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DISTANCES IN GRAPHS OF PERMUTATIONS

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ABSTRACT. We study the distance between permutations in three different settings which are related to DNA and quantum entanglements. We construct a graphs where the vertices correspond to permutations of enhanced permutations and edges are defined by adjacent permutations to define distances. Numerous bounds and a recursion formula are given for these distances. These distances are then related to distances in the Braid group.

1. Introduction

In this paper, we shall relate the distance between permutations using canonical definitions of distance in terms of Japanese ladders and generalized Japanese ladders. Namely, we construct various graphs from the final position of permutations and use the distance in the graphs to study distances between permutations. Given the natural connection between permutations, enhanced permutations, and the Braid group, we produce a distance between elements of this group based on the previously defined distance. Permutations also have numerous applications in mathematical biology, see [10] for various examples of this connection. For specific examples about the connection to DNA see [9] and [2].

We begin with the necessary definitions concerning permutations. A permutation of a set is a bijection from a set to itself. In this paper, we always assume that the set is $\{1,2,\dots,n\}$. That is, we are only concerned with permutations of finite sets. It is well known that there are $n!$ distinct permutations on this set which form a group under functional composition and we denote this group $\mathscr{S}_n$. Moreover, it is also well known that any finite group is necessarily a subgroup of this group for some n.

In general, permutations are written in the standard orbit notation where $(a_1,a_2,\dots,a_s)$ indicates, a_i is mapped to a_{i+1} for $1 \leq s-1$ and a_s is mapped to a_1. In addition to this standard notation, we shall also use the notation of a final state of a permutation. Namely, for a given permutation σ we write $[a_1,a_2,\dots,a_n]$ as the final state where $\sigma(a_i)=i$. For example, the permutation $(1,2,3)(4,6)$ has

$$1 \to 2, 2 \to 3, 3 \to 1, 4 \to 6, 5 \to 5, 6 \to 4.$$

This permutation has final state $[3,1,2,6,5,4]$ since $\sigma(3)=1, \sigma(1)=2, \sigma(2)=3, \sigma(6)=4, \sigma(5)=5$, and $\sigma(4)=6$. It will become apparent later why this is known as the final state.

Recall that a transposition is a permutation that interchanges two elements and fixes the remaining elements, for example the transposition (a,b) where $a \neq b$ interchanges a and b and leaves the remaining elements fixed. It is well known that every permutation can be written as a product of transpositions and that every permutation can be written as either evenly many transpositions or oddly many transpositions. These permutations are defined to be even or odd respectively.

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We shall study distances between permutations in a corresponding graph. We now give some standard definitions from graph theory.

Definition 1. *A graph* (V, E) *is a set of vertices* V *and a set of edges* E*, where an edge is of the form* $\{a, b\}$ *where* $a \neq b$.

By graph we mean a simple graph, meaning there are no multiple loops (that is, E is a set and not a multi-set) and there are no loops (an edge that connects a vertex to itself). A path in a graph (V, E) is a set $v_1, v_2, \ldots, v_{k+1}$ where $\{v_i, v_{i+1}\} \in E$. This path is said to have length k. The distance between two vertices in a graph is the length of the shortest path between two vertices. The degree of a vertex is the number of edges on that vertex. A graph is said to be regular if the degree of every vertex is the same. The eccentricity of a vertex is the length of the largest path with that vertex as the initial point in the graph. The diameter of a graph is the largest eccentricity of any vertex in the graph.

2. Permutations

Japanese ladders are a traditional technique used to construct a bijective map from a set to itself. They have been used to describe interesting mathematics; for example, they were related to Markov chains in [12] and in [7] and in [8] they were used to describe interesting mathematical games. Their connection to the braid group and quantum mechanics was described in [1]. The connection between permutations and DNA is described in [4].

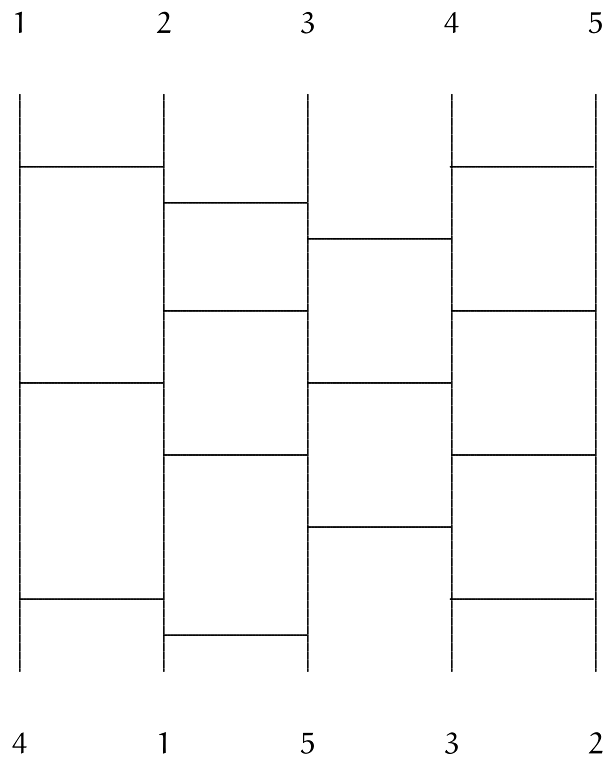
We shall use the standard notation to denote permutations and all permutations here will be read right to left. We begin with the definition of a Japanese ladder.

