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ON STABILITY OF QUADRATIC LIE HOM-DERS IN LIE BANACH ALGEBRAS

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ABSTRACT. In this paper, we study the notion of a quadratic Lie hom-der in a Lie Banach algebra associated with the functional equation

$$\text{(P)} \qquad f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)=f(x)+f(y)+4f(z),$$

which was first introduced by Park et al. [15]. We also present a relation between the functional equation (P) and the quadratic functional equation on certain groups. Finally, we prove some stability results of the quadratic Lie hom-ders in Lie Banach algebras by using Hyers' direct method.

1. Introduction and preliminaries

The notion of the *stability of functional equations* seems to be originated by Ulam [25] in 1940. He raised a problem concerning the stability of group homomorphisms. In fact, he proposed the following problem: *If a function from a group into a metric group is an approximate group homomorphism, does there exist a group homomorphism close to it (in a particular sense)?* In order to answer Ulam's problem, Hyers [10] was the first author who considered a stability result related to the additive (or Cauchy) functional equation $f(x+y) = f(x)+f(y)$ in Banach spaces. Since then, many authors generalized Hyers' result in various ways and it is worth mentioning some interesting results in [2, 7, 8, 9, 13, 17, 19, 22] for further references.

As it is known that the additive functional equation is well-known in this area, one of the interesting functional equations is the *quadratic functional equation*. Every function satisfying the quadratic functional equation is called a *quadratic function*, that is, if f is a quadratic function then it satisfies

$$f(x+y)+f(x-y) = 2f(x)+2f(y). \tag{1.1}$$

Aczél [1] (see also [23]) proved that a general solution of (1.1) in real linear spaces relates to a biadditive function. Indeed, if f is a quadratic function, then there exists a biadditive function B such that $f(x) = B(x,x)$. Skof [24] was the first author who investigated the stability of the functional equation (1.1) in Banach spaces. Some improvements and generalizations can be seen in [5, 6].

In 2006, Park et al. [15] investigated the functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)=f(x)+f(y)+4f(z) \tag{1.2}$$

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in a Banach space and proved some stability results based on the notion of Găvruta [9]. Also, they proved that any even mapping f satisfying $f(0)=0$ and (1.2) is quadratic. It fact, they proposed the following result [15, Lemma 2.1]

Lemma 1.1. ([15, Lemma 2.1]) *Suppose that X and Y are linear spaces. If $f:X\to Y$ is an even mapping that satisfies $f(0)=0$ and (1.2) for all $x,y,z\in X$, then f is indeed a quadratic mapping.*

Jang and Park [11] proved the stability of $*$-derivations and of quadratic $*$-derivations on Banach $*$-algebras. In addition, they also presented some results related to the superstability of $*$-derivations and of quadratic $*$-derivations. More interesting results concerning the stability of homomorphisms, quadratic mappings, and derivations on certain types of algebras can be seen in [3, 4, 16, 18, 20, 21]. Some stability results based on a fixed point approach of quadratic Lie $*$-derivations were investigated by Kang and Kho [12].

In the sequel, $\mathbb{R},\mathbb{C}$, and $\mathbb{N}_0$ denote the sets of real numbers, of complex numbers, and of non-negative integers, respectively. Moreover, B^A stands for the class of all mappings from a non-empty set A into a non-empty set B.

2. Solution and stability of the functional equation (1.2)

In this section, we organize our paper as follows.

- At first step, we improve Lemma 1.1 [15, Lemma 2.1] by showing that the result still holds in certain groups and we also relax some conditions. Moreover, the quadratic homogeneity of quadratic mappings will be presented.
- By applying Hyers' method, we prove the existence of a quadratic mapping arising from an approximately quadratic mapping related to the functional equation (1.2).

A commutative group $(G,+)$ is said to be *uniquely 2-divisible* if for each element $x\in G$ there exists uniquely $x'\in G$ such that $2x'=x$ where $2x':=x'+x'$. In this case, we define $\frac{x}{2}:=x'$. By defining $\frac{x}{2^0}:=x$ for all $x\in G$, it is easy to see by the association and commutativity laws of the group that

$$\frac{x}{2^n}\pm\frac{y}{2^n}=\frac{x\pm y}{2^n}\quad\text{for all } x,y\in G \text{ and } n\in\mathbb{N}_0.$$

To improve Lemma 1.1, we assume that $(G,+)$ is a uniquely 2-divisible group and $(H,+)$ is a group.

For convenience, we define an operator $\Delta: H^G\to H^{G^3}$ by

$$\Delta f(x,y,z):=f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)$$
$$-f(x)-f(y)-4f(z)\quad\text{for all } x,y,z\in G,$$

for any mapping $f:G\to H$.

Lemma 2.1. *Suppose that $(H,+)$ is a commutative non-trivial group (that is, $H\neq\{0\}$) in which each of its element is of order different from 2. Then a mapping $f:G\to H$ satisfies*

$$\Delta f(x,y,z)=0\quad\textit{for all } x,y,z\in G \tag{2.1}$$

if and only if f is quadratic.

