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VON NEUMANN-SCHATTEN P-FRAMES FOR OPERATORS

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ABSTRACT. In the present paper, we introduce von Neumann-Schatten K-p-frames and von Neumann-Schatten K^*-atomic systems, where K is a bounded operator on a separable Banach space, and we collect some relationships between these two concepts. Also, von Neumann-Schatten K-p-frames are characterized in terms of a range inclusion property and it is shown that they are stable under different kinds of perturbations and the action of some bounded operators. Moreover, the notions of K-duals and K^*-duals are introduced, characterized and some of their properties are obtained.

1. Introduction

Let $\mathscr{H}$ be a separable Hilbert space. A sequence $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ is a frame if there exist constants $0 < A_{\mathscr{F}} \leq B_{\mathscr{F}} < \infty$ such that

$$A_{\mathscr{F}}\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B_{\mathscr{F}}\|f\|^2,$$

for each $f \in \mathscr{H}$. The sequence $\mathscr{F}$ is called a Bessel sequence for $\mathscr{H}$ if only the second inequality is required.

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [10] in the context of nonharmonic Fourier series. Frames in a Hilbert space are generalizations of orthonormal bases and usually provide non-unique representations for each element of the Hilbert space, i.e., for every frame $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathscr{H}$, there exists at least one frame $\{g_i\}_{i=1}^{\infty}$ such that

$$f = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i, \qquad \forall f \in \mathscr{H}.$$

Many generalizations of frames have been introduced. Sun in [25] introduced the concept of g-frames in Hilbert spaces. G-frames are natural generalizations of frames which cover many other recent generalizations of frames such as bounded quasi-projections [13, 14], fusion frames [5], outer frames and pseudoframes [18]. Aldroubi, Sun and Tang [1] introduced the concept of p-frames for a Banach space $\mathscr{X}$. Their main results were about p-frames for $L^p(\mathbb{R}^d)$. They obtained series expansions in shift-invariant spaces of $L^p(\mathbb{R}^d)$. Christensen and Stoeva [8] further discussed some properties of p-frames for a general Banach space $\mathscr{X}$. In [24], the authors unified p-frames and g-frames to introduce the notion of von Neumann-Schatten p-frames. In fact, the concept of von Neumann-Schatten p-frames generalizes almost all of the concepts related to frames. As an important class

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of von Neumann-Schatten p-frames, Hilbert-Schmidt frames have been considered by some mathematicians [2, 21, 22]. L. Găvruţa introduced in [15] the notion of atomic systems for a bounded operator K defined on a Hilbert space $\mathscr{H}$. Indeed, a Bessel sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ is said to be an atomic system for K if there is some positive number C such that for every $f \in \mathscr{H}$ there exists some $a_f = \{a_i^f\}_{i=1}^{\infty} \in \ell^2(\mathbb{N})$ with $\|a_f\|_{\ell^2} \leq C\|f\|$ and $Kf = \sum_{i=1}^{\infty} a_i^f f_i$. It is shown in [15] that the sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ is an atomic system for K if and only if $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{H}$, i.e., there exist two positive numbers A and B such that

$$A\|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in \mathscr{H}$.

As we see, ordinary frames and also atomic systems for subspaces introduced in [12] are special cases of K-frames. It is also stated in [15] that a necessary and sufficient condition for a Bessel sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ to be a K-frame is the existence of a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that

$$Kf = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i, \qquad \forall f \in \mathscr{H}.$$

Hence, a K-frame allows us to reconstruct every element of $R(K)$, the range of K (which is not necessarily closed). This property can be useful in application, especially when the reconstruction of the elements in the range of a bounded operator on a Hilbert space $\mathscr{H}$ is needed and it is not necessary to reconstruct every element of $\mathscr{H}$.

After the definition of K-frames in Hilbert spaces, many generalizations of them were presented in both Hilbert and Banach spaces, see [9, 16, 19, 27, 28]. This paper addresses the theory of von Neumann-Schatten p-frames. First, motivated by [15] and in order to reconstruct elements from the range of a bounded linear operator K on a separable Banach space, we introduce K-frames and K^*-atomic systems for von Neumann-Schatten p-Bessel sequences. Then we study the relationship between them and obtain some characterizations of von Neumann-Schatten K-p-frames. Because the study of duals is important to obtain nice properties for von Neumann-Schatten K-p-frames, inspired by [20], K-duals and K^*-duals for von Neumann-Schatten p-Bessel sequences are introduced and studied. We obtain some characterizations and properties of them and it is shown that they share many useful properties with their corresponding notions in Hilbert spaces. Moreover, motivated by [4, 6, 7, 17, 21, 26], we obtain some sufficient conditions under which von Neumann-Schatten K-p-frames are stable under small pertubations. It is worthwhile to mention that Hilbert-Schmidt K-frames, as an class of von Neumann-Schatten K-p-frames, are obtained from our results.

2. Preliminaries

In this section, we have a brief review of the definitions and basic properties of von Neumann-Schatten p-Bessel sequences and von Neumann-Schatten p-frames ([23] can be used as a reference.). Throughout this paper, $\mathscr{X}$ is a separable Banach space, $\mathscr{X}^*$ is the dual space of $\mathscr{X}$ and $\mathscr{X}^{**}$ is the dual space of $\mathscr{X}^*$. As we know, $\mathscr{X}$ can be considered as a subspace of $\mathscr{X}^{**}$. We denote the set of all bounded, linear operators from a Banach space $\mathscr{X}$ to a Banach space $\mathscr{Y}$ by $B(\mathscr{X}, \mathscr{Y})$ and $B(\mathscr{X}, \mathscr{X})$ is denoted by $B(\mathscr{X})$. If $T \in B(\mathscr{X}, \mathscr{Y})$, then T^* is the adjoint operator of T and $R(T)$ is the range of T. Also,