Definition 2. *A Japanese ladder is a representation of a permutation by*

$$\prod_{i \in A} (i, i+1),$$

where A *is an ordered list of elements of* $\{1, 2, \ldots, n\}$.

Notice that all of the transpositions in this product are of adjacent elements. In other words, a Japanese ladder is writing a permutation in terms of transpositions of adjacent elements. Visually each transposition in a Japanese ladder corresponds to a rung in its physical description. Consider the following representation of a Japanese ladder.



Pictorially, we see the permutation as:

$$1 \to 2$$
$$2 \to 5$$
$$3 \to 4$$
$$4 \to 1$$
$$5 \to 3$$

The rungs of this ladder can be read as

$$(2,3)(4,5)(1,2)(3,4)(4,5)(2,3)(1,2)(3,4)(4,5)(2,3)(3,4)(2,3)(4,5)(1,2).$$

In cycle form this permutation is $(1,2,5,3,4)$. The final state of this permutation is $[4,1,5,3,2]$. It is clear why the final state is used for this permutation since that is way the Japanese ladder ends.

It is easy to see that any permutation has a description as a Japanese ladder. Any transposition (a,b) can be written as a Japanese ladder as follows:

$$(a,b) = (a,a+1)(a+1,a+2)\cdots(b-1,b)(b-2)(b-1)\cdots(a+1,a+2)(a,a+1).$$

Then since every permutation can be written as a product of transpositions, and each transposition can be written as a product of Japanese ladders, then each permutation has a representation as a Japanese ladder. This representation is in no way a minimal representation in terms of number of rungs. In general, it is far from the minimal number of rungs needed.

We can now describe a graph on the set of transpositions, in which the path distance will relate to the minimal number of rungs for a given permutation representation as a Japanese ladder.

Definition 3. *Let Γ_n be the graph where the vertices are the possible arrangements of $\{1,2,3,\dots,n\}$ and two vertices are connected if one can be obtained from the other by permuting two adjacent coordinates.*

In other words, the set of vertices is the set of all final states possible using the set $\{1,2,\dots,n\}$ where two vertices are connected by an edge if one final state can be obtained from the other by adding a single rung to the Japanese ladder. For instance, the final state $[2,3,4,5,1]$ would be connected to the final states $[3,2,4,5,1]$, $[2,4,3,5,1]$, $[2,3,5,4,1]$, $[2,3,4,1,5]$ since these are the 4 final states obtained by switching the positions 1 and 2, 2 and 3, 3 and 4, and 4 and 5 respectively.

Theorem 1. *The graph Γ_n is a regular graph on $n!$ vertices and each vertex has degree $n-1$. The number of edges is $\frac{n!(n-1)}{2}$.*

Proof. It is well known that there are $n!$ permutations of a set of size n which gives the number of vertices. There are $n-1$ adjacent transpositions that are possible to act on a given ordering, namely $(1,2),(2,3),\dots,((n-1),n)$. Therefore, each vertex has degree $n-1$. Since it is regular, we apply the formula $2|E| = |V|d$, where d is the degree of each vertex to get the number of edges. □

It is easy to see that diameter of the graph Γ_n is equal to the eccentricity of any vertex in the graph. This is because any permutation beginning with $[1,2,\dots,n]$ can be written as a permutation of any initial ordering by simply renaming the n elements.

Given a final state $[a_1,a_2,\dots,a_n]$ of a permutation, an inversion is a pair a_i,a_j where $i<j$ and $a_i > a_j$. For example, the final state $[5,3,4,1,2]$ contains 8 inversions. Namely, the inversions are

$$(5,4),(5,3),(5,2),(5,1),(3,1),(3,2),(4,1),(4,2).$$

It is well known that the distance to $[1,2,\dots,n]$ is the number of inversions, see [7] for example.

Lemma 1. *The diameter of the graph Γ_n is $\frac{(n)(n-1)}{2}$.*

Proof. The maximum number of inversions for a final state is $\frac{(n)(n-1)}{2}$. The final state $[n,n-1,n-2,n-3,\dots,2,1]$ contains this many inversions. This gives the result. □

This leads to the following theorem.