Proof. ($\Longrightarrow$) Recall from (2.1) that the mapping f satisfies: for all $x, y, z \in G$,

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)=f(x)+f(y)+4f(z). \tag{2.2}$$

Letting $x = y = z = 0$ in (2.2), we have that $4f(0) = 6f(0)$. It follows from the cancellation law that $2f(0) = 0$. If $f(0) \neq 0$, then the order of $f(0)$ is 2 which is impossible. Hence, we can conclude that $f(0) = 0$. Replacing x, y, z by $2x, 2y, 0$ in (2.2), respectively, we have

$$\frac{2x \pm 2y}{2} = \frac{2x}{2} \pm \frac{2y}{2} = \frac{x+x}{2} \pm \frac{y+y}{2} = \left(\frac{x}{2}+\frac{x}{2}\right) \pm \left(\frac{y}{2}+\frac{y}{2}\right) = x \pm y$$

and hence

$$2f(x+y)+2f(x-y) = f(2x)+f(2y) \quad \text{for all } x, y \in G.$$

Letting $y = z = 0$ and replacing x by $2x$ in (2.2), we obtain that

$$4f(x) = f(2x) \quad \text{for all } x \in G.$$

It follows that

$$2f(x+y)+2f(x-y) = f(2x)+f(2y) = 4f(x)+4f(y) \quad \text{for all } x, y \in G.$$

Since $(H,+)$ is commutative, we see that

$$2\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right) = 0 \quad \text{for all } x, y \in G.$$

This implies that f is quadratic.

($\Longleftarrow$) Since f is quadratic, one gets that $2f(0) = 4f(0)$ and then $2f(0) = 0$ and hence $f(0) = 0$. We also have that

$$f(2x) = 4f(x) \quad \text{for all } x \in G.$$

If we replace x by $\frac{x}{2}$, we have that $2\left(\frac{x}{2}\right) = x$ and

$$f(x) = 4f\left(\frac{x}{2}\right) \quad \text{for all } x \in G.$$

Let $x, y, z \in G$ be given. We consider

$$\begin{aligned}
&2f\left(\frac{x+y}{2}+z\right)+2f\left(\frac{x+y}{2}-z\right)+2f\left(\frac{x-y}{2}+z\right)+2f\left(\frac{x-y}{2}-z\right)\\
&=\left(4f\left(\frac{x+y}{2}\right)+4f(z)\right)+\left(4f\left(\frac{x-y}{2}\right)+4f(z)\right)\\
&=\left(f\left(x+y\right)+4f(z)\right)+\left(f\left(x-y\right)+4f(z)\right)\\
&=\left(f(x+y)+f(x-y)\right)+8f(z)\\
&=2f(x)+2f(y)+8f(z).
\end{aligned}$$

Therefore, f satisfies (2.1). $\square$

We directly obtain the following corollary from Lemma 2.1. Moreover, it is related to [15, Lemma 2.1] with a weaker condition.

Corollary 2.2. *Suppose that X and Y are linear spaces. Then $f : X \to Y$ satisfies (2.1) if and only if it is quadratic.*

By making use of Lemma 2.1, we obtain the following result.

Lemma 2.3. *Suppose that $(Y, \|\cdot\|)$ is a Banach space, $f : G \to Y$ is a mapping, and $\varphi : G^3 \to [0,\infty)$ is a function satisfying*

$$\|\Delta f(x,y,z)\| \leq \varphi(x,y,z) \quad \text{for all } x,y,z \in G. \tag{2.3}$$

If $\varphi : G^3 \to [0,\infty)$ additionally satisfies the following condition

$$\underline{\Phi}(x,y,z) := \sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \quad \text{for all } x,y,z \in G,$$

then there exists a quadratic mapping $F : G \to Y$ such that

$$\|F(x) - f(x)\| \leq \underline{\Phi}(x,0,0) \quad \text{for all } x \in G. \tag{2.4}$$

Moreover, if there exist a quadratic mapping $Q : G \to Y$ and a constant $K \geq 0$ such that

$$\|Q(x) - f(x)\| \leq K\underline{\Phi}(x,0,0) \quad \text{for all } x \in G, \tag{2.5}$$

then Q and F are identical.

Proof. It follows from (2.3) that $\varphi(0,0,0) = 0$. So, we have that $f(0) = 0$.

It can be easily obtained by letting $y = z = 0$ in (2.3) that

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x,0,0) \quad \text{for all } x \in G. \tag{2.6}$$

We now define a sequence (f_n) by, for each $n \in \mathbb{N}_0$,

$$f_n(x) := 4^n f\left(\frac{x}{2^n}\right) \quad \text{for all } x \in G.$$

Note that $f_0 = f$. Let $n, m \in \mathbb{N}_0$ be such that $n < m$. For each $x \in G$, the triangle inequality of the norm on Y implies that

$$\begin{aligned}
\|f_n(x) - f_m(x)\| &\leq \sum_{k=n}^{m-1} \left\|4^k f\left(\frac{x}{2^k}\right) - 4^{k+1} f\left(\frac{x}{2^{k+1}}\right)\right\| \\
&= \sum_{k=n}^{m-1} 4^k \left\|f\left(\frac{x}{2^k}\right) - 4f\left(\frac{\frac{x}{2^k}}{2}\right)\right\| \\
&\leq \sum_{k=n}^{m-1} 4^k \varphi\left(\frac{x}{2^k}, 0, 0\right).
\end{aligned} \tag{2.7}$$

Since $(Y, \|\cdot\|)$ is a Banach space, we have that the sequence $(f_n(x))$ is a Cauchy sequence for every $x \in G$. This allows us to define $F : G \to Y$ as follows:

$$F(x) := \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in G.$$

Next, we prove that F is a quadratic mapping. Let $x,y,z \in G$ be given. We see that

$$\|\Delta F(x,y,z)\| = \lim_{n\to\infty} \left\| 4^n \Delta f\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\right\| \leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0.$$

Hence, $\Delta F(x,y,z) = 0$ for all $x,y,z \in G$. Lemma 2.1 asserts that F is quadratic. Moreover, by letting $n = 0$ and $m \to \infty$ in (2.7) we have that (2.4) holds.

