RIGIDITY OF PROPER HOLOMORPHIC SELF-MAPPING OF SOME UNBOUNDED WEAKLY PSEUDOCONVEX HARTOGS DOMAIN

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ABSTRACT. This paper is concerned with the rigidity of proper holomorphic self-mapping of some unbounded weakly pseudoconvex domain, called the generalized Fock-Bargmann-Hartogs domain, which is defined as a Hartogs domain fibered over $\mathbb{C}^n$ with the fiber being a generalized complex ellipsoid. We develop a new technique to show that any proper holomorphic self-mapping of generalized Fock-Bargmann-Hartogs domain must be an automorphism without the restriction of the dimension of each fibre. As a main contribution, we partly solve the rigidity problems for proper holomorphic self-mapping of generalized Fock-Bargmann-Hartogs domain.

CONTENTS

1. INTRODUCTION

Let X and Y be two topological spaces. A continuous map $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is compact in X for every compact $K \subset Y$. We shall study proper holomorphic maps $F : \Omega \to \Omega'$, where Ω and Ω' are regions in $\mathbb{C}^n$ and $\mathbb{C}^k$, respectively. Especially, the case $k = n$ has attracted a lot of attention.

The study of the structure of proper holomorphic self-mappings of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ had greatly promoted the development of this field. When $n = 1$, it is well-known that all the proper holomorphic self-mappings of the disk $\mathbb{D}$ are precisely the finite Blaschke products. But for $n \geq 2$, the situation becomes quite different.

In 1977, Alexander [1] proved the following fundamental result.

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Theorem A (Alexander [1]). *Any proper holomorphic self-mapping of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ $(n \geq 2)$ is an automorphism of $\mathbb{B}^n$.*

Inspired by Alexander's result, there are many important results concerning bounded pseudoconvex domains with smooth boundary (e.g., Bedford-Bell [2] and Diederich-Fornaess [8]). Furthermore, the holomorphic mapping $f(z_1, z_2) := (z_1, z_2^2)$ gives rise to a proper holomorphic mapping between $\{|z_1|^2 + |z_2|^4 < 1\}$ and $\{|z_1|^2 + |z_2|^2 < 1\}$ in $\mathbb{C}^2$. Obviously, these domains are bounded pseudoconvex with smooth real-analytic boundaries, but the mapping f is branched and is not biholomorphic. This counterexample suggests us to discover some interesting bounded weakly pseudoconvex domains D_1, D_2 in $\mathbb{C}^n$ $(n \geq 2)$ such that any proper holomorphic mapping from D_1 to D_2 is a biholomorphism.

Once we consider bounded pseudoconvex domains without boundary regularity, the situation will face a serious analytical difficulty. In 1984, motivated by the works of Bell [3] and Tumanov-Henkin [26], Henkin-Novikov [12] proved that any proper holomorphic self-mapping on an irreducible bounded symmetric domain of rank ≥ 2 is an analytic automorphism. Moreover, by using the methods in Mok-Tsai [18], Tsai [22] and Mok-Ng-Tu [19], Tu [23] showed some rigidity results of proper holomorphic mappings between equidimensional bounded symmetric domains. For more studies of proper holomorphic mappings, please refer to Dini-Primicerio [9, 10], Landucci [17] and Tu [24].

We should point out that very little seems to be known about the rigidity of proper holomorphic mapping between the unbounded pseudoconvex domains. Those indicate that it is of a great interest to find some unbounded pseudoconvex domains such that the associated proper holomorphic self-mapping becomes an automorphism.

Recently, an unbounded Hartogs domain called Fock-Bargmann-Hartogs domain has been attracted a lot of attention. The Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ is defined by

$$D_{n,m}(\mu) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\} \quad (\mu > 0),$$

where $\|\cdot\|$ is the standard Hermitian norm. One can easily check that the Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ are unbounded strongly pseudoconvex domains in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary. In 2014, Tu-Wang [25] firstly established the rigidity of the proper holomorphic mappings between two equidimensional Fock-Bargmann-Hartogs domains as follows.

Theorem B (Tu-Wang [25]). *If $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ are two equidimensional Fock-Bargmann-Hartogs domains with $m \geq 2$ and f is a proper holomorphic mapping of $D_{n,m}(\mu)$ into $D_{n',m'}(\mu')$, then f is a biholomorphism between $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$.*

There are also other beautiful works on this domain. In 2013, Yamamori [27] gave an explicit formula for the Bergman kernels of the Fock-Bargmann-Hartogs

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domains in terms of the polylogarithm functions. In 2014, Kim-Ninh-Yamamori [13] determined the automorphism group of the Fock-Bargmann-Hartogs domains by checking that the Bergman kernel ensures revised Cartan's theorem. Later on, Bi-Feng-Tu [4] obtained an explicit formula for the Bergman kernel of the weighted Hilbert space of square integrable holomorphic functions on $D_{n,m}(\mu)$, and what's more, used the explicit expression to prove the existence of balanced metrics for a class of Fock-Bargmann-Hartogs domains. Furthermore, Bi-Su-Tu [5] calculated explicitly the Kobayashi pseudometric for the Fock-Bargmann-Hartogs domains and established the Schwarz lemma at the boundary for holomorphic mappings between the non-equidimensional Fock-Bargmann-Hartogs domains. The reader is also referred to Kim-Yamamori [14], Kim-Yamamori-Zhang [15], Kodama [16], Yang-Bi [28] and references therein.