Theorem 2. *In Γ_n the number of vertices at distance i from a given vertex is equal to the number of vertices at distance $\frac{(n)(n-1)}{2} - i$.*

Proof. We shall consider the distance from any final state to $[1,2,\dots,n]$. For a given permutation σ let R_σ be the number of inversions. A reverse inversion is a pair a_i,a_j where $i>j$ and $a_i > a_j$. Let L_σ be the number of reverse inversions. It is immediate that $L_\sigma + R_\sigma = \frac{(n-1)(n-2)}{2}$ which is the total number of possible inversions. Therefore, for each final state distance i from $[1,2,\dots,n]$ there is a final state distance $\frac{(n)(n-1)}{2} - i$ from $[1,2,\dots,n]$. This gives the result. □

Let g_i^n be the number of vertices of distance i from a vertex in Γ_n.

We have the following easy lemma.

Lemma 2. *For all* n*, we have* $g_1^n = n-1$.

Proof. There are $n-1$ adjacent transpositions possible for Γ_n. They are the transpositions

$$(1,2),(2,3),(3,4),\dots,(n-1,n).$$

Therefore, there are $n-1$ permutations distance 1 from any given permutation. □

Lemma 3. *For all* n*, we have* $g_2^n = \frac{(n-1)(n-2)}{2} + (n-2)$.

Proof. There are $n-1$ adjacent transpositions. The number of ways of choosing two distinct adjacent transpositions is $C(n-1,2) = \frac{(n-1)(n-2)}{2}$. If two adjacent transpositions commute then we only want to count them once since $(a,b)(c,d) = (c,d)(a,b)$ if a,b,c,d are distinct. However, if they do not commute then $(a,b)(b,c) = (b,c,a)$ and $(b,c)(a,b) = (a,c,b)$. Of these $(n-2)$ have an element in common (and hence do not commute). Therefore, the number is

$$\frac{(n-1)(n-2)}{2} - (n-2) + 2(n-2) = \frac{(n-1)(n-2)}{2} + (n-2).$$

□

Example 1. *For* $n=2$, $\frac{(n-1)(n-2)}{2} + (n-2) = 0$, *for* $n=3$, $\frac{(n-1)(n-2)}{2} + (n-2) = 2$, *for* $n=4$, $\frac{(n-1)(n-2)}{2} + (n-2) = 5$, *for* $n=5$, $\frac{(n-1)(n-2)}{2} + (n-2) = 9$, *and for* $n=6$, $\frac{(n-1)(n-2)}{2} + (n-2) = 14$.

We are now able to give a recursive formula for the number of vertices distance i from any vertex in Γ_n. Using this recursion we can determine g_i for all i for a given n.

Theorem 3. *We have the following recursion:*

$$g_i^{n+1} = \sum_{j=0}^{n} g_{i-j}^n, \tag{1}$$

where $g_i^n = 0$ *if* $i < 0$ *or* $i > n$.

Proof. If a final state $[a_1, a_2, \dots, a_n]$ on n elements has i inversions, then placing $n+1$ at the end, namely the final state $[a_1, a_2, \dots, a_n, n+1]$ gives i inversions. Placing $n+1$ before the j-the position to give

$$[a_1, a_2, \dots, a_{n-j}, n+1, a_{n-j+1}, \dots, a_n]$$

gives $i+j$ inversions. This gives the result. □

This theorem allows us to compute all g_i.

Example 2. *As an example of the theorem, we have*

$$g_6^{12} = g_6^{11} + g_5^{11} + g_4^{11} + g_3^{11} + g_2^{11} + g_1^{11} + g_0^{11}$$

and

$$g_9^5 = g_9^4 + g_8^4 + g_7^4 + g_6^4 + g_5^4.$$

We can use this theorem to determine g_i^n for some small values of n.

i	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
0	1	1	1	1	1	1
1		1	2	3	4	5
2			2	5	9	14
3			1	6	15	29
4				5	20	49
5				3	22	71
6				1	20	90
7					15	101
8					9	101
9					4	90
10					1	71
11						49
12						29
13						14
14						5
15						1

Notice the symmetry from Theorem 2 and that the sum of every column is $n!$.

2.1. ***Braids.*** In this subsection, we shall give an important application of the results presented here concerning distance of permutations. We begin by recalling some basic definitions of the Braid group. This group was first explicitly defined in [3] by Emil Artin. Complete descriptions of this group can be found in [6] and [11].

The braid group, denoted by $\mathscr{B}_n$, is generated by the elements $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ along with with the following defining relations:

(B1): $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| \geq 2$;

(B2): $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-2$.