Finally, suppose that $Q : G \to Y$ is a quadratic mapping and K is a real number such that (2.5) holds. Let $x \in G$ and $n \in \mathbb{N}_0$ be given. Since F and Q are quadratic, we see that $F(x) = 4^n F\left(\frac{x}{2^n}\right)$ and $Q(x) = 4^n Q\left(\frac{x}{2^n}\right)$. So, we have

$$\begin{aligned} \|F(x) - Q(x)\| &= \left\| 4^n F\left(\frac{x}{2^n}\right) - 4^n Q\left(\frac{x}{2^n}\right)\right\| \\ &\leq \left\| 4^n F\left(\frac{x}{2^n}\right) - 4^n f\left(\frac{x}{2^n}\right)\right\| + \left\| 4^n Q\left(\frac{x}{2^n}\right) - 4^n f\left(\frac{x}{2^n}\right)\right\| \\ &\leq (1+K) 4^n \underline{\Phi}\left(\frac{x}{2^n},0,0\right). \end{aligned} \tag{2.8}$$

By letting $n \to \infty$ in (2.8), we can see that $F(x) = Q(x)$.

This completes the proof. □

Remark 1. According to Lemma 2.3, we have the following observations.

1) As in the proof of the lemma, the result still holds if we only assume that

$$\underline{\Phi}(x,0,0) := \sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n},0,0\right) < \infty \text{ and } \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0$$

for all $x,y,z \in G$.

2) The last sentence of the lemma says that: *there exists a unique quadratic mapping $F : G \to Y$ such that* (2.4) *holds true*.

To complement Lemma 2.3, we present the following result.

Lemma 2.4. *Suppose that $(Y, \|\cdot\|)$ is a Banach space, $f : G \to Y$ is a mapping, and $\varphi : G^3 \to [0,\infty)$ is a function satisfying*

$$\|\Delta f(x,y,z)\| \leq \varphi(x,y,z) \quad \textit{for all } x,y,z \in G.$$

If $\varphi : G^3 \to [0,\infty)$ additionally satisfies the following condition

$$\overline{\Phi}(x,y,z) := \sum_{n=1}^{\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z) < \infty \quad \textit{for all } x,y,z \in G,$$

then there exists a quadratic mapping $F : G \to Y$ such that

$$\|F(x) - f(x)\| \leq \overline{\Phi}(x,0,0) \quad \textit{for all } x \in G.$$

Moreover, if there exist a quadratic mapping $Q : G \to Y$ and a constant $K \geq 0$ such that

$$\|Q(x) - f(x)\| \leq K\overline{\Phi}(x,0,0) \quad \textit{for all } x \in G,$$

then Q and F are identical.

Proof. We see from (2.6) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(2x,0,0) \quad \text{for all } x \in G.$$

We now define a sequence (f_n) by, for each $n \in \mathbb{N}_0$,

$$f_n(x) := \frac{1}{4^n} f(2^n x) \quad \text{for all } x \in G.$$

Let $n, m \in \mathbb{N}_0$ be such that $n < m$. For each $x \in G$, we see that

$$\begin{aligned} \|f_n(x) - f_m(x)\| &\leq \sum_{k=n}^{m-1} \left\| \frac{1}{4^k} f(2^k x) - \frac{1}{4^{k+1}} f(2^{k+1} x) \right\| \\ &= \sum_{k=n}^{m-1} \frac{1}{4^k} \left\| f\left(2^k x\right) - 4 f\left(2(2^k x)\right) \right\| \\ &\leq \sum_{k=n}^{m-1} \frac{1}{4^{k+1}} \varphi\left(2(2^k), 0, 0\right) = \sum_{k=n+1}^{m} \frac{1}{4^k} \varphi\left(2^k x, 0, 0\right). \end{aligned}$$

It follows that the sequence $(f_n(x))$ is Cauchy for all $x \in G$. So, we define $F : G \to H$ by

$$F(x) := \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in G.$$

Since the rest of the proof is quite similar to that of Lemma 2.3, our result is proved. □

Remark 2. According to Lemma 2.4, the result still holds if we only assume that

$$\overline{\Phi}(x,0,0) := \sum_{n=1}^{\infty} \frac{1}{4^n} \varphi(2^n x, 0, 0) < \infty \text{ and } \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in G$.

We end this section with the result concerning the *quadratic homogeneity* of a quadratic mapping. Suppose that X and Y are complex linear spaces. By a *quadratic homogeneous* mapping, we mean a mapping $f : X \to Y$ which satisfies the identity

$$f(\lambda x) = \lambda^2 f(x) \quad \text{for all } x \in X \text{ and all } \lambda \in \mathbb{C}.$$

Example 2.5. Consider that following two examples:

(1) For any complex algebra X, a mapping $X \ni x \mapsto x^2$ is quadratic homogeneous;
(2) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x) := \overline{x}^2$ for all $x \in \mathbb{C}$. It is easy to verify that f is a quadratic mapping which is *not* quadratic homogeneous.

To prove our main theorems, the following result is needed. For the sake of the completeness of this paper, we also present its proof which is based on the idea in [12, 14].