$\mathbb{C}$ and $\mathbb{N}$ are the field of all complex numbers and the set of all natural numbers, respectively. p is a number with $1 < p < \infty$ and q is the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\mathscr{H}$ be a separable Hilbert space. For a compact operator $\mathscr{A} \in B(\mathscr{H})$, let $s_1(\mathscr{A}) \geq s_2(\mathscr{A}) \geq \ldots \geq 0$ denote the singular values of $\mathscr{A}$, i.e., the eigenvalues of the positive operator $|\mathscr{A}| = (\mathscr{A}^*\mathscr{A})^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. For $1 \leq p < \infty$, the von Neumann-Schatten p-class $\mathscr{C}_p$ is defined as the set of all compact operators $\mathscr{A}$ for which $\sum_{i=1}^{\infty} s_i^p(\mathscr{A}) < \infty$. For $\mathscr{A} \in \mathscr{C}_p$, the von Neumann-Schatten p-norm of $\mathscr{A}$ is defined by

$$\|\mathscr{A}\|_p = \left(\sum_{i=1}^{\infty} s_i^p(\mathscr{A})\right)^{\frac{1}{p}} = (\mathbf{tr}|\mathscr{A}|^p)^{\frac{1}{p}}, \tag{2.1}$$

where $\mathbf{tr}$ is the trace functional which is defined by $\mathbf{tr}(\mathscr{A}) = \sum_{e\in\mathscr{E}} \langle \mathscr{A}e, e\rangle$, where $\mathscr{E}$ is any orthonormal basis of $\mathscr{H}$. $\mathscr{C}_p$ is a Banach space with respect to this norm. Also, $\mathscr{C}_1$ is the trace class and $\mathscr{C}_2$ is the Hilbert-Schmidt class. Note that an operator $\mathscr{A}$ is in $\mathscr{C}_p$ if and only if $\mathscr{A}^p \in \mathscr{C}_1$. It is proved that $\mathscr{C}_p$ is a two sided $*$-ideal of the C^*-algebra $B(\mathscr{H})$ and the finite rank operators are dense in $\mathscr{C}_p$. Also, if $\mathscr{A} \in \mathscr{C}_p$ and $\mathscr{B} \in \mathscr{C}_q$, then $\mathscr{A}\mathscr{B} \in \mathscr{C}_1$, $\mathbf{tr}(\mathscr{A}\mathscr{B}) = \mathbf{tr}(\mathscr{B}\mathscr{A})$, and $\|\mathscr{A}\mathscr{B}\|_1 \leq \|\mathscr{A}\|_p\|\mathscr{B}\|_q$, whenever $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, if $1 \leq p \leq p' < \infty$, then $\mathscr{C}_p \subseteq \mathscr{C}_{p'}$ and $\|\mathscr{A}\|_p \geq \|\mathscr{A}\|_{p'}$. It is well-known that the space $\mathscr{C}_2$ with respect to the inner product $\langle \mathscr{A}, \mathscr{B}\rangle = \mathbf{tr}(\mathscr{B}^*\mathscr{A})$ is a Hilbert space.

For $1 \leq p < \infty$, the Banach space $\oplus\mathscr{C}_p$ is defined by

$$\oplus\mathscr{C}_p = \left\{ \{\mathscr{A}_i\}_{i=1}^{\infty} : \quad \mathscr{A}_i \in \mathscr{C}_p \text{ and } \|\{\mathscr{A}_i\}_{i=1}^{\infty}\| = \left(\sum_{i=1}^{\infty} \|\mathscr{A}_i\|_p^p\right)^{\frac{1}{p}} < \infty \right\}.$$

In particular, $\oplus\mathscr{C}_2$ is a Hilbert space with the inner product $\langle \{\mathscr{A}_i\}_{i=1}^{\infty}, \{\mathscr{B}_i\}_{i=1}^{\infty}\rangle = \sum_{i=1}^{\infty} \mathbf{tr}(\mathscr{B}_i^*\mathscr{A}_i)$.

Definition 2.1. A countable family $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ of bounded linear operators from a separable Banach space $\mathscr{X}$ to $\mathscr{C}_p$ is a von Neumann-Schatten p-frame for $\mathscr{X}$ with respect to $\mathscr{H}$ if there exist constants $A_{\mathscr{G}}, B_{\mathscr{G}} > 0$ such that

$$A_{\mathscr{G}}\|f\| \leq \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B_{\mathscr{G}}\|f\|, \quad \forall f \in \mathscr{X}.$$

It is called a von Neumann-Schatten p-Bessel sequence with bound $B_{\mathscr{G}}$ if the second inequality is satisfied.

Lemma 2.2. [24] *If $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for $\mathscr{X}$, then $\mathscr{X}$ is reflexive.*

Definition 2.3. Let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$. Then its analysis operator is defined by $U_{\mathscr{G}} : \mathscr{X} \to \oplus\mathscr{C}_p$, $U_{\mathscr{G}}(f) = \{\mathscr{G}_i(f)\}_{i=1}^{\infty}$. Furthermore, $T_{\mathscr{G}} : \oplus\mathscr{C}_q \to \mathscr{X}^*$ defined by $T_{\mathscr{G}}(\{\mathscr{A}_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathscr{A}_i\mathscr{G}_i$ is called the synthesis operator (note that this operator is defined using the existence of an isometric isomorphism from $\mathscr{C}_q$ onto $\mathscr{C}_p^*$. For more information, see [24]).

Theorem 2.4. [24] *A sequence $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ is a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$ with a bound $B_{\mathscr{G}}$ if and only if its synthesis operator $T_{\mathscr{G}}$ is a bounded operator with $\|T_{\mathscr{G}}\| \leq B_{\mathscr{G}}$. In this case, $T_{\mathscr{G}} = U_{\mathscr{G}}^*$ and if $\mathscr{X}$ is reflexive, then $U_{\mathscr{G}} = T_{\mathscr{G}}^*$ and $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for $\mathscr{X}$ if and only if the operator $T_{\mathscr{G}}$ is bounded and onto.*

Let us recall the concept of K-frames for a Hilbert space $\mathscr{H}$.