The first relation is known as far-commutativity and the second relation is called the braid relation. Additionally, there is also the trivial relation, $\sigma_i(\sigma_i)^{-1} = e = (\sigma_i)^{-1}\sigma_i$, where e is n straight strands without any crossings (that is, the identity). We may represent the braid σ_i as the i^{th} strand crossing over the $(i+1)^{\text{st}}$. In general, this group is non-abelian and infinite.

By enumerating each strand, one can associate permutations with braids. This can be accomplished by mapping each generator, σ_i to the transposition $(i, i+1)$. Notice that this is an adjacent permutation.

The kernel of the homomorphism that maps $\mathscr{B}_n$ to $\mathscr{S}_n$ that is given by $\sigma_i \to (i, i+1)$ is $\mathscr{P}_n$, the pure braid group, where $\mathscr{P}_n$ consists of the braids in which each strand starts and ends in the same position.

In general, we have the following short exact sequence:

$$1 \longrightarrow \mathscr{P}_n \longrightarrow \mathscr{B}_n \longrightarrow \mathscr{S}_n \longrightarrow 1.$$

Define the map $\Phi_n : \mathscr{B}_n \to \mathscr{S}_n$ to be the map given above in the short exact sequence. It is quite natural to define the final position of the braid in the same way that the final position was defined for permutations, which makes it easy to determine the corresponding permutation to the element of the braid group.

Definition 4. *Let α, β be two elements of $\mathscr{B}_n$. Then $d_p(\alpha, \beta)$ is defined as the distance in the graph Γ_n between $\Phi_n(\alpha)$ and $\Phi_n(\beta)$.*

This definition means that between two elements in the braid group α and β, the minimal number of elements needed to transform α into an element of β's equivalence class in $\mathscr{B}_n/\mathscr{P}_n$ is their distance. Given this, we can define the following distance as well.

Definition 5. *Let A, B be two elements of $\mathscr{B}_n/\mathscr{P}_n$, then $D_p(A, B)$ is the distance in the graph Γ_n between $\Phi_m(\alpha)$ and $\Phi_n(\beta)$ where $\alpha \in A$ and $\beta \in B$.*

We can combine the results obtained earlier to give the following theorem.

Theorem 4. *Let α, β be two elements in $\mathscr{B}_n$, and $A, B \in \mathscr{B}_n/\mathscr{P}_n$. Then we have:*

- $d_p(\alpha, \beta) \leq \frac{n(n-1)}{2}$, $D_p(A, B) \leq \frac{n(n-1)}{2}$.
- *The number of cosets in $\mathscr{B}_n/\mathscr{P}_n$ distance i from a given coset is equal to the number of cosets distance $\frac{n(n-1)}{2} - i$ from that coset.*
- *The number of cosets in $\mathscr{B}_n/\mathscr{P}_n$ distance 1 is $n-1$ and distance 2 is $\frac{(n-1)(n-2)}{2} + (n-2)$.*
- *If G_i is the number of cosets in $\mathscr{B}_n/\mathscr{P}_n$ distance i from a given coset, then*

$$G_i^{n+1} = \sum_{j=0}^{n} G_{i-j}^{n}, \tag{2}$$

where $G_i^n = 0$ if $i < 0$ or $i > n$.

Proof. The proof follows from Lemma 1, Theorem 2, Lemma 2, Lemma 3, and Theorem 3. □

3. Circular Permutations

In this section, we consider circular permutations. They are called circular because we add the transposition $(n, 1)$ to the list of adjacent permutations. In other words, we allow for the Japanese ladder to wrap around a cylinder. These permutations are related to the evolutionary distance in certain bacteria as their DNA is circular. See [5] for a description of this relation. One might think of this as the numbers $1, 2, 3, \ldots, n$ arranged in a circle and trying to get to another arrangement by switching adjacent elements in the circle. Essentially, regular permutations are arranging people at a lunch counter, hence the final position $[a_1, a_2, \ldots, a_n]$, whereas circular permutations arrange people at a circular table. The addition of this single transposition makes an enormous difference in terms finding minimal paths from one permutation to another. It often significantly reduces the length of the minimal path. We begin with the definition of the corresponding graph.

Definition 6. *Let Δ_n be the graph where the vertices are the the possible arrangements of $\{1, 2, 3, \ldots, n\}$ and two vertices are connected if one can be obtained from the other by permuting two adjacent coordinates or permuting 1 and n.*

The following is immediate.

Proposition 1. *The graph Γ_n is a subgraph of the graph Δ_n.*

Proof. The vertices of Γ_n and Δ_n are the same, that is they correspond to the permutations. However, there are more edges in Δ_n, but any edge in Γ_n is still an edge in Δ_n. This gives the result. □

We next determine the degree and the number of edges in Γ_n.