Lemma 2.6. *Suppose that X is a complex Banach space and $F : X \to X$ is a quadratic mapping that satisfies*

$$F(\lambda x) = \lambda^2 F(x) \quad \text{for all } x \in X \text{ and all } \lambda \in \mathbf{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Suppose that there exists a sequence of mappings (f_n) such that

$$F(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in X.$$

If, for any $n \in \mathbb{N}_0$ and for each fixed $x \in X$, the mapping $\mathbb{R} \ni r \mapsto f_n(rx)$ is continuous on $\mathbb{R}$, then F is quadratic homogeneous.

Proof. Let $x \in X$ be fixed and $\rho : X \to \mathbb{R}$ be any bounded linear functional. For $n \in \mathbb{N}_0$, we define two functions $\eta, \gamma_n : \mathbb{R} \to \mathbb{R}$ by

$$\eta(r) := \rho(F(rx)) \quad \text{and} \quad \gamma_n(r) := \rho(f_n(rx)) \quad \text{for all } r \in \mathbb{R}.$$

For any $r \in \mathbb{R}$, we see from the continuity of ρ that

$$\lim_{n\to\infty} \gamma_n(r) = \lim_{n\to\infty} \rho(f_n(rx)) = \rho\left(\lim_{n\to\infty} f_n(rx)\right) = \rho(F(rx)) = \eta(r).$$

Observe that η is measurable (in fact, η is the limit of the sequence of measurable functions). It is noted that η is a quadratic function on $\mathbb{R}$. It follows from [14, Corollary 1 and Theorem 1] that η is of the form

$$\eta(r) = r^2\eta(1) \quad \text{for all } r \in \mathbb{R}.$$

We finally prove that F is quadratic homogeneous. To see this, let $\lambda \in \mathbb{C} \setminus \{0\}$ be given. Consider

$$\begin{aligned}\rho(F(\lambda x)) &= \rho\left(F\left(|\lambda| \cdot \frac{\lambda}{|\lambda|}x\right)\right) = \frac{\lambda^2}{|\lambda|^2}\eta(|\lambda|)\\ &= \lambda^2\eta(1) = \lambda^2\rho(F(x)) = \rho\left(\lambda^2 F(x)\right).\end{aligned}$$

Since $x \in X$ and ρ are arbitrarily given, we have that F is a quadratic homogeneous mapping. □

3. Stability results of quadratic Lie hom-ders

Now, we are in a position to introduce the notion of a *quadratic Lie hom-der* in a complex Lie algebra.

Definition 3.1. Suppose that $\mathscr{L}$ is a complex Lie algebra. A quadratic mapping $\delta : \mathscr{L} \to \mathscr{L}$ is called a $(\mathbf{q_1}, \mathbf{q_2})$-*quadratic Lie hom-der* if δ is quadratic homogeneous, $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ are quadratic mappings, and the following identity is valid:

$$[\delta(x), \delta(y)] = [\mathbf{q_1}(x), \delta(y)] + [\delta(x), \mathbf{q_2}(y)] \quad \text{for all } x, y \in \mathscr{L}.$$

Example 3.2. Suppose that $\mathscr{A}$ is a complex algebra. It is well-known that $\mathscr{A}$ is a Lie algebra under the commutator bracket defined by $[x, y] := xy - yx$. The mapping $\delta : \mathscr{L} \to \mathscr{L}$, defined by $\delta(x) := 2x^2$ for all $x \in \mathscr{L}$, is a $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der where $\mathbf{q_1}(x) = \mathbf{q_2}(x) := x^2$ for all $x \in \mathscr{L}$.

Inspired by the above example, we have the following result.

Lemma 3.3. *Suppose that $\mathscr{L}$ is a Lie algebra and $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ are quadratic mappings. If $\mathbf{q_1}$ and $\mathbf{q_2}$ are quadratic homogeneous, then the mapping $\delta := \mathbf{q_1} + \mathbf{q_2}$ is a $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der if and only if $[\mathbf{q_1}, \mathbf{q_2}] = [\mathbf{q_2}, \mathbf{q_1}]$ (that is, $[\mathbf{q_1}(x), \mathbf{q_2}(y)] = [\mathbf{q_2}(x), \mathbf{q_1}(y)]$ for all $x, y \in \mathscr{L}$).*

Proof. It is clear that δ is a quadratic mapping. Let $x, y \in \mathscr{L}$ be given. We see that

$$\begin{aligned}[\delta(x), \delta(y)] &= [\mathbf{q_1}(x) + \mathbf{q_2}(x), \mathbf{q_1}(y) + \mathbf{q_2}(y)] \\ &= [\mathbf{q_1}(x), \mathbf{q_1}(y) + \mathbf{q_2}(y)] + [\mathbf{q_2}(x), \mathbf{q_1}(y) + \mathbf{q_2}(y)] \\ &= [\mathbf{q_1}(x), \delta(y)] + [\mathbf{q_2}(x), \mathbf{q_1}(y)] + [\mathbf{q_2}(x), \mathbf{q_2}(y)]\end{aligned}$$

and

$$\begin{aligned}&[\mathbf{q_1}(x), \delta(y)] + [\delta(x), \mathbf{q_2}(y)] \\ &= [\mathbf{q_1}(x), \delta(y)] + [\mathbf{q_1}(x) + \mathbf{q_2}(x), \mathbf{q_2}(y)] \\ &= [\mathbf{q_1}(x), \delta(y)] + [\mathbf{q_1}(x), \mathbf{q_2}(y)] + [\mathbf{q_2}(x), \mathbf{q_2}(y)].\end{aligned}$$