In this paper, we mainly consider a class of weakly pseudoconvex domains called the generalized Fock-Bargmann-Hartogs domains $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ which is defined by

$$D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) = \{(z, w_{(1)}, \cdots, w_{(l)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_l} : \sum_{j=1}^{l} \left\|w_{(j)}\right\|^{p_j} < e^{-\mu\|z\|^2}\} \quad (\mu > 0),$$

where $\mathbf{n} = (n_1, \cdots, n_l),\ w_{(j)} = (w_{j1}, \cdots, w_{jn_j}) \in \mathbb{C}^{n_j}$, in which n_j is a non-negative integer for $1 \leq j \leq l$. Obviously, each generalized Fock-Bargmann-Hartogs domain $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ is an unbounded non-hyperbolic weakly pseudoconvex domain. Compared to the Fock-Bargmann-Hartogs domains, the boundary of generalized Fock-Bargmann-Hartogs domain is more complicated and lacks certain regularity and homogeneity. Hence, we will encounter a serious analytical difficulty in dealing with the rigidity of proper holomorphic mappings.

In 2018, Bi-Tu [6] made a try to obtain the rigidity result of proper holomorphic self-mappings of generalized Fock-Bargmann-Hartogs domains under some extra codimensional conditions. More precisely, they proved the following result.

Theorem C. *Suppose $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ is a generalized Fock-Bargmann-Hartogs domain with*

$$\min\{n_{1+\epsilon}, n_2, \cdots, n_l, n_1 + \cdots + n_l\} \geq 2.$$

Then any proper holomorphic self-mapping of $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ must be an automorphism.

Here we use the following notation:

$$\epsilon = \begin{cases} 1, & p_1 = 2 \\ 0, & p_1 > 2 \end{cases}.$$

Furthermore, they also pointed out that the condition $n_1 + \cdots + n_l \geq 2$ can not be removed, which it can be seen from the following counterexample:

$$F : (z, w_{(1)}) \to (\sqrt{2}z, w_{(1)}^2)$$

is a proper holomorphic self-mapping of $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$, but it is branched and is not an automorphism. But it is still unknown whether this condition in Theorem **C** can be weaken to $n_1 + \cdots + n_l \geq 2$. This is exactly what we want to do in this paper.

To this end, we firstly consider the generalized Fock-Bargmann-Hartogs domain with the special case $n_1 = n_2 = \cdots = n_l = 1$. In this case the generalized Fock-Bargmann-Hartogs domain can be rewritten as follows:

$$D^p_{n,m}(\mu) := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{m} |w_j|^{p_j} < e^{-\mu\|z\|^2}\}.$$

Throughout this paper we always assume that $p_j > 1$ for each j.

Remark 1.1. *In Bi-Tu's case, the non-strongly pseudoconvex boundary points are contained in some complex analytic set of complex codimension at least* 2*. This fact plays a key role in proving the rigidity results for the proper holomorphic mapping. However, in our case the non-strongly pseudoconvex part of* $bD^p_{n,m}(\mu)$ *is contained in some complex analytic set of complex codimension* 1*. It means that this analytic set is big enough to contain the zero locus of the complex Jacobian of the proper holomorphic self-mapping.*

This obstruction forces us to explore a quite different method to study the rigidity of proper holomorphic self-mapping of $D^p_{n,m}(\mu)$. In this paper, we develop a new technique to prove the following result.

Theorem 1.2. *Let* $F = (f, g)$ *be a proper holomorphic self-mapping of the generalized Fock-Bargmann-Hartogs domain* $D^p_{n,m}(\mu)$*. If* $m \geq 2$*, then* F *is an automorphism of* $D^p_{n,m}(\mu)$*.*

As a consequence, we can partly solve the rigidity problems for proper holomorphic self-mapping of $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$.

Corollary 1.3. *Suppose* $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ *is a generalized Fock-Bargmann-Hartogs domain with*

$$n_1 + \cdots + n_l \geq 2.$$

Then any proper holomorphic self-mapping of $D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ *must be an automorphism.*

We now summarize the ideas behind the proof of Theorem 1.2. At first, by the standard way we have the holomorphic extension for the proper holomorphic mapping F. Then we are going to calculate the zero locus of the complex Jacobian of the proper holomorphic self-mapping for which may intersect with $bD^p_{n,m}(\mu)$. There may have two different cases. One is that there exists a strongly pseudoconvex point in the intersection, and the image for this boundary point will lie on the non-strongly pseudoconvex part. This is the case we considered in the Lemma 2.4. The other one is that the intersection is contained in the non-strongly pseudoconvex part. We will show that neither of these is true, and this means that the zero locus cannot cross the boundary.