Definition 2.5. [15] Let $K \in B(\mathscr{H})$. A sequence $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{H}$ if there exist constants $A_{\mathscr{F}}, B_{\mathscr{F}} > 0$ such that

$$A_{\mathscr{F}}\|K^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B_{\mathscr{F}}\|f\|^2.$$

The constants $A_{\mathscr{F}}$ and $B_{\mathscr{F}}$ are called the lower K-frame bound and the upper K-frame bound of $\mathscr{F}$, respectively. $\mathscr{F}$ is called a tight K-frame if the following condition is satisfied:

$$A_{\mathscr{F}}\|K^*f\|^2 = \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2.$$

Theorem 2.6. [15] *A sequence $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{H}$ if and only if $\mathscr{F}$ is a Bessel sequence and $R(K) \subseteq R(T_{\mathscr{F}})$, where $T_{\mathscr{F}}$ is the synthesis operstor of $\mathscr{F}$.*

The notion of duality for K-frames was introduced in [20].

Definition 2.7. [20] Let $\mathscr{F} = \{f_i\}_{i=1}^{\infty} \subseteq \mathscr{H}$ be a Bessel sequence. A Bessel sequence $\{g_i\}_{i=1}^{\infty}$ is called a K-dual of $\mathscr{F}$ if

$$Kf = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i \qquad (f \in \mathscr{H}).$$

The following two theorems will play a key role in the proofs of our main results.

Theorem 2.8. [11] *Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ be Banach spaces and let $D \in B(\mathscr{X}, \mathscr{Y})$ and $E \in B(\mathscr{X}, \mathscr{Z})$. Then the following conditions are equivalent:*

(i) *$D = FE$ for some continuous linear transformation $F : R(E) \to \mathscr{Y}$.*
(ii) *$\|D(x)\| \leq M\|E(x)\|$, for some $M \geq 0$ and each $x \in \mathscr{X}$.*
(iii) *$R(D^*) \subseteq R(E^*)$.*

Theorem 2.9. [3] *Let $T \in B(\mathscr{X}, \mathscr{Y})$. If $S \in B(\mathscr{X}, \mathscr{Z})$ with $R(S^*) \subseteq R(T^*)$ and $\overline{R(T)}$ is complemented, then there exists some $V \in B(\mathscr{Y}, \mathscr{Z})$ such that $S = VT$.*

3. von Neumann-Schatten K-p-frames

In this section, we introduce and study von Neumann-Schatten K-p-frames, von Neumann-Schatten K^*-atomic systems, von Neumann-Schatten K-duals and von Neumann-Schatten K^*-duals.

Definition 3.1. Let $K \in B(\mathscr{X})$. A sequence $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ with respect to $\mathscr{H}$ if there exist constants $A_{\mathscr{G}}, B_{\mathscr{G}} > 0$ such that

$$A_{\mathscr{G}}\|Kf\| \leq \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B_{\mathscr{G}}\|f\|, \quad \forall f \in \mathscr{X}.$$

The sequence $\mathscr{G}$ is called a tight von Neumann-Schatten K-p-frame if the following condition is satisfied:

$$A_{\mathscr{G}}\|Kf\| = \left(\sum_{i=1}^{\infty}\|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}}.$$

Remark 3.2. In general, a von Neumann-Schatten K-p-frame may not be a von Neumann-Schatten p-frame although it is always a von Neumann-Schatten p-Bessel sequence. For some special operators K such as $K = I_{\mathscr{X}}$ or when K is bounded below, a von Neumann-Schatten K-p-frame is a von Neumann-Schatten p-frame.

Now, some examples of von Neumann-Schatten K-p-frames which are not von Neumann-Schatten p-frames are presented.

Example 3.3. Let $\mathscr{X} = \oplus\mathscr{C}_p$. Define $K : \oplus\mathscr{C}_p \to \oplus\mathscr{C}_p$ by

$$K(\{\mathscr{A}_i\}_{i=1}^{\infty}) = \begin{cases} \mathscr{A}_i & i = 2k, \\ 0 & i = 2k+1. \end{cases}$$

Obviously, $K \in B(\oplus\mathscr{C}_p)$ and $\|K\| \leq 1$.
Now, for each $i \in \mathbb{N}$, define

$$\mathscr{G}_i : \oplus\mathscr{C}_p \to \mathscr{C}_p, \quad \mathscr{G}_i(\mathscr{A}) = P_iK(\mathscr{A}),$$

where P_i is the coordinate operator on $\oplus\mathscr{C}_p$. Clearly, $\{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\oplus\mathscr{C}_p, \mathscr{C}_p)$ and for each $\mathscr{A} = \{\mathscr{A}_i\}_{i=1}^{\infty} \in \oplus\mathscr{C}_p$, we have

$$\left(\sum_{i=1}^{\infty}\|\mathscr{G}_i(\mathscr{A})\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}\|P_iK(\mathscr{A})\|_p^p\right)^{\frac{1}{p}} = \|K(\mathscr{A})\|.$$

Therefore, $\{\mathscr{G}_i\}_{i=1}^{\infty}$ is a tight von Neumann-Schatten K-p-frame for $\mathscr{X}$. But $\{\mathscr{G}_i\}_{i=1}^{\infty}$ is not a von Neumann-Schatten p-frame for $\mathscr{X}$. To see this, suppose that $\{e_i\}_{i=1}^{\infty}$ is an orthonrmal basis for $\mathscr{H}$. Consider $A_1 = e_1 \otimes e_1$, and

$$\{\mathscr{A}_i\}_{i=1}^{\infty} = \begin{cases} \mathscr{A}_1 & i = 1, \\ 0 & i \neq 1. \end{cases}$$

Clearly, $\{\mathscr{A}_i\}_{i=1}^{\infty} \in \oplus\mathscr{C}_p$ and $\|\{\mathscr{A}_i\}_{i=1}^{\infty}\| = \|\mathscr{A}_1\|_p = 1$. But $\sum_{i=1}^{\infty}\|\mathscr{G}_i(\{\mathscr{A}_i\}_{i=1}^{\infty})\|_p^p = 0$.