So, we have that

$$[\delta(x), \delta(y)] = [\mathbf{q_1}(x), \delta(y)] + [\delta(x), \mathbf{q_2}(y)] \iff [\mathbf{q_2}(x), \mathbf{q_1}(y)] = [\mathbf{q_1}(x), \mathbf{q_2}(y)].$$

This completes the proof. $\square$

Example 3.4. Let $\mathbf{u} := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \in \mathscr{L} := M_2(\mathbb{C})$. We define $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ by

$$\mathbf{q_1}(x) := x^2 \quad \text{and} \quad \mathbf{q_2}(x) := [x, [\mathbf{u}, x]] \quad \text{for all } x \in \mathscr{L}.$$

We see that $\mathbf{q_1}$ is a quadratic mapping. Now, we prove that $\mathbf{q_2}$ is quadratic. To verify this, let $x, y \in \mathscr{L}$ be given. By using the bilinearity of $[\cdot, \cdot]$, we see that:

$$\begin{aligned}\mathbf{q_2}(x+y) &= [x+y, [\mathbf{u}, x+y]] \\ &= [x, [\mathbf{u}, x+y]] + [y, [\mathbf{u}, x+y]] \\ &= [x, [\mathbf{u}, x] + [\mathbf{u}, y]] + [y, [\mathbf{u}, x] + [\mathbf{u}, y]], \\ \mathbf{q_2}(x-y) &= [x-y, [\mathbf{u}, x-y]] \\ &= [x, [\mathbf{u}, x-y]] - [y, [\mathbf{u}, x-y]] \\ &= [x, [\mathbf{u}, x] - [\mathbf{u}, y]] + [y, [\mathbf{u}, y] - [\mathbf{u}, x]].\end{aligned}$$

This implies that $\mathbf{q_2}$ is a quadratic mapping. We also see that $\mathbf{q_1}, \mathbf{q_2}$ are quadratic homogeneous.

Next, we show that there is no $\mathbf{u}' \in M_2(\mathbb{C})$ such that $x_0^2 = \mathbf{q_2}(x_0) = [x_0, [\mathbf{u}, x_0]]$ where $x_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. To verify this, suppose that such a $\mathbf{u}'$ exists and we write $\mathbf{u} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider the following computation:

$$\begin{aligned}[\mathbf{u}', x_0] &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}, \\ [x_0, [\mathbf{u}', x_0]] &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} - \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.\end{aligned}$$

It follows that

$$x_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = [x_0, [\mathbf{u}', x_0]].$$

This shows that $x_0^2 \neq [x_0, [\mathbf{u}', x_0]]$ for all $\mathbf{u}' \in \mathscr{L}$. It follows that $\mathbf{q_1} \neq \mathbf{q_2}$.

It is not hard to see that there exists $x \in \mathscr{L}$ such that $[\mathbf{q_1}(x), \mathbf{q_2}(x)] \neq [\mathbf{q_2}(x), \mathbf{q_1}(x)]$ (this implies that $[\mathbf{q_1}, \mathbf{q_2}] \neq [\mathbf{q_2}, \mathbf{q_1}]$). By Lemma 3.3, we see that $\delta := \mathbf{q_1} + \mathbf{q_2}$ is a quadratic mapping which is *not* a $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der. In particular, this shows that there exists a quadratic mapping which is not the (certain $\mathbf{q_1}, \mathbf{q_2}$) quadratic Lie hom-der.

From now on, we assume that $\mathscr{L}$ is a complex Lie Banach algebra.

In the rest of this paper, to make the simplicity, we make the following conventions.

- A function $\varphi : \mathscr{L}^3 \to [0, \infty)$ belongs to the class $\mathfrak{C}^i(\mathscr{L})$ (i=1,2) if it satisfies

$$\mathscr{G}(\varphi, i)(x,y,z) := \sum_{n=\frac{1-(-1)^{i+1}}{2}}^{\infty} 4^{(-1)^{i+1}n} \varphi\left(2^{(-1)^i n}x, 2^{(-1)^i n}y, 2^{(-1)^i n}z\right) < \infty$$

  for all $x,y,z \in \mathscr{L}$.
- A mapping $f : \mathscr{L} \to \mathscr{L}$ is of class $\mathscr{R}(\mathscr{L})$ if the mapping $\mathbb{R} \ni r \mapsto f(rx)$ is continuous on $\mathbb{R}$, for each fixed $x \in \mathscr{L}$.

It is easy to see that if φ belongs to $\mathfrak{C}^1(\mathscr{L})$ and $\widetilde{\varphi}$ belongs to $\mathfrak{C}^2(\mathscr{L})$, then

$$\sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{4^n} \widetilde{\varphi}\left(2^n x, 2^n y, 2^n z\right) < \infty$$

for all $x,y,z \in \mathscr{L}$, respectively.

Now, inspired by the idea of Găvruta [9], we are in a position to prove our main theorems.

In this section, we further define an operator $\blacktriangle : \mathscr{L}^{\mathscr{L}} \times \mathbf{T}^1 \to \mathscr{L}^{\mathscr{L}^3}$ by

$$\begin{aligned}\blacktriangle(f,\lambda)(x,y,z) := f\left(\frac{x+y}{2} + \lambda z\right) + f\left(\frac{x+y}{2} - \lambda z\right) + f\left(\frac{x-y}{2} + \lambda z\right) + f\left(\frac{x-y}{2} - \lambda z\right) \\ - f(x) - f(y) - 4\lambda^2 f(z) \quad \text{for all } x,y,z \in \mathscr{L},\end{aligned}$$

for any mapping $f : \mathscr{L} \to \mathscr{L}$ and $\lambda \in \mathbf{T}^1$. It is easy to see that $\blacktriangle(f,1) = \Delta f$. Hence, we shall use the notation Δf instead of $\blacktriangle(f,1)$.

