

From this, we readily derive (2.5).

Now to prove (2.6), start by setting $t = 2$ in Theorem 2.2 to get

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_2(n)q^n &= \frac{q}{(q^2;q)_\infty} \sum_{r=0}^{\infty} q^{\binom{r+2}{2}} \\ &= \frac{q}{(q^2;q)_\infty}(\psi(q)-1) = (q-q^2)\frac{\psi(q)}{(q;q)_\infty} - \frac{q}{(q^2;q)_\infty},\end{aligned} \tag{4.18}$$

where we again invoked (2.3) between the expressions in the first and second lines above. Now, another application of (3.1) gives us

$$(q-q^2)\frac{\psi(q)}{(q;q)_\infty} = (q-q^2)\sum_{n=0}^{\infty} \sigma\text{mex}(n)q^n = \sum_{n=0}^{\infty}\{\sigma\text{mex}(n-1) - \sigma\text{mex}(n-2)\}q^n. \tag{4.19}$$

Also, observe that $\dfrac{q}{(q^2;q)_\infty}$ is the generating function for partitions with exactly one 1. Combining this knowledge along with (4.19), then substituting in (4.18) and comparing the coefficients of $q^n$ at the two extremes furnishes us the required identity:

$$\Delta_2(n) + p(n|\textit{exactly one } 1) = \sigma\text{mex}(n-1) - \sigma\text{mex}(n-2).$$

□

*Proof of Proposition 2.4.* We begin by taking a note of the structure of partitions $\pi$ of $n$ with infinitely long mex sequences. This happens precisely when no integer greater than the minimal excludant can occur as a part in $\pi$. Hence these partitions must be 'gap-free' with smallest part 1, i.e., every part between 1 and the largest part must also occur as parts. Denoting the set of such partitions of $n$ by $\mathcal{P}^*(n)$, we hence see that $\mathcal{P}^*(n) \subset \mathcal{M}_r(n)$ for all positive integers $r$. Thus,

$$p_r^{\text{mex}}(n) \geq |\mathcal{P}^*(n)|, \quad \forall\, r \geq 1. \tag{4.20}$$

Next, we claim that $p_n^{\text{mex}}(n) = |\mathcal{P}^*(n)|$. Suppose that $\mu \in \mathcal{M}_n(n)$ and $\text{mex}(\mu) = r(\geq 1)$. Then, $r, r+1, \ldots, r+n-1$ do not occur in $\mu$. But an integer $m \geq r+n$ also cannot occur

19 Apr 2023 11:57:58 PDT
220812-Eyyunni Version 3 - Submitted to Rocky Mountain J. Math.

in $\mu$, a partition of $n$, because $r+n \geq n+1$. Thus the parts in $\mu$, possibly with repetitions, are $1, 2, \dots, r-1$. (each of them occurs at least once since $\operatorname{mex}(\mu) = r$) This means that $\mu \in \mathcal{P}^*(n)$ and we conclude $\mathcal{M}_n(n) \subset \mathcal{P}^*(n)$, which gives us $p_n^{\mathrm{mex}}(n) = |\mathcal{P}^*(n)|$. As already observed in (2.7) in Section 2, we know that for fixed $n$, $p_r^{\mathrm{mex}}(n)$ is a non-increasing function of $r$. Therefore, using this along with (4.20) gives us

$$|\mathcal{P}^*(n)| \leq p_r^{\mathrm{mex}}(n) \leq p_n^{\mathrm{mex}}(n) = |\mathcal{P}^*(n)|, \quad \forall\, r \geq n.$$

So, for all $r \geq n$, we have showed that $p_r^{\mathrm{mex}}(n) = |\mathcal{P}^*(n)|$. We next show that if $r < n$, then $p_r^{\mathrm{mex}}(n) > |\mathcal{P}^*(n)|$. Assume that $n > 1$ as the proposition is readily seen to hold for $n = 1$. Consider the partition $\mu_0 = n$ of $n$, which has the mex sequence $(1, 2, \dots, n-1)$ of length $n-1$. Since $r \leq n-1$, we deduce that $\mu_0 \in \mathcal{M}_r(n)$. But note that as $n > 1$ we have that $\mu_0 \notin \mathcal{P}^*(n)$, and consequently $p_r^{\mathrm{mex}}(n) > |\mathcal{P}^*(n)|$. Thus, we have established that $p_r^{\mathrm{mex}}(n) = |\mathcal{P}^*(n)| \iff r \geq n$. The proof of the proposition follows because $\mathcal{P}^*(n)$ is equinumerous with the set of distinct parts partitions of $n$, as can be seen by the bijection of conjugation between the two sets. □