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The paper is organized as follows. In Section 2, we will exploit the boundary structure of $D^p_{n,m}(\mu)$ and then we also prove that the proper holomorphic self-mapping F of $D^p_{n,m}(\mu)$ can be holomorphically extended to $\overline{D^p_{n,m}(\mu)}$. Moreover we also exhibit some useful conclusions to facilitate our proofs. In Section 3, we will give the proof of our main results by using micro-local analysis technique.

2. Preliminaries

2.1. **The boundary structure.** To begin, we give the boundary structure of $D^p_{n,m}(\mu)$, which is comprised of

$$bD^p_{n,m}(\mu) = b_0 D^p_{n,m}(\mu) \cup b_1 D^p_{n,m}(\mu) \cup b_2 D^p_{n,m}(\mu),$$

where

$$b_0 D^p_{n,m}(\mu) := \Big\{(z,w) \in \mathbb{C}^{n+m} : \sum_{j=1}^{m} |w_j|^{p_j} = e^{-\mu\|z\|^2},\ |w_j| \neq 0,\ 1+\epsilon \leq j \leq m\Big\},$$

$$b_1 D^p_{n,m}(\mu) := \bigcup_{j=1+\epsilon}^{m} \Big\{(z,w) \in \mathbb{C}^{n+m} : \sum_{j=1}^{m} |w_j|^{p_j} = e^{-\mu\|z\|^2},\ |w_j| = 0, p_j > 2\Big\},$$

and

$$b_2 D^p_{n,m}(\mu) := \bigcup_{j=1+\epsilon}^{m} \Big\{(z,w) \in \mathbb{C}^{n+m} : \sum_{j=1}^{m} |w_j|^{p_j} = e^{-\mu\|z\|^2},\ |w_j| = 0, 2 > p_j > 1\Big\}.$$

By the above definition, we easily have the following result.

Proposition 2.1 (Bi-Tu [6], Proposition 3.1). (1) *The boundary $b_0 D^p_{n,m}(\mu)$ is a real analytic hypersurface in $\mathbb{C}^{n+m}$ and $D^p_{n,m}(\mu)$ is strongly pesudoconvex at all points of $b_0 D^p_{n,m}(\mu)$.*
(2) *$D^p_{n,m}(\mu)$ is weakly pesudoconvex but not strongly pesudoconvex at any point of $b_1 D^p_{n,m}(\mu)$, and is only C^1-smooth at any point of $b_2 D^p_{n,m}(\mu)$.*

2.2. **Holomorphic automorphism and holomorphic extension.** To simplify our discussions, it will be very convenient for us if we know the explicit forms for the automorphism group of generalized Fock-Bargmann-Hartogs domain. In fact, Bi-Tu [6] showed the following result.

Lemma 2.2 (Bi-Tu [6], Theorem 1.6)**.** *The automorphism group* $\mathrm{Aut}(D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu))$ *is generated by the following mappings:*

$\varphi_A : (z, w_{(1)}, \cdots, w_{(l)}) \longmapsto (zA, w_{(1)}, \cdots, w_{(l)});$

$\varphi_D : (z, w_{(1)}, \cdots, w_{(l)}) \longmapsto (z, (w_{(\sigma(1))}, \cdots, w_{(\sigma(l))})D);$

$\varphi_a : (z, w) \longmapsto (z + a, w_{(1)}(e^{-2\mu<z,a>-\mu\|a\|^2})^{\frac{1}{p_1}}, \cdots, w_{(l)}(e^{-2\mu<z,a>-\mu\|a\|^2})^{\frac{1}{p_l}}),\ (a \in \mathbb{C}^{n_0}),$

where $A \in \mathcal{U}(n_0)$*,* $\sigma \in S_l$ *is a permutation such that* $n_{\sigma(j)} = n_j$*,* $p_{\sigma(j)} = p_j$ $(1 \le j \le l)$*, and*

$$D = \begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_l \end{pmatrix},$$

in which $\Gamma_i \in \mathcal{U}(n_i)$ *(*$1 \le i \le l$*).*

By using standard arguments developed in Bi-Tu [6], we are also able to show that a proper holomorphic self-mapping of generalized Fock-Bargmann-Hartogs domains extends holomorphically to its closures.