Theorem 3.5. *Suppose that $\varphi, \varphi_1, \varphi_2 : \mathscr{L}^3 \to [0,\infty)$ are functions and $f, q_1, q_2 : \mathscr{L} \to \mathscr{L}$ are mapping fulfilling the following three inequalities*

$$\begin{aligned}\|\blacktriangle(f,\lambda)(x,y,z)\| &\le \varphi(x,y,z); \\ \|\Delta q_1(x,y,z)\| &\le \varphi_1(x,y,z); \\ \|\Delta q_2(x,y,z)\| &\le \varphi_2(x,y,z)\end{aligned}$$

for all $x,y,z \in \mathscr{L}$ and all $\lambda \in \mathbf{T}^1$. Suppose that f, q_1 and q_2 also satisfy

(3.1) $$\big\|[f(x), f(y)] - [q_1(x), f(y)] - [f(x), q_2(y)]\big\| \le \Psi(x,y) \quad \text{for all } x,y \in \mathscr{L},$$

for some control function $\Psi : \mathscr{L}^2 \to [0,\infty)$ having the property

$$\lim_{n\to\infty} 16^n \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad \text{for all } x,y \in \mathscr{L}.$$

If $f \in \mathscr{R}(\mathscr{L})$ and $\phi, \phi_1, \phi_2 \in \mathfrak{C}^1(\mathscr{L})$, then there exist uniquely three quadratic mappings $\delta, \mathbf{q}_1, \mathbf{q}_2 : \mathscr{L} \to \mathscr{L}$ such that δ is a $(\mathbf{q}_1, \mathbf{q}_2)$-quadratic Lie hom-der and

$$\|\delta(x) - f(x)\| \leq \mathscr{G}(\varphi, 1)(x,0,0); \tag{3.2}$$

$$\|\mathbf{q}_1(x) - q_1(x)\| \leq \mathscr{G}(\varphi_1, 1)(x,0,0); \tag{3.3}$$

$$\|\mathbf{q}_2(x) - q_2(x)\| \leq \mathscr{G}(\varphi_2, 1)(x,0,0) \tag{3.4}$$

for all $x \in \mathscr{L}$.

Proof. Since $\varphi, \varphi_1, \varphi_2 \in \mathfrak{C}^1(\mathscr{L})$, there exist uniquely quadratic mappings $\delta, \mathbf{q}_1$ and $\mathbf{q}_2$ satisfying (3.2), (3.3), and (3.4), respectively. We also see that

$$\delta\left(\frac{x+y}{2} + \lambda z\right) + \delta\left(\frac{x+y}{2} - \lambda z\right) + \delta\left(\frac{x-y}{2} + \lambda z\right) + \delta\left(\frac{x-y}{2} - \lambda z\right)$$
$$= \delta(x) + \delta(y) + 4\lambda^2 \delta(z)$$

for all $x,y,z \in \mathscr{L}$ and all $\lambda \in \mathbf{T}^1$. Letting $x = y = 0$ in the previous equation, one gets that

$$\delta(\lambda z) = \lambda^2 \delta(z) \quad \text{for all } z \in \mathscr{L} \text{ and all } \lambda \in \mathbf{T}^1.$$

By using Lemma 2.6, we show that δ is quadratic homogeneous. For each $n \in \mathbb{N}_0$, we define a sequence $(f_n) \subset \mathscr{L}^{\mathscr{L}}$ by

$$f_n(x) := 4^n f\left(\frac{x}{2^n}\right) \quad \text{for all } x \in \mathscr{L}.$$

By Lemma 2.3, it is easy to see that $\delta = \lim_{n\to\infty} f_n$. Since $f \in \mathscr{R}(\mathscr{L})$, we have by Lemma 2.6 that δ is quadratic homogeneous as desired.

Finally, it remains to show that δ is a $(\mathbf{q_1}, \mathbf{q}_2)$-quadratic Lie hom-der. Given $x, y \in \mathscr{L}$ and $n \in \mathbb{N}_0$, we see from (3.1) that

$$\left\| \left[4^n f\left(\frac{x}{2^n}\right), 4^n f\left(\frac{y}{2^n}\right)\right] - \left[4^n q_1\left(\frac{x}{2^n}\right), 4^n f\left(\frac{y}{2^n}\right)\right] - \left[4^n f\left(\frac{x}{2^n}\right), 4^n q_2\left(\frac{y}{2^n}\right)\right] \right\|$$
$$= 16^n \left\| \left[f\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right] - \left[q_1\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right] - \left[f\left(\frac{x}{2^n}\right), q_2\left(\frac{y}{2^n}\right)\right] \right\|$$
$$\leq 16^n \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right). \tag{3.5}$$

Again, by Lemma 2.3 we also have

$$\delta(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right), \ \mathbf{q}_1(x) = \lim_{n\to\infty} 4^n q_1\left(\frac{x}{2^n}\right) \text{ and } \mathbf{q}_2(x) = \lim_{n\to\infty} 4^n q_2\left(\frac{x}{2^n}\right).$$

The proof is finished by letting $n \to \infty$ in (3.5). □

As a consequence of Theorem 3.1, we obtain the following two corollaries.