*Proof of Theorem 2.5.* Suppose the minimal excludant in a partition is $k+1$ with the integers $k+2, \dots, k+r$ also not occurring as parts. The integers $k+r+1$ and upwards may or may not occur as parts. Also, note that $k$ is a non-negative integer (as the minimal excludant can be 1). Then we can begin to write the generating function for $p_r^{\mathrm{mex}}(n)$ in the following manner:

$$\begin{aligned}
\sum_{n=0}^{\infty} p_r^{\mathrm{mex}}(n) q^n &= \sum_{k=0}^{\infty} \frac{q^1}{1-q^1} \times \cdots \times \frac{q^k}{1-q^k} \times \frac{1}{1-q^{k+r+1}} \times \dots \quad \text{to } \infty \\
&= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q;q)_k} \cdot \frac{1}{(q^{k+r+1};q)_\infty} \\
&= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(q^{k+1};q)_r}{(q;q)_\infty} \\
&= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(q;q)_{k+r}}{(q;q)_k} q^{k(k+1)/2} \\
&= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(q;q)_r (q^{r+1};q)_k}{(q;q)_k} q^{k(k+1)/2} \\
&= \frac{1}{(q^{r+1};q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{r+1};q)_k}{(q;q)_k} q^{k(k-1)/2} \cdot q^k \\
&= \frac{1}{(q^{r+1};q)_\infty} \lim_{A \to 0} \sum_{k=0}^{\infty} \frac{(-1/A;q)_k (q^{r+1};q)_k}{(q;q)_k (0;q)_k} (Aq)^k. \qquad (4.21)
\end{aligned}$$

Note that the sum in (4.21) can be written as ${}_2\phi_1 \left[ \begin{matrix} -1/A, & q^{r+1} \\ & 0 \end{matrix} ; q, Aq \right]$, using the notation in (3.4). It then changes as follows, by setting $a = -1/A,\ b = q^{r+1}, c = 0$ and $z = Aq$ in

Heine's transformation (3.3),

$$\begin{aligned}\frac{1}{(q^{r+1};q)_\infty}\lim_{A\to 0}\sum_{k=0}^{\infty}\frac{(-1/A;q)_k(q^{r+1};q)_k}{(q;q)_k(0;q)_k}(Aq)^k &= \frac{1}{(q^{r+1};q)_\infty}\lim_{A\to 0}\frac{(q^{r+1};q)_\infty(-q;q)_\infty}{(Aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(Aq;q)_n}{(q;q)_n(-q;q)_n}q^{(r+1)n}\\ &= (-q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{(r+1)n}}{(q;q)_n(-q;q)_n}\\ &= (-q;q)_\infty\sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}q^{(r+1)n}. \qquad (4.22)\end{aligned}$$

The sum in (4.22) can be written as a $q$-product by first replacing $q$ by $q^2$ in $q$-binomial theorem (3.2), and then setting $a=0$ and $z=q^{r+1}$ in it. This gives us

$$\sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}q^{(r+1)n}=\frac{1}{(q^{r+1};q^2)_\infty}.$$

Putting this in (4.22), we finally obtain the following from (4.21):

$$\sum_{n=0}^{\infty}p_r^{\mathrm{mex}}(n)q^n=(-q;q)_\infty\frac{1}{(q^{r+1};q^2)_\infty}=\frac{1}{(q;q^2)_\infty}\cdot\frac{1}{(q^{r+1};q^2)_\infty},$$

where the rightmost equality follows by Euler's partition theorem $(-q;q)_\infty = 1/(q;q^2)_\infty$. □

*Proof of Corollary 2.6.* We proceed with the proof in two directions depending on the parity of $r$. Recall from Theorem 2.5 that the generating function for $p_r^{\mathrm{mex}}(n)$ is $\dfrac{1}{(q;q^2)_\infty(q^{r+1};q^2)_\infty}$.

If $r$ **is odd:** In this case, the numbers $r+1,\ r+3,\dots$ are all even and hence $\dfrac{1}{(q;q^2)_\infty}\times\dfrac{1}{(q^{r+1};q^2)_\infty}$ is the generating function for partitions where no even part less than $r$ is allowed. Hence, $p_r^{\mathrm{mex}}(n)=p_e^{>r}(n)$.