Lemma 2.3 (Bi-Tu [6], Lemma 2.7)**.** *Suppose that* $f : D^p_{n,m}(\mu) \to D^p_{n,m}(\mu)$ *is a proper holomorphic self-mapping of generalized Fock-Bargmann-Hartogs domain. Then* f *can be holomorphically extended to a neighborhood of the closure* $\overline{D^p_{n,m}(\mu)}$*.*

2.3. **The image of the strongly pesudoconvex part.**

Lemma 2.4. *Let* $F = (f_1, \cdots, f_n, g_1, \cdots, g_m) : M \to D^p_{n,m}(\mu)$*,* $M \subset \mathbb{C}^{n+m}$ *open, be a holomorphic mapping. Suppose, there are two relatively open sets* $U_1 \subset bM$ *and* $U_2 \subset bD^p_{n,m}(\mu)$*, and that let* U_1 *be strongly pesudoconvex . Furthermore, suppose that* F *extends holomorphically to* $M \cup U_1$*, and* $F(U_1) \subset U_2$ *with* $F(U_1) \cap b_1 D^p_{n,m}(\mu) \neq \emptyset$*. Then* F *must be a constant.*

Remark 2.5. *If all* $p_j \ge 2$*, the generalized Fock-Bargmann-Hartogs domain* $D^p_{n,m}(\mu)$ *will have* $\mathcal{C}^2$*-smooth boundary. Then such result can be easily obtained by the determinant of the Levi-form of the defining function (refer to Bell* [2]*).*

Proof. Suppose that F is a non-constant holomorphic mapping. As the statement is purely local at the points of U_1, we assume that $U_1 = \{(u, v) \in \mathbb{C}^{n+m-1} \times \mathbb{C} : \rho(u, v) = 0\}$, where

$$\rho(u, v) = 2\mathrm{Re}v + \|u\|^2 + |v|^2 + o(\|u\|^2 + |v|^2),$$

$\|u\|^2 = \sum_{k=1}^{n+m-1} |u_k|^2$, and $\|u\|^2 + |v|^2 < \varepsilon$, for some $\varepsilon > 0$. Moreover, suppose that

$$Q = F(0) \in b_1 D^p_{n,m}(\mu).$$

Thus we have

$$\det F'(0) = 0.$$

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Then taking the composite mapping $\phi \circ F$ where ϕ is an automorphism mapping of $D^p_{n,m}(\mu)$ if necessary, we may assume that

$$Q = (0, w_1, \cdots, w_m).$$

Without loss of generality, in the following we can assume that $w_1 = \cdots = w_k = 0$ $(1 \le k < m)$, and the rest components are non-zero. Consider the hyperplane defined by

$$L := \{(u, v) \in \mathbb{C}^{n+m-1} \times \mathbb{C} : v = 0\}.$$

Since F can be holomorphically extended to $M \cup U_1$, there exists a neighborhood U of the origin 0 such that F is holomorphic on U. Let V be a set defined by

$$V := U \cap \{\mathbb{C}^{n+m} \setminus M\}.$$

Notice that $r \circ F(u, v) \ge 0$ and $r \circ F(0) = 0$ for $(u, v) \in V \cap L$, where

$$r(z, w) = \sum_{j=1}^{m} |w_j|^{p_j} - e^{-\mu\|z\|^2}.$$

It means that $r \circ F(u, v)$ attains the minimal value at 0 for $(u, v) \in V \cap L$. Choosing a smooth curve $\gamma(t) : [-1, 1] \to V \cap L$ with $\gamma(0) = 0$ and $\gamma'(0) = (\alpha, 0)$, for any $\alpha \in \mathbb{C}^{n+m-1}$. Then we have

$$0 = \frac{dr(F(\gamma(t)))}{dt} |_{t=0} = \operatorname{Re}\left(\sum_{j=k+1}^{m} \sum_{s=1}^{n+m-1} p_j w_j^{\frac{p_j}{2}-1} \overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial u_s} \frac{d\gamma_s}{dt} |_{t=0} \right).$$

A direct computation implies that

$$\sum_{j=k+1}^{m} p_j w_j^{\frac{p_j}{2}-1} \overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial u_s}(0) = 0, \quad 1 \le s \le n+m-1. \tag{2.1}$$

Now we consider the subharmonic function $r(F(u, v))$ on a neighborhood U of 0. For any $(u, v) \in \overline{M} \cap U$, $r(F(u, v)) \le r(F(0))$. Hence by Hopf lemma, we have

$$\lim_{t \to 0^+} \frac{r(F(0)) - r(F(-t\nabla\rho(0)))}{t} = \operatorname{Re}\left(\sum_{j=k+1}^{m} p_j w_j^{\frac{p_j}{2}-1} \overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial v}(0) \right) > 0.$$

That is

$$\operatorname{Re}\left(\sum_{j=k+1}^{m} p_j w_j^{\frac{p_j}{2}-1} \overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial v}(0) \right) > 0. \tag{2.2}$$

Furthermore since $\det F'(0) = 0$, by (2.1) we have

$$\det F'(0) = \left(\sum_{j=k+1}^{m} p_j w_j^{\frac{p_j}{2}-1} \overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial v}(0) \right) \times \det \left(\frac{\partial \widehat{F}}{\partial \zeta}(0) \right),$$

where $\widehat{F} = (f_1, \cdots, f_n, g_1, g_2, \cdots, g_{m-1})$ and $\zeta = (u_1, \cdots, u_{n+m-1})$.