Example 3.4. Let $\mathscr{X} = \mathscr{H}$ be a separable Hilbert space and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Fix $M \in \mathbb{N}$ and define $K \in B(\mathscr{H})$ by

$$K(e_j) = \begin{cases} 0, & 1 \leq j \leq M, \\ e_j, & j > M. \end{cases}$$

It is easy to see that for each $x \in \mathscr{H}$, $\|Kx\| \leq \|x\|$. Now, for each $i \in \mathbb{N}$, define

$$\mathscr{G}_i : \mathscr{H} \to \mathscr{C}_p \qquad \mathscr{G}_i(f) = \frac{1}{2^i}K(f) \otimes e_1.$$

Obviously, $\{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{H},\mathscr{C}_p)$. For each $f \in \mathscr{H}$, we have

$$\begin{aligned}\Big(\sum_{i=1}^{\infty}\|\mathscr{G}_i(f)\|_p^p\Big)^{\frac{1}{p}} &= \Big(\sum_{i=1}^{\infty}\frac{1}{2^{pi}}\|Kf\|^p\|e_1\|^p\Big)^{\frac{1}{p}}\\ &= \|Kf\|\Big(\sum_{i=1}^{\infty}\frac{1}{2^{pi}}\Big)^{\frac{1}{p}}\\ &= \frac{\|Kf\|}{(2^p-1)^{\frac{1}{p}}}.\end{aligned}$$

Therefore, $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{H}$. If we take $f = e_1$, then $\|f\| = 1$ but $\sum_{i=1}^{\infty}\|\mathscr{G}_i(f)\|_p^p = 0$. Thus $\mathscr{G}$ is not a von Neumann-Schatten p-frame for $\mathscr{H}$.

The next proposition characterizes von Neumann-Schatten K-p-frames in terms of a range inclusion property. This characterization is analogous to the one presented for Hilbert space K-frames in [15, Theorem 4].

Proposition 3.5. *Let $K \in B(\mathscr{X})$. Then a sequence $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ if and only if $\mathscr{G}$ is a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$ and $R(K^*) \subseteq R(T_{\mathscr{G}})$, where $T_{\mathscr{G}}$ is the synthesis operator of $\mathscr{G}$.*

Proof. A sequence $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ with bounds $A_{\mathscr{G}}$ and $B_{\mathscr{G}}$ if and only if

$$A_{\mathscr{G}}\|Kf\| \leq \left(\sum_{i=1}^{\infty}\|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B_{\mathscr{G}}\|f\|, \quad \forall f \in \mathscr{X},$$

which is equivalent to say that

$$A_{\mathscr{G}}\|Kf\| \leq \|U_{\mathscr{G}}(f)\| \leq B_{\mathscr{G}}\|f\|, \quad \forall f \in \mathscr{X},$$

where $U_{\mathscr{G}}$ is the analysis operator of $\mathscr{G}$. Now, the result follows from Theorem 2.4 and Theorem 2.8. □

Corollary 3.6. *Let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$ and $T \in B(\mathscr{X})$ with $R(T^*) \subseteq R(K^*)$. Then $\mathscr{G}$ is a von Neumann-Schatten T-p-frame for $\mathscr{X}$.*

Definition 3.7. Let $K \in B(\mathscr{X})$. A countable family $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X},\mathscr{C}_p)$ is a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$ if the following conditions hold:

(i) the series $\sum_{i=1}^{\infty}\mathscr{A}_i\mathscr{G}_i$ converges for each $\{\mathscr{A}_i\}_{i=1}^{\infty} \in \oplus\mathscr{C}_q$;

(ii) there exists some positive number C such that for each $x^* \in \mathscr{X}^*$, there is a sequence $\mathscr{A}_{x^*} = \{\mathscr{A}_i^{x^*}\}_{i=1}^{\infty} \in \oplus\mathscr{C}_q$ with $\|\mathscr{A}_{x^*}\| \leq C\|x^*\|$ and $K^*x^* = \sum_{i=1}^{\infty}\mathscr{A}_i^{x^*}\mathscr{G}_i$.

In the rest of this section, some relationships between von Neumann-Schatten K-p-frames and von Neumann-Schatten K^*-atomic systems are collected. We first show that every von Neumann-Schatten K^*-atomic system is a von Neumann-Schatten K-p-frame. In fact, the next theorem generalizes the result obtained in the implication (i) $\Rightarrow$ (ii) of [15, Theorem 3].

Theorem 3.8. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$. Then $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$.*