Theorem 5. *For $n > 2$, the graph Δ_n is a regular graph on $n!$ vertices and each vertex has degree n. The number of edges is $\frac{n!(n)}{2}$.*

Proof. It is well known that there are $n!$ permutations of a set of size n which gives the number of vertices. There are n circular adjacent transpositions that are possible to act on a given ordering, namely $(1,2),(2,3),\ldots((n-1),n),(n,1)$. Therefore, each vertex has degree n. Since it is a regular graph, we apply the formula $2|E| = |V|d$, where d is the degree of each vertex to get the number of edges. □

For $n = 2$, the graph Δ_2 has $2! = 2$ vertices, but only one edge since $(1,2) = (2,1)$. In this case, $\Delta_n = \Gamma_n$.

Theorem 6. *Given a final state $[a_1, a_2, \ldots, a_n]$, let D_g be the distance to the identity in Γ_n and let D_d be the distance to the identity in Δ_n. Then $D_d \leq D_g$.*

Proof. Any path in Γ_n is still a path in Δ_n. This gives the result. □

In general, we cannot make this bound a strict inequality. For example, the final state $[1,2,4,3,5,6]$ has distance 1 from the identity in both graphs. More interestingly, $[2,5,1,3,4]$ has distance 4 from the identity in both graphs.

The following corollary is an immediate consequence of this theorem.

Corollary 1. *The diameter of Δ_n is less than or equal to the diameter of Γ_n.*

Let d_i^n be the number of vertices of distance i from a vertex in Γ_n.

We have the following easy lemma.

Lemma 4. *For all $n > 2$, we have $d_1^n = n$.*

Proof. There are n adjacent transpositions possible for Δ_n. □

When $n = 2$, both $(1,2)$ and $(2,1)$ are the same transposition.

Lemma 5. *For $n \geq 4$, we have*

$$d_2^n = \frac{n(n-3)}{2} + 2n.$$

Proof. We have that there are n adjacent transpositions. Then, there are $n-3$ transpositions that are disjoint from a given transpositions, so there are $\frac{n(n-3)}{2}$ permutations of the form $(a,b)(c,d)$, where a,b,c,d are distinct. Then for each of the n transpositions, there are 2 that are adjacent (meaning they are of the form $(a,b),(b,c)$ where $a \neq c$ (hence they do not commute)). This gives there are $2n$ permutations of this form, that is fix the first one and there are two choices for the second. □

Example 3. *For $n = 4$, $\frac{n(n-3)}{2} + 2n = \frac{4}{2} + 2(4) = 10$ and for $n = 5$, $\frac{n(n-3)}{2} + 2n = \frac{5(2)}{2} + 2(5) = 15$.*

The next theorem determines the diameter of the graph. Note that it is quite different than the diameter of Γ_n.

Theorem 7. *The diameter of* Δ_n *is* $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.

Proof. The maximum that can be reached is when each number in the initial position is as at least as far from their desired position by moving the left and wrapping around as they are from the right. The initial positions of these are

$$\lfloor \frac{n}{2} \rfloor + 1 \ \ \lfloor \frac{n}{2} \rfloor + 2 \ \ \cdots \ \ n \ \ 1 \ \ 2 \ \ 3 \ \ \lfloor \frac{n}{2} \rfloor.$$

The distance to the identity for this is the same in Δ_n as it is in Γ_n. We can simply count the number of inversions to get that its distance to the origin is $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$. □

Example 4. *For* $n = 8$ *the initial position would be*

$$5 \ 6 \ 7 \ 8 \ 1 \ 2 \ 3 \ 4.$$

For $n = 9$ *the initial position would be*

$$5 \ 6 \ 7 \ 8 \ 9 \ 1 \ 2 \ 3 \ 4.$$

As an example, the diameter of Γ_{10} is 45 and the diameter of Δ_{10} is 25 and the diameter of Γ_{10} is 4950 and the diameter of Δ_{10} is 2500. If $\mathrm{diam}(\Gamma_n)$ is the diameter of Γ_n and $\mathrm{diam}(\Delta_n)$ is the diameter of Δ_n then we have

$$\lim_{n\to\infty} \frac{\mathrm{diam}(\Delta_n)}{\mathrm{diam}(\Gamma_n)} = \lim_{n\to\infty} \frac{n^2/4}{n^2/2} = \frac{1}{2}.$$

We can now give the following computation results of the number of permutations of distance i from a given permutation in Δ_n.

i	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	1	1	1	1	1
1		1	3	4	5
2			2	10	15
3				9	32
4					42
5					23
6					2

We note that again the sum of every column is $n!$. However, unlike the table for g_i^n, this table is not symmetric. Moreover, there is not a recursive relation for the values.