Then (2.2) yields that

$$\det \left(\frac{\partial \widehat{F}}{\partial \zeta}(0) \right)_{(n+m-1)\times(n+m-1)} = 0.$$

Thus together with (2.1), there exist constants $\lambda_1, \cdots, \lambda_{n+m-1}$ not all equal to 0 such that

$$\lambda_1 \frac{\partial F}{\partial u_1}(0) + \cdots + \lambda_{n+m-1} \frac{\partial F}{\partial u_{n+m-1}}(0) = 0. \tag{2.3}$$

Since U_1 is strongly pesudoconvex , we can choose some points q_t near the origin in U_1 with the following form (refer to Fornæss [11])

$$q_t = (\delta\lambda_1 t^{\frac{1}{2}} + O(t), \cdots, \delta\lambda_{n+m-1} t^{\frac{1}{2}} + O(t), -t) \in U_1 \quad (0 < t \ll 1),$$

where δ satisfying

$$|\delta|^2 \sum_{s=1}^{n+m-1} |\lambda_s|^2 = 2. \tag{2.4}$$

Since F is holomorphic on U_1, by the Taylor expansion we have

$$\begin{aligned} f_j(q_t) &= \sum_{s=1}^{n+m-1} \frac{\partial f_j}{\partial u_s}(0)(\delta\lambda_s t^{\frac{1}{2}} + O(t)) + \frac{\partial f_j}{\partial v}(0)(-t) + O(t) \\ &= O(t) \qquad (1 \le j \le n), \end{aligned} \tag{2.5}$$

where the second equality is obtained by (2.3). Similarly,

$$g_j(q_t) = O(t) \qquad (1 \le j \le k). \tag{2.6}$$

For $k+1 \le j \le m$, we have

$$\begin{aligned} g_j(q_t) =& w_j + \sum_{s=1}^{n+m-1} \frac{\partial g_j}{\partial u_s}(0)(\delta\lambda_s t^{\frac{1}{2}} + O(t)) + \frac{\partial g_j}{\partial v}(0)(-t) \\ &+ \sum_{i=1}^{n+m-1} \sum_{s=1}^{n+m-1} \frac{\partial g_j}{\partial u_i \partial u_s}(0)(\delta\lambda_i t^{\frac{1}{2}} + O(t))(\delta\lambda_s t^{\frac{1}{2}} + O(t)) + o(t) \\ =& w_j + \frac{\partial g_j}{\partial v}(0)(-t) + \ell t + o(t), \end{aligned} \tag{2.7}$$

where ℓ is a constant depending on δ.

Following by the assumption $F(U_1) \subset U_2 \subset bD^p_{n,m}(\mu)$, $F(q_t) \in bD^p_{n,m}(\mu)$. This means that

$$\sum_{j=1}^{k} |g_j(q_t)|^{p_j} = \exp\left(-\mu \sum_{j=1}^{n} |f_j(q_t)|^2\right) - \sum_{j=k+1}^{m} |g_j(q_t)|^{p_j}. \tag{2.8}$$

However, on the one hand, by (2.6) we have

$$\sum_{j=1}^{k} |g_j(q_t)|^{p_j} = O(t^{\min_k\{p_j\}}) = o(t). \tag{2.9}$$

On the other hand, together with (2.5) and (2.7), we learn

$$\begin{aligned}&\exp\left(-\mu\sum_{j=1}^{n}|f_j(q_t)|^2\right)-\sum_{j=k+1}^{m}|g_j(q_t)|^{p_j}\\ &=-t\mathrm{Re}\Big(\sum_{j=k+1}^{m}p_jw_j^{\frac{p_j}{2}-1}\overline{w}_j^{\frac{p_j}{2}}\frac{\partial g_j}{\partial v}(0)\Big)+\ell' t+o(t)\\ &=\alpha t+o(t),\end{aligned}\tag{2.10}$$

where ℓ' is a constant related to δ.

Combining (2.2) with (2.4), we can choose appropriate δ to satisfy that $\alpha \neq 0$. Hence, together with (2.9), (2.10) and (2.8) leads to a contradiction. This implies that F must be a constant. The proof is completed. □

2.4. **Conclusions on complex analytic sets.** In this part, we introduce some important facts on the complex analytic sets.

Lemma 2.6 (Chirka [7], §7.4 Theorem 3). *A pure p-dimensional analytic subset $A \subset \mathbb{C}^n$ is algebraic if and only if it is contained, after some unitary change of coordinates, in a domain $D : \|z''\| < C(1 + \|z'\|)^s$, where $z = (z', z''), z' = (z_1, \cdots, z_p)$, and C, s are certain constants.*

The following formula for the dimension of the intersection of two algebraic sets is also needed in our proof.