Proof. Let $x^* \in \mathscr{X}^*$. Since $\mathscr{G}$ is a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$, there exist a sequence $\{\mathscr{A}_i^{x^*}\}_{i=1}^{\infty} \in \oplus\mathscr{C}_q$ and a constant $C > 0$ such that $\|\{\mathscr{A}_i^{x^*}\}_{i=1}^{\infty}\| \leq C\|x^*\|$ and $K^*x^* = \sum_{i=1}^{\infty} \mathscr{A}_i^{x^*}\mathscr{G}_i$. Now, for each $x \in \mathscr{X}$, we have

$$\begin{aligned}
\|Kx\| &= \sup_{x^*\in\mathscr{X}^*, \|x^*\|\leq 1} \left|x^*(Kx)\right| = \sup_{x^*\in\mathscr{X}^*, \|x^*\|\leq 1} \left|K^*x^*(x)\right| \\
&= \sup_{x^*\in\mathscr{X}^*, \|x^*\|\leq 1} \left|\sum_{i=1}^{\infty} \mathscr{A}_i^{x^*}\mathscr{G}_i(x)\right| \\
&\leq \sup_{x^*\in\mathscr{X}^*, \|x^*\|\leq 1} \left(\sum_{i=1}^{\infty} \|\mathscr{A}_i^{x^*}\|_q^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(x)\|_p^p\right)^{\frac{1}{p}} \\
&\leq \sup_{x^*\in\mathscr{X}^*, \|x^*\|\leq 1} C\|x^*\| \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(x)\|_p^p\right)^{\frac{1}{p}} = C\left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(x)\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, consider the operator

$$\begin{aligned}
T : \oplus\mathscr{C}_q &\longrightarrow \mathscr{X}^* \\
\{\mathscr{A}_i\}_{i=1}^{\infty} &\mapsto \sum_{i=1}^{\infty} \mathscr{A}_i\mathscr{G}_i.
\end{aligned}$$

By the Banach-Steinhaus theorem, T is a bounded operator. It follows from Theorem 2.4 that $\mathscr{G}$ is also a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$. Therefore, $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$. □

As we know, every closed subspace of a Hilbert space is complemented but this does not necessarily hold for subspaces of a Banach space. The following theorem shows that if $\overline{R(U_{\mathscr{G}})}$ is complemented, where $\mathscr{G}$ is a von Neumann-Schatten K-p-frame, then $\mathscr{G}$ is also a von Neumann-Schatten K^*-atomic system. Indeed, the next result is a more general version of the implication (ii) $\Rightarrow$ (i) in [15, Theorem 3].

Theorem 3.9. *Suppose that $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ and $\overline{R(U_{\mathscr{G}})}$ is complemented. Then $\mathscr{G}$ is a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$.*

Proof. The von Neumann-Schatten K-p-frame $\mathscr{G}$ is a von Neumann-Schatten p-Bessel sequence. It follows from Theorem 2.4 that $T_{\mathscr{G}}$ is a well-defined linear operator. Hence, the first condition of Definition 3.7 is satisfied. Since $\mathscr{G}$ is a von Neumann-Schatten K-p-frame, it follows from Proposition 3.5 that $R(K^*) \subseteq R(T_{\mathscr{G}})$ and because $\overline{R(U_{\mathscr{G}})}$ is complemented, by Theorem 2.9, there exists some $V \in B(\oplus\mathscr{C}_p, \mathscr{X})$ such that $K = VU_{\mathscr{G}}$, so $K^* = T_{\mathscr{G}}V^*$. For each $x^* \in \mathscr{X}^*$, $V^*(x^*) \in \oplus\mathscr{C}_q$ and $\|V^*(x^*)\| \leq \|V^*\|\|x^*\|$, also we get

$$K^*x^* = T_{\mathscr{G}}V^*(x^*) = \sum_{i=1}^{\infty} P_i(V^*(x^*))\mathscr{G}_i.$$

Considering $C := \|V^*\|$ and $\mathscr{A}_i^{x^*} := P_i(V^*(x^*))$, where P_i is the coordinate operator on $\oplus\mathscr{C}_q$, we obtain that $\mathscr{G}$ is a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$. □

Definition 3.10. Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$ and $\mathscr{F} = \{\mathscr{F}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}^*, \mathscr{C}_q)$ be a von Neumann-Schatten q-Bessel sequence for $\mathscr{X}^*$. The sequence $\mathscr{G}$ is called a von Neumann-Schatten K-dual for $\mathscr{F}$ if

$$Kf = \sum_{i=1}^{\infty} \mathscr{G}_i(f)\mathscr{F}_i, \qquad (f \in \mathscr{X}),$$

and $\mathscr{F}$ is called a von Neumann-Schatten K^*-dual for $\mathscr{G}$ if

$$K^*g = \sum_{i=1}^{\infty} \mathscr{F}_i(g)\mathscr{G}_i, \qquad (g \in \mathscr{X}^*).$$

Remark 3.11. With the assumptions of Definition 3.10, it is easy to check that if $\mathscr{X}$ is a reflexive Banach space, then $\mathscr{F}$ is a von Neumann-Schatten K^*-dual for $\mathscr{G}$ if and only if $\mathscr{G}$ is a von Neumann-Schatten K-dual for $\mathscr{F}$.

Example 3.12. Let $\mathscr{X} = \oplus\mathscr{C}_p$. Suppose that $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ and $K \in B(\mathscr{X})$ are the sequence and the linear operator defined in Example 3.3, respectively. As we saw, $\mathscr{G}$ is a von Neumann-Schatten K-p-frame. Now, for each $i \in \mathbb{N}$, suppose that $\mathscr{F}_i$ is the coordinate operator on $\oplus\mathscr{C}_q$ ($\mathscr{F}_i : \oplus\mathscr{C}_q \to \mathscr{C}_q$). Obviously, $\{\mathscr{F}_i\}_{i=1}^{\infty} \subseteq B(\oplus\mathscr{C}_q, \mathscr{C}_q)$. It is easy to see that $\{\mathscr{F}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten q-Bessel sequence for $\oplus\mathscr{C}_q$ and $\sum_{i=1}^{\infty} \mathscr{G}_i(\mathscr{A})\mathscr{F}_i = K\mathscr{A}$, for each $\mathscr{A} \in \oplus\mathscr{C}_p$ and $\sum_{i=1}^{\infty} \mathscr{F}_i(\mathscr{A})\mathscr{G}_i = K^*\mathscr{A}$, for each $\mathscr{A} \in \oplus\mathscr{C}_q$. Hence, $\mathscr{F}$ is a von Neumann-Schatten K^*-dual of $\mathscr{G}$ and $\mathscr{G}$ is a von Neumann-Schatten K-dual of $\mathscr{F}$.