4. Enhanced Permutations

While permutations have a natural connection to the braid group. There is another map that has an even more precise connection to the braid group, that is, enhanced permutations. While in a usual adjacent transposition two items switch places, in an enhanced permutation, not only do they switch places, but like in the braid group, we keep track of which goes over the other and which goes under. Since we are noting this geometric characterization, these maps have a natural connection to quantum states as well.

We shall now consider these enhanced permutations and build a third graph based on them.

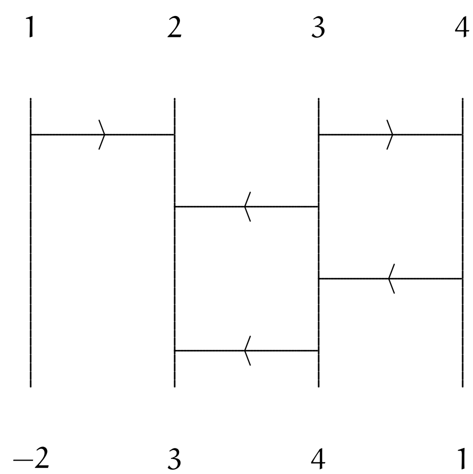
Definition 7. *An enhanced permutation is a permutation where* $\sigma : \{1, 2\dots, n\} \to \{\pm 1, \pm 2, \pm 3, \dots, \pm n\}$ *and the product of the signs must be* $(-1)^{par(\sigma)}$, *where* $par(\sigma) = 1$ *if* σ *is even and* $par(\sigma) = -1$ *if* σ *is odd.*

The fact that the product of the signs is $(-1)^{par(\sigma)}$ is vital to this definition. Since each adjacent permutation switches the sign of a single element, we need the number of negatives to be the same parity as the permutation. That is, the element that goes under the other is multiplied by a -1 in an enhanced adjacent permutation.

We write the final state of an enhanced permutation by $[\pm a_1, \pm a_2, \dots, \pm a_n]$. Each adjacent transposition is written as $\langle i, i+1\rangle$ or $\langle i+1, i\rangle$ where $\langle a, b\rangle$ is the function that is defined as $\tau(a) = b$ and $\tau(b) = -a$. Unless regular transpositions where $(a, b) = (b, a)$ in this case $\langle a, b\rangle \neq \langle b, a\rangle$ which is why we need to introduce new notation.

One can think of this as a Japanese ladder where the rungs have arrows on them and traversing a rung in the direction of the arrow keeps the sign and traversing a rung in the opposite direction of the rung multiplies the element by a -1.

Consider the following enhanced Japanese ladder.



This can be written as a function σ where $\sigma(1) = 4$, $\sigma(2) = -1$, $\sigma(3) = 2$ and $\sigma(4) = 3$. Here we write the final state as $[-2, 3, 4, 1]$. We note that there are oddly many rungs and oddly many negatives in the final state.

Definition 8. *Let* Π_n *be the graph where the vertices are the possible arrangements of* $\{\pm 1, \pm 2, \pm 3, \dots, \pm n\}$ *and the product of the signs must be* $(-1)^{par(\sigma)}$ *and two vertices are connected if one can be obtained from the other by a signed adjacent transposition.*

Theorem 8. *The number of vertices of* Π_n *is* $n!2^{n-1}$.

Proof. There are $\frac{n!}{2}$ even permutations and there are $C(n,0)+C(n,2)+\cdots+C(n,g)$ possible arrangements of negatives where $g=n-1$ if n is odd and $g=n$ if n is even.

There are $\frac{n!}{2}$ odd permutations and there are $C(n,1)+C(n,3)+\cdots+C(n,k)$ possible arrangements of negatives where $k=n-1$ if n is even and $k=n$ if n is odd.

This gives $\frac{n!}{2}(\sum_{i=0}^{n} C(n,i)) = \frac{n!}{2}2^n = n!2^{n-1}$. □

We note that in the graph Π_n the number of vertices is much higher than in the previous two graphs.

Theorem 9. *The graph Π_n is a regular graph with degree $2(n-1)$. The number of edges is*

$$n!2^{n-1}(n-1).$$

Proof. Between rung i and $i+1$ there are two possible enhanced adjacent transpositions namely $\langle i,i+1\rangle$ and $\langle i+1,i\rangle$. Therefore, the degree of each vertex is $2(n-1)$.

Since is is regular we apply the formula $2|E|=|V|d$, where d is the degree of each vertex to get $\frac{n!2^{n-1}2(n-1)}{2} = n!2^{n-1}(n-1)$. □

Let p_i^n be the number of vertices of distance i from a vertex in Π_n.