Lemma 2.7 (see Shafarevich §6.2 Th. 6 [21]). *Let $X, Y \subset \mathbb{P}^N$ be irreducible quasiprojective varieties with $\dim X = n$ and $\dim Y = m$. Then any (nonempty) component Z of $X \cap Y$ has $\dim Z \geq n + m - N$.*

3. The proof of main theorem

Proof of Theorem 1.2. Let $F = (f_1, \cdots, f_n, g_1, \cdots, g_m) : D^p_{n,m}(\mu) \to D^p_{n,m}(\mu)$ be a proper holomorphic self-mapping. By Lemma 2.3, F extends holomorphically to a neighborhood V of $\overline{D^p_{n,m}(\mu)}$ with $F(bD^p_{n,m}(\mu)) \subset bD^p_{n,m}(\mu)$. We define

$$S := \{\xi \in V : J_F(\xi) = 0\},$$

where $J_F(\xi) = \det(\frac{\partial F_i}{\partial \xi_j})$ is the complex Jacobian determinant of $F(\xi)$ $(\xi \in V)$.

Assume that $S \cap bD^p_{n,m}(\mu) = \emptyset$. Let $M := S \cap D^p_{n,m}(\mu)$, then we have

$$M \subset D^p_{n,m}(\mu), \quad \overline{M} \cap bD^p_{n,m}(\mu) = \emptyset.$$

Suppose that $M \neq \emptyset$, then M is a complex analytic set in $\mathbb{C}^{n+m}$. For any $(z, w) \in M$, one can see that

$$|w_m| \leq e^{-\mu\|z\|^2} \leq 1 \leq 1 + \|(z, w')\|,$$

where $w = (w', w_m)$. Then by Lemma 2.6, we know that M is an algebraic set of $\mathbb{C}^{n+m}$.

Suppose M_1 is an irreducible component of M. Then $\overline{M_1}$ is a projective algebraic set where $\overline{M_1}$ denotes the closure of M_1 in $\mathbb{P}^{n+m}$, and $\dim \overline{M_1} = n + m - 1$. Let

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$[\xi, z, w]$ be the homogeneous coordinate in $\mathbb{P}^{n+m}$. Then we can embed $\mathbb{C}^{n+m}$ into $\mathbb{P}^{n+m}$ as the affine piece $U_0 = \{[\xi, z, w] \in \mathbb{P}^{n+m},\ \xi \neq 0\}$ by $(z, w) \hookrightarrow [1, z, w]$. Then we get

$$D^p_{n,m}(\mu) \cap U_0 = \left\{[\xi, z, w],\ \xi \neq 0,\ \sum_{j=1}^{m} \frac{|w_j|^{p_j}}{|\xi|^{p_j}} < e^{-\mu\frac{\|z\|^2}{|\xi|^2}}\right\}.$$

Let $H := \{\xi = 0\} \subset \mathbb{P}^{n+m}$. Now we consider another affine piece $U_1 = \{[\xi, z, w] \in \mathbb{P}^{n+m},\ z_1 \neq 0\}$ with affine coordinate $(\zeta, t, s) = (\zeta, t_2, \cdots, t_n, s)$. Let $t' = (1, t_2, \cdots, t_n)$. Since

$$\sum_{j=1}^{m} \frac{|w_j|^{p_j}}{|\xi|^{p_j}} = \sum_{j=1}^{m} \frac{|w_j|^{p_j}}{|z_1|^{p_j}} \frac{|z_1|^{p_j}}{|\xi|^{p_j}} = \sum_{j=1}^{m} \frac{|s_j|^{p_j}}{|\zeta|^{p_j}},$$

and

$$\exp\left(-\mu\frac{\|z\|^2}{|\xi|^2}\right) = \exp\left(-\mu\frac{\|z\|^2}{|z_1|^2}\frac{|z_1|^2}{|\xi|^2}\right) = \exp\left(-\mu\frac{1 + |t_2|^2 + \cdots + |t_n|^2}{|\zeta|^2}\right),$$

we have

$$\begin{aligned} &D^p_{n,m}(\mu) \cap U_0 \cap U_1 \\ =&\left\{(\zeta, t_2, \cdots, t_n, s) \in \mathbb{C}^{n+m},\ \sum_{j=1}^{m} \frac{|s_j|^{p_j}}{|\zeta|^{p_j}} < e^{-\mu\frac{\|t'\|^2}{|\zeta|^2}}\right\}. \end{aligned} \tag{3.1}$$

Let $M' = \overline{M_1} \cap U_1$ and $H_1 = H \cap U_1 = \{\zeta = 0\}$. For any $u \in M' \cap H_1$, there exists a sequence of points $\{u_k\} \subset \overline{M_1} \cap ((U_0 \cap U_1)\backslash H_1)$ such that $u_k \to u$ $(k \to \infty)$. The formula (3.1) implies

$$\|s_j(u_k)\|^{p_j} \leq |\zeta(u_k)|^{p_j} e^{-\mu\frac{\|t'\|^2}{|\zeta(u_k)|^2}} \quad (1 \leq j \leq m). \tag{3.2}$$