The following proposition says that if a von Neumann-Schatten p-Bessel sequence $\mathscr{G}$ possesses a von Neumann-Schatten K^*-dual, then $\mathscr{G}$ and its K^*-dual are von Neumann-Schatten K-p-frame and von Neumann-Schatten K^*-q-frame, respectively. A special case of this result can be found in the implication (iii) ⇒ (ii) of [15, Theorem 3].

Proposition 3.13. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$ and $\mathscr{F} = \{\mathscr{F}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}^*, \mathscr{C}_q)$ be a von Neumann-Schatten q-Bessel sequence for $\mathscr{X}^*$. If $\mathscr{F}$ is a von Neumann-Schatten K^*-dual of $\mathscr{G}$, then $\mathscr{F}$ and $\mathscr{G}$ are von Neumann-Schatten K^*-q-frame for $\mathscr{X}^*$ and von Neumann-Schatten K-p-frame for $\mathscr{X}$, respectively.*

Proof. Suppose that $B_{\mathscr{G}}$ is an upper bound of $\mathscr{G}$. Then, for each $g \in \mathscr{X}^*$, we have

$$\begin{aligned}
\|K^*g\| &= \left\|\sum_{i=1}^{\infty} \mathscr{F}_i(g)\mathscr{G}_i\right\| \\
&= \sup_{f\in\mathscr{X},\|f\|\le 1} \left|\sum_{i=1}^{\infty} \mathscr{F}_i(g)\mathscr{G}_i(f)\right| \\
&\le \sup_{f\in\mathscr{X},\|f\|\le 1} \left(\sum_{i=1}^{\infty} \|\mathscr{F}_i(g)\|_q^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&\le B_{\mathscr{G}} \left(\sum_{i=1}^{\infty} \|\mathscr{F}_i(g)\|_q^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore $\mathscr{F}$ is a a von Neumann-Schatten K^*-q-frame for $\mathscr{X}^*$.
Suppose that $B_{\mathscr{F}}$ is an upper bound of $\mathscr{F}$. Then, for each $f \in \mathscr{X}$, we have

$$\begin{aligned}
\|Kf\| &= \sup_{x^*\in\mathscr{X}^*,\|x^*\|\le 1} \left|x^*(Kf)\right| \\
&= \sup_{x^*\in\mathscr{X}^*,\|x^*\|\le 1} \left|K^*x^*(f)\right| \\
&= \sup_{x^*\in\mathscr{X}^*,\|x^*\|\le 1} \left|\sum_{i=1}^{\infty} \mathscr{F}_i(x^*)\mathscr{G}_i(f)\right| \\
&\le \sup_{x^*\in\mathscr{X}^*,\|x^*\|\le 1} \left(\sum_{i=1}^{\infty} \|\mathscr{F}_i(x^*)\|_q^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&\le B_{\mathscr{F}} \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$. □

Now, using Theorems 3.8, 3.9 and Proposition 3.13, we obtain the following result which is a generalization of Theorem 3 in [15] to von Neumann-Schatten K-p-frames, K^*-atomic systems and K^*-duals in Banach spaces.

Theorem 3.14. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten p-Bessel sequence for $\mathscr{X}$. If $\overline{R(U_{\mathscr{G}})}$ is complemented, then the following are equivalent:*

(i) *$\mathscr{G}$ is a von Neumann-Schatten K^*-atomic system for $\mathscr{X}$.*
(ii) *$\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$.*
(iii) *There exists a von Neumann-Schatten q-Bessel sequence $\mathscr{F} = \{\mathscr{F}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}^*, \mathscr{C}_q)$ such that $\mathscr{F}$ is a K^*-dual of $\mathscr{G}$.*

Proof. The equivalence of (i) and (ii) can be obtained using Theorem 3.8 and Theorem 3.9.
(ii) $\Rightarrow$ (iii). Suppose that $\mathscr{G}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$. By the argument we used

in the proof of Theorem 3.9, there exists some $V \in B(\oplus\mathscr{C}_p, \mathscr{X})$ such that $K = VU_{\mathscr{G}}$, so $K^* = T_{\mathscr{G}}V^*$. Consider $\mathscr{F}_i := P_i(V^*)$, where P_i is the coordinate operator on $\oplus\mathscr{C}_q$, for each $i \in \mathbb{N}$. For each $x^* \in \mathscr{X}^*$, $V^*(x^*) \in \oplus\mathscr{C}_q$ and $\left(\sum_{i=1}^{\infty} \|P_i(V^*(x^*))\|_q^q\right)^{\frac{1}{q}} = \|V^*(x^*)\| \le \|V^*\|\|x^*\|$, also we have

$$K^*x^* = T_{\mathscr{G}}V^*(x^*) = \sum_{i=1}^{\infty} P_i(V^*(x^*))\mathscr{G}_i = \sum_{i=1}^{\infty} \mathscr{F}_i(x^*)\mathscr{G}_i.$$

This means that $\{\mathscr{F}_i\}_{i=1}^{\infty}$ is a K^*-dual of $\mathscr{G}$.
We can get the implication (iii) $\Rightarrow$ (ii) by Proposition 3.13. □

Corollary 3.15. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$. If the analysis operator of $\mathscr{G}$ is surjective, then $\mathscr{G}$ has a K^*-dual.*

Several characterizations for duals of a Hilbert space frame appeared in [7]. Now, motivated by the results obtained in [7], we give a characterization for von Neumann-Schatten K^*-duals of a von Neumann-Schatten K-p-frame.