Lemma 6. *For all n, we have $p_1^n = 2(n-1)$.*

Proof. There are $n-1$ adjacent transpositions possible and 2 possible directions for each of these. Therefore, there are $2(n-1)$ vertices that are distance 1 from any vertex in for Π_n. □

We can now determine the number of vertices that are distance 2 from any given vertex.

Lemma 7. *For all n, we have $p_2^n = (\frac{(n-1)(n-2)}{2}+(n-2))(2^2)+(n-1)$.*

Proof. As before, in Lemma 3, there are $\frac{(n-1)(n-2)}{2}+(n-2)$ ways of writing 2 distinct adjacent transpositions giving different permutations. Each of these has 2^2 ways of placing the arrows. Additionally, there are $n-1$ ways of writing $\langle i,i+1\rangle\langle i,i+1\rangle$ which leave the elements fixed but changes the signs on two of the elements. Note that $\langle i,i+1\rangle\langle i,i+1\rangle = \langle i+1,i\rangle\langle i+1,i\rangle$ and $\langle i,i+1\rangle\langle i+1,i\rangle$ is the identity. □

Example 5. *For $n=2$, $(\frac{(n-1)(n-2)}{2}+(n-2))(2^2)+(n-1)=1$. For $n=3$, $(\frac{(n-1)(n-2)}{2}+(n-2))(2^2)+(n-1)=10$.*

Proposition 2. *The enhanced transposition with final state $[-1,2,3,4,\ldots,n-1,-n]$ has distance $2(n-1)$ from $[1,2,\ldots,n]$.*

Proof. The shortest path is formed by $\langle 1,2\rangle\langle 1,2\rangle\langle 2,3\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 3,4\rangle\ldots\langle n-1,n\rangle\langle n-1,n\rangle$. □

This proposition is interesting since it shows an example of a permutation that would have distance 0 from $[1,2\ldots,n]$ in Γ_n.

Lemma 8. *The diameter of the graph Π_n is* $\max\{\frac{(n)(n-1)}{2}, \frac{(n-2)(n-3)}{2}+2(n-1)\}$.

Proof. Consider a Japanese ladder that is not enhanced. If the adjacent transpositions $(i, i+1)$ occurs for each i, $1 \leq i \leq n-1$, then by changing the directions of the first occurrence of each of these in a given enhanced ladder built on this Japanese ladder, will result in $2(n-1)$ changes in the signs in the final position which is the total amount.

As we saw in Proposition 2, given each of these $n-1$ adjacent transpositions that is not used, 2 enhanced rungs need to be added to get the proper signs. Given any enhanced ladder on $2, 3, \ldots, n-1$, if 1 and n are not a part of any rung, then it may require an additional $2(n-1)$ rungs to change their signs.

Therefore, if no changes to the signs are necessary or every adjacent transposition occurs, then the maximum distance is $\frac{n(n-1)}{2}$. However, if the signs need to be changed and not every adjacent transposition occurs, then it is the maximum of the number of regular permutations on $n-2$ rungs (n rungs excluding 1 and n) which is $\frac{(n-2)(n-3)}{2}$ plus the number which is needed to make the necessary sign changes which is $2(n-1)$.

Since it is not always true that one of these numbers is larger than the other, we have that the maximum distance in this graph is the maximum of $\frac{n(n-1)}{2}$ and $\frac{(n-2)(n-3)}{2} + 2(n-1)$.

□

We give the values of p_i^n for small values of n in the following table.

i	$n=1$	$n=2$	$n=3$
0	1	1	1
1		2	4
2		1	11
3			7
4			1

We note that the sum of each column is $n!2^{n-1}$.

We can now summarize our results for the three graphs in the following table.

Graph	vertices	edges	degree	diameter
Γ	$n!$	$n!(n-1)/2$	$n-1$	$n(n-1)/2$
Δ	$n!$	$n!(n)/2$	n	$\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.
Π	$n!2^{n-1}$	$n!2^{n-1}(n-1)$	$2(n-1)$	$\max\{\frac{(n)(n-1)}{2}, \frac{(n-2)(n-3)}{2} + 2(n-1)\}$

4.1. *Braids and Enhanced Permutations.* We can now show results in terms of the Braid group from our study of enhanced permutations.

Let $\mathscr{E}_n$ be the group of signed permutations that are realizable as signed ladders. That is, this is the group of functions that are described as the vertices of the graph Π_n. Therefore, $|E_n| = n!2^{n-1}$.