Since $u \in H_1$, we have $\zeta(u) = 0$ and $\zeta(u_k) \to 0$ $(k \to \infty)$. Therefore we get $\|s_j(u_k)\|^{2p_j} \leq 0$ $(1 \leq j \leq m)$ as $k \to \infty$. Thus

$$M' \cap H_1 \subset \{\zeta = 0,\ s_1 = \cdots = s_m = 0\}, \tag{3.3}$$

which implies that $\dim(M' \cap H_1) \leq n - 1$. By Lemma 2.7, we have

$$n - 1 \geq \dim(M' \cap H_1) \geq \dim M' + \dim H_1 - n - m \geq \dim M' - 1. \tag{3.4}$$

This means $\dim M' \leq n$, and thus $n + m - 1 = \dim M' \leq n$ which contradicts to the fact $m \geq 2$.

Hence, $S \cap bD^p_{n,m}(\mu) \neq \emptyset$. Furthermore, take an irreducible component S' of S with $S' \cap bD^p_{n,m}(\mu) \neq \emptyset$. Let $E_{S'} = S' \cap bD^p_{n,m}(\mu)$, then $E_{S'}$ contains a real submanifold of real dimension $2(n + m) - 3$. Otherwise, S' cannot be separated by $E_{S'}$. This means that we can find a complex analytic subset $M \subset S \cap D^p_{n,m}(\mu)$ satisfied $\overline{M} \cap bD^p_{n,m}(\mu) = \emptyset$ and $\dim \overline{M} = n + m - 1$, which leads to a contradiction.

Combining results in Pinčuk [20] and Lemma 2.4, we learn that

$$S \cap b_0 D^p_{n,m}(\mu) = \emptyset.$$

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This implies that

$$S \cap bD^p_{n,m}(\mu) \subset \bigcup_{j=1+\epsilon}^{m} \{w_j = 0\} \bigcap bD^p_{n,m}(\mu).$$

So consider the real dimension of the above two parts, and then by the identity principle, we conclude that there exists j_0 such that

$$\{w_{j_0} = 0\} \cap D^p_{n,m}(\mu) \subset S.$$

Without loss of generality, we assume that $j_0 = m$.

Notice that by Lemma 2.4, we know that

$$F^{-1}(b_1 D^p_{n,m}(\mu) \cup b_2 D^p_{n,m}(\mu)) \subset b_1 D^p_{n,m}(\mu) \cup b_2 D^p_{n,m}(\mu).$$

More precisely,

$$F^{-1}\left(\bigcup_{j=1+\epsilon}^{m} \{w_j = 0\} \bigcap bD^p_{n,m}(\mu)\right) \subset \bigcup_{j=1+\epsilon}^{m} \{w_j = 0\} \bigcap bD^p_{n,m}(\mu).$$

This implies that for every j, $1+\epsilon \le j \le m$, there exists an index of $\{1+\epsilon, \cdots, m\}$, for example, k, and an open subset V of $\{w_k = 0\} \cap bD^p_{n,m}(\mu)$, with $g_j \mid_V = 0$. It follows that $g_j \mid_{w_k=0} \equiv 0$ by the identity principle. Furthermore, the correspondence $j \to k$ is injective. Otherwise, suppose that for $j_1 \neq j_2$, $j_1, j_2 \in \{1+\epsilon, \cdots, m\}$ such that

$$g_{j_1} \mid_{w_k=0} \equiv g_{j_2} \mid_{w_k=0} \equiv 0.$$

However, this is a contradiction to that F is a proper holomorphic mapping, as the above equality implies the image set is contained in a coordinate subspace of codimension 2. Hence, for $w_m = 0$, we assume that $g_{k_0} \mid_{w_m=0} \equiv 0$. Then there exists a holomorphic function $h_1(z, w)$ such that

$$g_{k_0}(z, w) = w_m h_1(z, w).$$

As $\{w_m = 0\} \cap D^p_{n,m}(\mu) \subset S$, we have $\det F'(z, w) \mid_{w_m=0} = 0$, that is

$$\left| \begin{array}{cccc} * & * & * & * \\ \frac{\partial g_{k_0-1}}{\partial z_k} & \frac{\partial g_{k_0-1}}{\partial w_1} & \cdots & \frac{\partial g_{k_0-1}}{\partial w_m} \\ 0 & \cdots & 0 & h_1 \\ \frac{\partial g_{k_0+1}}{\partial z_k} & \frac{\partial g_{k_0+1}}{\partial w_1} & \cdots & \frac{\partial g_{k_0+1}}{\partial w_m} \\ * & * & * & * \end{array} \right|_{w_m=0} = 0. \tag{3.5}$$