Corollary 3.6. *Suppose that $\varphi : \mathscr{L}^3 \to [0,\infty)$ and $\psi : \mathscr{L}^2 \to [0,\infty)$ are function, and there are real numbers $L, L' > 1$ such that*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{4}\varphi(x,y,z) \quad \textit{for all } x,y,z \in \mathscr{L},$$

$$\psi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L'}{16}\psi(x,y) \quad \textit{for all } x,y \in \mathscr{L}.$$

Suppose that $f : \mathscr{L} \to \mathscr{L}$ is a mapping such that $f \in \mathscr{R}(\mathscr{L})$ and satisfies the inequality

$$\|\blacktriangle(f,\lambda)(x,y,z)\| \le \varphi(x,y,z) \quad \text{for all } x,y,z \in \mathscr{L} \text{ and all } \lambda \in \mathbf{T}^1.$$

If there exist two quadratic mappings $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ such that

$$\big\|[f(x), f(y)] - [\mathbf{q_1}(x), f(y)] - [f(x), \mathbf{q_2}(y)]\big\| \le \psi(x,y) \quad \text{for all } x,y \in \mathscr{L},$$

then there exists a unique $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der $\delta : \mathscr{L} \to \mathscr{L}$ such that

$$\|\delta(x) - f(x)\| \le \frac{1}{1-L}\varphi(x,0,0) \quad \text{for all } x \in \mathscr{L}.$$

Proof. We can see by induction on $\mathbb{N}_0$ that

$$\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \le \frac{L^n}{4^n}\varphi(x,y,z) \quad \text{and} \quad \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \frac{L'^n}{16^n}\psi(x,y)$$

for all $x,y,z \in \mathscr{L}$. It follows that

$$\sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) \le \sum_{n=0}^{\infty} L^n \varphi(x,y,z) = \frac{1}{1-L}\varphi(x,y,z) < \infty,$$

$$\lim_{n\to\infty} 16^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n\to\infty} L'^n \psi(x,y) = 0.$$

By choosing $\varphi_1 = \varphi_2 := \mathbf{0}$ and $\Psi := \psi$, the result follows from Theorem 3.1. □

Corollary 3.7. *Suppose that $h : \mathscr{L} \to [0,\infty)$ is a given function and $f : \mathscr{L} \to \mathscr{L}$ is a mapping such that $f \in \mathscr{R}(\mathscr{L})$ and*

$$\|\blacktriangle(f,\lambda)(x,y,z)\| \le h(x) + h(y) + h(z) \quad \text{for all } x,y,z \in \mathscr{L} \text{ and all } \lambda \in \mathbf{T}^1.$$

Suppose that there exist two quadratic mappings $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ such that

$$\big\|[f(x), f(y)] - [\mathbf{q_1}(x), f(y)] - [f(x), \mathbf{q_2}(y)]\big\| \le (h(x))^2 + (h(y))^2$$

for all $x,y \in \mathscr{L}$. If there exists a real number $\nu > 2$ such that $h(rx) = |r|^\nu h(x)$ for all $r \in \mathbb{R}$, then there exists a unique $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der $\delta : \mathscr{L} \to \mathscr{L}$ such that

$$\|\delta(x) - f(x)\| \le \frac{2^\nu}{2^\nu - 4} h(x) \quad \text{for all } x \in \mathscr{L}.$$

Proof. We first let $\varphi(x,y,z) := h(x) + h(y) + h(z)$ for all $x,y,z \in \mathscr{L}$. For each $n \in \mathbb{N}_0$ and $x,y,z \in \mathscr{L}$, one can see that

$$\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) = h\left(\frac{x}{2^n}\right) + h\left(\frac{y}{2^n}\right) + h\left(\frac{z}{2^n}\right) = \frac{1}{2^{\nu n}}\varphi(x,y,z).$$

Since $\nu > 2$, we get that

$$\sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) = \sum_{n=0}^{\infty} 2^{(2-\nu)n}\varphi(x,y,z) = \frac{2^\nu}{2^\nu - 4}\varphi(x,y,z).$$

Since $\mathbf{q_1}$ and $\mathbf{q_2}$ are quadratic, one can define $\varphi_1 = \varphi_2 := \mathbf{0}$. Also, we define $\Psi : \mathscr{L}^2 \to [0,\infty)$ by $\Psi(x,y) := (h(x))^2 + (h(y))^2$ for all $x,y \in \mathscr{L}$. It is easy to see that $2 - \nu < 0$ and hence

$$\lim_{n\to\infty} 16^n \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n\to\infty} 4^{n(2-\nu)}\Psi(x,y) = 0.$$

Hence, the result follows from Theorem 3.5. □

A result naturally related to Theorem 3.1 is the following stability theorem.

Theorem 3.8. *Suppose that $\varphi, \varphi_1, \varphi_2 : \mathscr{L}^3 \to [0,\infty)$ are functions and $f, q_1, q_2 : \mathscr{L} \to \mathscr{L}$ are mapping satisfying $f(0) = q_1(0) = q_2(0) = 0$ and*

$$\|\blacktriangle(f,\lambda)(x,y,z)\| \le \varphi(x,y,z),$$
$$\|\Delta q_1(x,y,z)\| \le \varphi_1(x,y,z),$$
$$\|\Delta q_2(x,y,z)\| \le \varphi_2(x,y,z)$$

for all $x,y,z \in \mathscr{L}$ and all $\lambda \in \mathbf{T}^1$. Suppose that f, q_1 and q_2 also satisfy

$$\big\|[f(x), f(y)] - [q_1(x), f(y)] - [f(x), q_2(y)]\big\| \le \Psi(x,y) \quad \text{for all } x,y \in \mathscr{L},$$

for some control function $\Psi : \mathscr{L}^2 \to [0,\infty)$ having the property

$$\lim_{n\to\infty} \frac{1}{16^n} \Psi(2^n x, 2^n y) = 0 \quad \text{for all } x,y \in \mathscr{L}.$$