Instead of mapping $\mathscr{B}_n$ to $\mathscr{S}_n$, it makes more sense to consider the mapping of $\mathscr{B}_n$ to $\mathscr{E}_n$. See [1] for a complete description of this approach. The kernel, $\mathscr{K}_n$, consists only of those braids whose underlying signed permutation is the identity. This means that not only is the underlying permutation the identity, but all signs are positive. The corresponding short exact sequence is:

$$1 \longrightarrow \mathscr{K}_n \longrightarrow \mathscr{B}_n \longrightarrow \mathscr{E}_n \longrightarrow 1.$$

Define the map $\Psi_n : \mathscr{B}_n \to \mathscr{E}_n$ to be the map given above in the short exact sequence. This leads to the following definition.

Definition 9. *Let* α, β *be two elements of* $\mathscr{B}_n$. *Then* $d_e(\alpha, \beta)$ *is defined as the distance in the graph* Π_n *between* $\Psi_n(\alpha)$ *and* $\Psi_n(\beta)$.

We note that earlier we defined the distance d_p, where the p stood for permutation, whereas here we use e to stand for enhanced permutation.

This definition means that between two elements in the braid group α and β, the minimal number of elements needed to transform α into an element of β's equivalence class in $\mathscr{B}_n/\mathscr{E}_n$ is their distance. As before, we can extend this.

Definition 10. *Let* A, B *be two elements of* $\mathscr{B}_n/\mathscr{E}_n$, *then* $D_e(A, B)$ *is the distance in the graph* Π_n *between* $\Psi_m(\alpha)$ *and* $\Psi_n(\beta)$ *where* $\alpha \in A$ *and* $\beta \in B$.

We can combine the results obtained earlier to give the following theorem.

Theorem 10. *Let* α, β *be two elements in* $\mathscr{B}_n$, *and* $A, B \in \mathscr{B}_n/\mathscr{E}_n$. *Then we have:*

- $d_e(\alpha, \beta) \leq \max\{\frac{(n)(n-1)}{2}, \frac{(n-2)(n-3)}{2} + 2(n-1)\}$
- $D_e(A, B) \leq \max\{\frac{(n)(n-1)}{2}, \frac{(n-2)(n-3)}{2} + 2(n-1)\}$.
- *The number of cosets in* $\mathscr{B}_n/\mathscr{P}_n$ *that are distance* 1 *under* D_e *from a given coset is* $2(n-1)$.
- *The number of cosets in* $\mathscr{B}_n/\mathscr{P}_n$ *that are distance* 2 *under* D_e *from a given coset is* $(\frac{(n-1)(n-2)}{2} + (n-2))(2^2) + (n-1)$.

Proof. The proof follows from Lemma 6, Lemma 7, and Lemma 8. □

References

[1] M. Allocca, S. Dougherty, J. F. Vasquez, *Signed Permutations and the Braid Group*, Rocky Mountain J. Math. **47**, no. 2, 391-402, 2017.

[2] M. Allocca, J. Graham, C. Price, S. Talbott, and J. Vasquez, Word length perturbations in certain symmetric presentations of dihedral groups. Discrete Appl. Math., **221**, 33-45, 2017.

[3] E. Artin: Theorie der Zöpfe, Hamburger Abhandlungen, **4**, 1925.

[4] V. Bafna and P. A. Pevzner, Sorting permutations by transpositions, In Proceedings of the Sixth Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, CA, 1995), 614623. ACM, New York, 1995.

[5] V. Bafna and P. A. Pevzner, Genome rearrangements and sorting by reversals. SIAM J. Comput., **25**, no. 2, 272-289, 1996.

[6] V. G. Bardakov, *The Virtual and Universal Braids*, Fund. Math., **181**, 1-18, 2004.

[7] S. T. Dougherty and J. Franko Vasquez, *Amidakuji and Games*, Journal of Recreational Mathematics, **37**, no. 1, 46-56, 2008.

[8] S. T. Dougherty and J. Franko Vasquez, Ladder Games, MAA Focus, June/July, 2001, page 25.

[9] A. Egri-Nagy, Volker Gebhardt, M. M. Tanaka, and A. R. Francis, Group- theoretic models of the inversion process in bacterial genomes. J. Math. Biol., **69**, no. 1, 243-265, 2014.

[10] G. Fertin, A. Labarre, I. Rusu, É. Tannier, and S. Vialette, Combinatorics of genome rearrangements. Computational Molecular Biology. MIT Press, Cambridge, MA, 2009.

[11] C. Kassel and V. Turaev: Braid Groups, Graduate Texts in Mathematics, **247**, Springer, New York, 2008.

[12] L. Lange and J. W. Miller: *A Random Ladder Game: Permutations, Eigenvalues, and Convergence of Markov Chains*, Mathematics Magazine, **23**, no. 5, 1992.

[13] V. Moulton and M. Steel, The 'butterfly effect' in Cayley graphs with applications to genomics. J. Math. Biol., **65**, no. (6-7), 1267-1284, 2012.

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