Now we set $\Phi(z, w') = (f(z, w', 0), g_1(z, w', 0), \cdots, \check{g}_{k_0}(z, w', 0), \cdots, g_m(z, w', 0))$, where $w' = (w_1, \cdots, w_{m-1})$ and $\check{g}_{k_0}$ means the k_0-element missing. One can check that $\Phi(z, w')$ is a proper holomorphic mapping from $D^p_{n,m}(\mu) \cap \{w_m = 0\}$ to $D^p_{n,m}(\mu) \cap \{w_j = 0\}$. This means that there exist an open set U of $D^p_{n,m}(\mu) \cap \{w_m = 0\}$ such that $\det \Phi'(z, w') \neq 0$ for $(z, w') \in U$. By (3.5), we have

$$h_1(z, w', 0) = 0, \qquad (z, w') \in U.$$

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This implies that $h_1(z,w) \equiv 0$ on $D^p_{n,m}(\mu) \cap \{w_m = 0\}$. Therefore, we can find another holomorphic function $h_2(z,w)$ such that

$$g_{k_0}(z,w) = w_m^2 h_2(z,w).$$

Set $q = (0,\cdots,0,1,0) \in bD^p_{n,m}(\mu)$, and after composing some automorphism mapping of $D^p_{n,m}(\mu)$ if necessary, we assume that $F(q) = (0, w_1, \cdots, w_m)$ with $w_{k_0} = 0$. Then consider some boundary points near q with the following form

$$q_t = (0,\cdots,0,1-t,w_0), \qquad 0 < t \ll 1.$$

Then we have

$$|w_0|^{p_m} = 1-(1-t)^{p_{m-1}} = p_{m-1}t + o(t), \tag{3.6}$$

and by Hopf lemma we have

$$\mathrm{Re}\left(\sum_{j=1}^m p_j w_j^{\frac{p_j}{2}-1}\overline{w}_j^{\frac{p_j}{2}} \frac{\partial g_j}{\partial w_{m-1}}(q)\right) > 0. \tag{3.7}$$

Since q_t is a boundary point, we obtain

$$|g_m(q_t)|^{p_m} = e^{-\mu\|f(q_t)\|^2} - \sum_{j=1}^{m-1} |g_j(q_t)|^{p_j}. \tag{3.8}$$

As F is holomorphic on q_t, by Taylor expansion we know

$$|g_m(q_t)|^{p_m} = |w_0|^{2p_m}(h_2(q_t))^{p_m} = O(t^2).$$

However, when we consider the series expansion of $e^{-\mu\|f(q_t)\|^2} - \sum_{j=1}^{m-1}|g_j(q_t)|^{p_j}$ at q, by (3.6) and (3.7), we can choose appropriate w_0 so that the coefficient of t is not equal to 0, which leads to a contradiction to (3.8).

Therefore, $S = \emptyset$ and thus F is unbranched. Since the generalized Fock-Bargmann-Hartogs domain $D^p_{n,m}(\mu)$ is simply connected, $F : D^p_{n,m}(\mu) \to D^p_{n,m}(\mu)$ is a biholomorphism. The proof is completed. □

Proof of Corollary 1.3. Let $f : D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \to D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ be a proper holomorphic self-mapping. Then f extends holomorphically to a neighborhood V of $\overline{D}^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ with $f(bD^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)) \subset bD^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$. We define

$$S := \{\xi \in V : J_f(\xi) = 0\},$$

where $J_f(\xi) = \det(\frac{\partial f_i}{\partial \xi_j})$ is the complex Jacobian determinant of $f(\xi)$ $(\xi \in V)$.

Let $\mathbf{J}$ be a set of the index j such that $n_j \geq 2$. If $\mathbf{J} = \emptyset$, then the result follows by Theorem 1.2. Therefore, we can assume that $\mathbf{J} \neq \emptyset$. Moreover, the proofs of Theorem 1.2 tell that that there exists an irreducible component S' of S such that

$$f\left(S' \cap D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)\right) \subset \bigcup_{j\in\mathbf{J}} Pr_j\left(D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)\right), \tag{3.9}$$

where $Pr_j\left(D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)\right) := D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \cap \{w'_{(j)} = 0\}$ for each $j \in \mathbf{J}$.

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However, codim$(S') = 1$, and

$$\text{codim} \bigcup_{j \in \mathbf{J}} Pr_j \left(D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \right) = \min_{j \in \mathbf{J}} n_j \geq 2.$$

Since $f : D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \to D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ is proper, this leads to a contradiction with (3.9). Hence, $S = \emptyset$. This implies that $f : D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \to D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$ is unbranched.

Since the generalized Fock-Bargmann-Hartogs domain is simply connected, we get that $f : D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu) \to D^{\mathbf{p}}_{n_0,\mathbf{n}}(\mu)$is a biholomorphism. The proof of Corollary 1.3 is completed.

□

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