If $f \in \mathscr{R}(\mathscr{L})$ and $\phi, \phi_1, \phi_2 \in \mathfrak{C}^2(\mathscr{L})$, then there exist uniquely three quadratic mappings $\delta, \mathbf{q}_1, \mathbf{q}_2 : \mathscr{L} \to \mathscr{L}$ such that δ is a $(\mathbf{q}_1, \mathbf{q}_2)$-quadratic Lie hom-der and

$$\|\delta(x) - f(x)\| \le \mathscr{G}(\varphi,2)(x,0,0),$$
$$\|\mathbf{q}_1(x) - q_1(x)\| \le \mathscr{G}(\varphi_1,2)(x,0,0),$$
$$\|\mathbf{q}_2(x) - q_2(x)\| \le \mathscr{G}(\varphi_2,2)(x,0,0)$$

for all $x \in \mathscr{L}$.

Proof. The proof is quite similar to the proof of Theorem 3.5. □

By using the similar methods proposed in the proofs of Corollaries 3.6 and 3.7, we obtain the following results as a direct consequence of Theorem 3.8

Corollary 3.9. *Suppose that $\varphi : \mathscr{L}^3 \to [0,\infty)$ and $\psi : \mathscr{L}^2 \to [0,\infty)$ are functions, and there are real numbers $L, L' > 1$ such that*

$$\varphi(2x,2y,2z) \le 4L\varphi(x,y,z) \quad \text{for all } x,y,z \in \mathscr{L},$$
$$\psi(2x,2y) \le 16L'\psi(x,y) \quad \text{for all } x,y \in \mathscr{L}.$$

Suppose that $f : \mathscr{L} \to \mathscr{L}$ is a mapping such that $f \in \mathscr{R}(\mathscr{L})$, $f(0) = 0$, and satisfies the inequality

$$\|\blacktriangle(f,\lambda)(x,y,z)\| \le \varphi(x,y,z) \quad \text{for all } x,y,z \in \mathscr{L} \text{ and all } \lambda \in \mathbf{T}^1.$$

If there exist two quadratic mappings $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ such that

$$\big\|[f(x), f(y)] - [\mathbf{q_1}(x), f(y)] - [f(x), \mathbf{q_2}(y)]\big\| \le \psi(x,y) \quad \text{for all } x,y \in \mathscr{L},$$

then there exists a unique $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der $\delta : \mathscr{L} \to \mathscr{L}$ such that

$$\|\delta(x) - f(x)\| \le \frac{L}{1-L}\varphi(x,0,0) \quad \text{for all } x \in \mathscr{L}.$$

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Corollary 3.10. *Suppose that $h_1, h_2, h_3 : \mathscr{L} \to [0,\infty)$ are functions and $f : \mathscr{L} \to \mathscr{L}$ is a mapping of class $\mathscr{R}(\mathscr{L})$ and satisfies the inequality*

$$\|\blacktriangle(f,\lambda)(x,y,z)\| \leq h_1(x)+h_1(y)+h_1(z) \quad \text{for all } x,y,z \in \mathscr{L} \text{ and all } \lambda \in \mathbf{T}^1.$$

and there exist two quadratic mappings $\mathbf{q_1}, \mathbf{q_2} : \mathscr{L} \to \mathscr{L}$ such that

$$\big\|[f(x), f(y)] - [\mathbf{q_1}(x), f(y)] - [f(x), \mathbf{q_2}(y)]\big\| \leq h_2(x)+h_3(y) \quad \text{for all } x,y \in \mathscr{L}.$$

If there exists a positive real number $\nu < 2$ such that $h_i(rx) = |r|^\nu h_i(x)$ for all $r \in \mathbb{R}$ (i=1,2,3), then there exists a unique $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der $\delta : \mathscr{L} \to \mathscr{L}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{2^\nu}{4-2^\nu} h_1(x) \quad \text{for all } x \in \mathscr{L}.$$

To end the this paper, we finally present the following hyperstability result.

Proposition 3.11. *Suppose that all the settings of either Corollary 3.6 or Corollary 3.9 are satisfied. If a mapping $f : \mathscr{L} \to \mathscr{L}$ satisfies $f(2x) = 4f(x)$ for all $x \in \mathscr{L}$, then the existed $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der δ and the mapping f are indentical. In particular, f is itself a $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der.*

Proof. It can be seen that

$$f(2^n x) = 4^n f(x) \quad \text{and} \quad f\left(\frac{x}{2^n}\right) = \frac{1}{4^n} f(x) \quad \text{for all } x \in \mathscr{L}.$$

Assume that all the hypotheses of Corollary 3.6 hold. Then the mapping $\delta : \mathscr{L} \to \mathscr{L}$, uniquely determined by

$$\delta(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \quad \text{for all } \in \mathscr{L},$$

is a $(\mathbf{q_1}, \mathbf{q_2})$-quadratic Lie hom-der. We easily see that

$$\delta(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 4^n \cdot \frac{1}{4^n} f(x) = f(x). \quad \text{for all } \in \mathscr{L}.$$

In the case that all the settings of Corollary 3.9 hold, we can similary prove the result. □

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