If $r$ **is even:** This time around, the integers $r+1,\ r+3,\dots$ are all odd and therefore, $\dfrac{1}{(q;q^2)_\infty}\times\dfrac{1}{(q^{r+1};q^2)_\infty}$ represents partitions into odd parts where parts greater than $r$ come in two colors. Thus, $p_r^{\mathrm{mex}}(n)=p_{o,2}^{>r}(n)$. □

*Proof of Corollary 2.7.* By Theorem 2.2, we know that the generating function for $\Delta_1(n)$ is $\dfrac{\psi(q)}{(q^2;q)_\infty}$. On the other hand, from Theorem 2.5, we have seen that $\dfrac{1}{(q;q^2)_\infty(q^3;q^2)_\infty}$ generates the numbers $p_2^{\mathrm{mex}}(n)$. Since $p_2^{\mathrm{mex}}(n)=\Delta_1(n)$, we get

$$\frac{\psi(q)}{(q^2;q)_\infty}=\frac{1}{(q;q^2)_\infty(q^3;q^2)_\infty},$$

which yields

$$\frac{\psi(q)}{(q;q)_\infty}=\frac{1}{(q;q^2)_\infty^2},$$

after dividing by $1-q$ on both sides. This finally gives

$$\psi(q)=\frac{(q;q)_\infty}{(q;q^2)_\infty^2}=\frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

□

## 5. Concluding Remarks

In this article, we introduced the concept of second minimal excludant in an integer partition. We found its generating function and linked it to minimal excludants via the function $\Delta_t(n)$. The generating function for $\Delta_t(n)$, when interpreted combinatorially, gave rise to nice identities connecting it to $\sigma\text{mex}(n)$ and certain restricted partition functions. We also defined the mex sequence of a partition and discovered an elegant $q$-product expression for the generating function of a related function, namely, $p_r^{\text{mex}}(n)$. And this gives rise to the following natural question:

**Question 1.** *It would be highly desirable to get a bijective proof of the identities for* $p_r^{\text{mex}}(n)$ *in Corollary 2.6.*

We have only skimmed the surface of mex sequences and it would be worth exploring other aspects of them.

Before closing, we would like to point out that other techniques have been fruitfully applied to the study of minimal excludants and related ideas. For instance, by using the theory of modular forms, Barman and Singh [5, 6], and Chakraborty and Ray [9] obtained appealing congruence properties and density results for mex-related partition functions.

**Acknowledgements.** We sincerely thank Bibekananda Maji and Subhash Chand Bhoria for helpful discussions. The third author is a SERB National Post Doctoral Fellow (NPDF) supported by the fellowship PDF/2021/001090 and would like to thank SERB for the same.

## References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, New York, 1998.

[2] G. E. Andrews and D. Newman, *Partitions and the minimal excludant*, Ann. Comb. **23** (2019), no. 2, 249–254.

[3] G. E. Andrews and D. Newman, *The minimal excludant in integer partitions*, J. of Int. Seq. **23** (2020), Article 20.2.3.

[4] C. Ballantine and M. Merca, *Combinatorial proof of the minimal excludant theorem*, Int. J. Number Theory **17** (2021), No. 08, 1765–1779.

[5] R. Barman and A. Singh, *Mex-related partition functions of Andrews and Newman*, J. of Int. Seq. **24** (2021), Article 21.6.3.

[6] R. Barman and A. Singh, *On mex-related partition functions of Andrews and Newman*, Res. Number Theory **7** (2021), Article 53.

[7] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library 34, American Mathematical Society, 2006.

[8] P. S. Kaur, S. C. Bhoria, P. Eyyunni and B. Maji, *Minimal excludant over partitions into distinct parts*, Int. J. Number Theory, **18**, No. 9 (2022), 2015–2028.

[9] K. Chakraborty and C. Ray, *Distribution of generalized Mex-related integer partitions*, Hardy-Ramanujan Journal, Hardy-Ramanujan Society (Special Commemorative volume in honour of Srinivasa Ramanujan), **43** (2021), 122–128.

[10] S. Chern, *Partitions and the maximal excludant*, Electron. J. Combin. **28** (2021), Issue 3, Article P3.13.

[11] A. S. Fraenkel and U. Peled, *Harnessing the unwieldy MEX function*, Games of No Chance 4. Math. Sci. Res. Inst. Publ., 63, pp. 77–94, Cambridge University Press, New York, 2015.

[12] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, 2nd ed., 2004.

[13] P. J. Grabner and A. Knopfmacher, *Analysis of some new partition statistics*, Ramanujan J. **12**(3) (2006), 439–454.

[14] J. A. Sellers and R. da Silva, *Parity considerations for the mex-related partition functions of Andrews and Newman*, J. of Int. Seq. **23** (2020), Article 20.5.7.

Prabh Simrat Kaur, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, Punjab - 147004, India.

*Email address*: `prabh.simrat17@gmail.com`

Meenakshi Rana, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, Punjab - 147004, India.

*Email address*: `mrana@thapar.edu`

Pramod Eyyunni, Discipline of Mathematics, Indian Institute of Technology Indore, Simrol, Indore, Madhya Pradesh - 453552, India.

*Email address*: `pramodeyy@iiti.ac.in, pramodeyy@gmail.com`