Theorem 3.16. *Let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$. If P_i is the coordinate operator on $\oplus\mathscr{C}_q$, then von Neumann-Schatten K^*-duals of $\mathscr{G}$ are precisely the families $\{\mathscr{F}_i\}_{i=1}^{\infty} = \{P_iM\}_{i=1}^{\infty}$, where $M : \mathscr{X}^* \to \oplus\mathscr{C}_q$ is a linear bounded operator with $K^* = T_{\mathscr{G}}M$.*

Proof. For every $x^* \in \mathscr{X}^*$, we have

$$\begin{aligned} \left\|\{P_iM(x^*)\}_{i=1}^{\infty}\right\| &= \|M(x^*)\| \\ &\le \|M\|\,\|x^*\|. \end{aligned}$$

This shows that $\{\mathscr{F}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten q-Bessel sequence. Also, for each $x^* \in \mathscr{X}^*$, we have

$$\begin{aligned} K^*(x^*) &= T_{\mathscr{G}}M(x^*) \\ &= T_{\mathscr{G}}\left(\{P_iM(x^*)\}_{i=1}^{\infty}\right) \\ &= \sum_{i=1}^{\infty} \mathscr{F}_i(x^*)\mathscr{G}_i. \end{aligned}$$

Therefore $\{\mathscr{F}_i\}_{i=1}^{\infty}$ is a K^*-dual of $\mathscr{G}$.
For the other implication, suppose that $\{\mathscr{F}_i\}_{i=1}^{\infty}$ is a K^*-dual of $\mathscr{G}$. Then $U_{\mathscr{F}} : \mathscr{X}^* \to \oplus\mathscr{C}_q$ is a bounded linear operator, $\{\mathscr{F}_i\}_{i=1}^{\infty} = \{P_iU_{\mathscr{F}}\}_{i=1}^{\infty}$ and it is clear that $K^* = T_{\mathscr{G}}U_{\mathscr{F}}$. □

The preservation of the frame properties under the action of bounded operators has a major impact on the applications of frames and their generalizations, see [5, 7, 25]. In the following proposition, we show that every bounded below operator which commutes with K preserves the properties of a von Neumann-Schatten K-p-frame.

Proposition 3.17. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$. If $T \in B(\mathscr{X})$ is bounded below and $KT = TK$, then $\{\mathscr{G}_iT\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$.*

Proof. Suppose that $A_{\mathscr{G}}$ and $B_{\mathscr{G}}$ are von Neumann-Schatten K-p-frame bounds for $\mathscr{G}$ and A_1 is the lower bound of T. It is clear that $\{\mathscr{G}_iT\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Bessel sequence. For each $f \in \mathscr{X}$, we have

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\|\mathscr{G}_iT(f)\|_p^p\right)^{\frac{1}{p}} &\geq A_{\mathscr{G}}\|KT(f)\| \\ &= A_{\mathscr{G}}\|TK(f)\| \\ &\geq A_1A_{\mathscr{G}}\|Kf\|.\end{aligned}$$

Therefore $\{\mathscr{G}_iT\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$. □

4. Perturbations of von Neumann-Schatten K-p-frames

In this section, we focus on perturbations of von Neumann-Schatten K-p-frames. It is shown that they are stable under small perturbations. Indeed, the stability of von Neumann-Schatten K-p-frames under some well-known kinds of perturbations is considered. These perturbations have been presented and studied for frames and some of their generalizations such as g-frames, fusion frames and Hilbert-Schmidt frames, see [4, 6, 7, 17, 21, 26].

Theorem 4.1. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$ and let $\mathscr{V} = \{\mathscr{V}_i\}_{i=1}^{\infty}$ be a sequence in $B(\mathscr{X}, \mathscr{C}_p)$. If $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max\{\lambda_1 + \frac{\mu}{A_{\mathscr{G}}}, \lambda_2\} < 1$ and*

$$\left(\sum_{i=1}^{n}\|\mathscr{G}_i(f)-\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq \lambda_1\left(\sum_{i=1}^{n}\|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} + \lambda_2\left(\sum_{i=1}^{n}\|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} + \mu\|Kf\|, \tag{4.1}$$

for every $f \in \mathscr{X}$ and $n \in \mathbb{N}$, then $\{\mathscr{V}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ with bounds

$$\frac{(1-\lambda_1)A_{\mathscr{G}}-\mu}{(1+\lambda_2)} \quad \text{and} \quad \frac{(1+\lambda_1)B_{\mathscr{G}}+\mu\|K\|}{(1-\lambda_2)}.$$

Proof. For every $f \in \mathscr{X}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n}\|\mathscr{G}_i(f)-\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} &= \left\|\{\mathscr{G}_i(f)-\mathscr{V}_i(f)\}_{i=1}^{n}\right\| \\ &\geq \left\|\{\mathscr{V}_i(f)\}_{i=1}^{n}\right\| - \left\|\{\mathscr{G}_i(f)\}_{i=1}^{n}\right\|.\end{aligned}$$

Using (4.1), we obtain that

$$\left\|\{\mathscr{V}_i(f)\}_{i=1}^{n}\right\| - \left\|\{\mathscr{G}_i(f)\}_{i=1}^{n}\right\| \leq \lambda_1\left(\sum_{i=1}^{n}\|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} + \lambda_2\left(\sum_{i=1}^{n}\|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} + \mu\|Kf\|.$$

Therefore

$$\begin{aligned}(1-\lambda_2)\left(\sum_{i=1}^{n}\|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} &\leq (1+\lambda_1)\left(\sum_{i=1}^{n}\|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} + \mu\|Kf\| \\ &\leq (1+\lambda_1)B_{\mathscr{G}}\|f\| + \mu\|K\|\|f\|,\end{aligned}$$

for every $n \in \mathbb{N}$. Hence

$$\left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq \frac{(1+\lambda_1)B_{\mathscr{G}} + \mu\|K\|}{(1-\lambda_2)}\|f\|.$$

Similarly using (4.1), we get

$$\left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \geq \frac{(1-\lambda_1)A_{\mathscr{G}} - \mu}{(1+\lambda_2)}\|Kf\|.$$

Thus $\{\mathscr{V}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$ with bounds

$$\frac{(1-\lambda_1)A_{\mathscr{G}} - \mu}{(1+\lambda_2)} \quad \text{and} \quad \frac{(1+\lambda_1)B_{\mathscr{G}} + \mu\|K\|}{(1-\lambda_2)}.$$

∎

Proposition 4.2. *Let $K \in B(\mathscr{X})$. Suppose that $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty} \subseteq B(\mathscr{X}, \mathscr{C}_p)$ is a von Neumann-Schatten K-p-frame, $\mathscr{P} = \{\mathscr{P}_i\}_{i=1}^{\infty}$ is a subset of $B(\mathscr{X}, \mathscr{C}_p)$ and $\{c_i\}_{i=1}^{\infty}$ is a sequence in ℓ^p. If there exist some $0 \leq \lambda_1, \lambda_2 < 1$ such that*

$$\left\|\mathscr{G}_i(f) - \mathscr{P}_i(f)\right\|_p \leq \lambda_1\left\|\mathscr{G}_i(f)\right\|_p + \lambda_2\left\|\mathscr{P}_i(f)\right\|_p + c_i\|Kf\|,$$

for each $f \in \mathscr{X}$ and $i \in \mathbb{N}$ with $(1-\lambda_1)A_{\mathscr{G}} > \|\{c_i\}_{i=1}^{\infty}\|_{\ell^p}$, then $\mathscr{P}$ is a von Neumann-Schatten K-p-frame.

Proof. For each $f \in \mathscr{X}$ and every finite subset Ω in $\mathbb{N}$, we have

$$\left\|\left\{\mathscr{G}_i(f) - \mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| \leq \lambda_1\left\|\left\{\mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| + \lambda_2\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| + \left\|\{c_i\}_{i\in\Omega}\right\|_{\ell^p}\|Kf\|,$$

so

$$\begin{aligned}\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| &\leq \left\|\left\{\mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| + \left\|\left\{\mathscr{P}_i(f) - \mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| \\ &\leq (1+\lambda_1)\left\|\left\{\mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| + \lambda_2\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| + \left\|\{c_i\}_{i\in\Omega}\right\|_{\ell^p}\|K\|\|f\|,\end{aligned}$$

consequently,

$$\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| \leq \frac{1}{1-\lambda_2}\left((1+\lambda_1)B_{\mathscr{G}} + \|K\|\left\|\{c_i\}_{i=1}^{\infty}\right\|_{\ell^p}\right)\|f\|.$$

This means that $\mathscr{P}$ is a von Neumann-Schatten p-Bessel sequence. If $\mathscr{G}$ is a von Neumann-Schatten K-p-frame with $(1-\lambda_1)A_{\mathscr{G}} > \|\{c_i\}_{i=1}^{\infty}\|_{\ell^p}$, then

$$\begin{aligned}\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| &\geq \left\|\left\{\mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| - \left\|\left\{\mathscr{P}_i(f) - \mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| \\ &\geq (1-\lambda_1)\left\|\left\{\mathscr{G}_i(f)\right\}_{i\in\Omega}\right\| - \lambda_2\left\|\left\{\mathscr{P}_i(f)\right\}_{i\in\Omega}\right\| - \left\|\{c_i\}_{i\in\Omega}\right\|_{\ell^p}\|Kf\|.\end{aligned}$$

Hence

$$\left\|\left\{\mathscr{P}_i(f)\right\}_{i=1}^{\infty}\right\| \geq \frac{1}{1+\lambda_2}\left((1-\lambda_1)A_{\mathscr{G}} - \left\|\{c_i\}_{i=1}^{\infty}\right\|_{\ell^p}\right)\|Kf\|.$$

This yields that $\mathscr{P}$ is a von Neumann-Schatten K-p-frame. ∎

Theorem 4.3. *Let $K \in B(\mathscr{X})$ and let $\mathscr{G} = \{\mathscr{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten K-p-frame for $\mathscr{X}$ and let $\{\mathscr{V}_i\}_{i=1}^{\infty}$ be a sequence in $B(\mathscr{X}, \mathscr{C}_p)$. If there exists some positive constant M, such that for each $f \in \mathscr{X}$,*

$$\left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f) - \mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq M \min\left\{\left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}}\right\}, \tag{4.2}$$

then the sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame for $\mathscr{X}$.

Proof. Suppose that (4.2) holds. For each $f \in \mathscr{X}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{n} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} &\leq \left\|\{(\mathscr{G}_i - \mathscr{V}_i)(f)\}_{i=1}^{n}\right\| + \left\|\{\mathscr{G}_i(f)\}_{i=1}^{n}\right\| \\
&\leq \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f) - \mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&\leq M\left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&= (M+1)\left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B_{\mathscr{G}}(M+1)\|f\|.
\end{aligned}$$

Therefore,

$$\left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B_{\mathscr{G}}(M+1)\|f\|.$$

Also, for each $f \in \mathscr{X}$, we get

$$\begin{aligned}
A_{\mathscr{G}}\|Kf\| &\leq \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f)\|_p^p\right)^{\frac{1}{p}} = \left\|\{\mathscr{G}_i(f)\}_{i=1}^{\infty}\right\| \\
&\leq \left\|\{(\mathscr{G}_i - \mathscr{V}_i)(f)\}_{i=1}^{\infty}\right\| + \left\|\{\mathscr{V}_i(f)\}_{i=1}^{\infty}\right\| \\
&= \left(\sum_{i=1}^{\infty} \|\mathscr{G}_i(f) - \mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&\leq M\left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}} \\
&= (M+1)\left(\sum_{i=1}^{\infty} \|\mathscr{V}_i(f)\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, $\mathscr{V} = \{\mathscr{V}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten K-p-frame. ∎

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