GENERALIZATION OF DEGENERACY SECOND MAIN THEOREM FOR MEROMORPHIC MAPPINGS FROM A p-PARABOLIC MANIFOLD TO A PROJECTIVE ALGEBRAIC VARIETY

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ABSTRACT. In [6], the author introduced the notion of "distributive constant" of a family of hypersurfaces with respect to a projective variety. Inspired by this thought, we will prove a general form of Second Main Theorem for meromorphic maps from p-Parabolic manifold into smooth projective variety intersecting with arbitrary families of hypersurfaces. It generalizes and improves previous results, especially for the case of the families of hypersurfaces in subgeneral position.

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1. Introduction and main results

Throughout this paper, we shall use the standard notationin the value distribution theory of meromorphic maps on parabolic manifolds (see[11],[13]). To state clearly our result, we need some notations and definitions as follows:

Definition 1. A Kähler complex manifold (M, ω) of dimension m is said to be a **p-Parabolic manifold** for $1 \leq p \leq m$ if there exists a plurisubharmonic function ϕ such that
(i) $\{\phi = -\infty\}$ is a closed subset of M with strictly lower dimension;
(ii) ϕ is smooth on the open dense set $M \setminus \{-\infty\}$ with $dd^c\phi \geq 0$, such that

$$(dd^c\phi)^{p-1} \wedge \omega^{m-p} \not\equiv 0 \text{ and } (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0.$$

Accordingly, we define

$$\tau := e^{\phi} \text{ and } \sigma := d^c\phi \wedge (dd^c\phi)^{p-1} \wedge \omega^{m-p}$$

where τ is nonnegative and it is called a *p-parabolic exhaustion* on M.

Note that m-parabolicity is just the classical notion of parabolicity and the parabolic manifold (see [11],[12]) has the affine algebraic variety as a prototype.

For any $r > 0$, we denote

$$M[r] := \{x \in M \mid \tau(x) \leq r^2\}, \quad M(r) := \{x \in M \mid \tau(x) < r^2\},$$

$$M\langle r\rangle := M[r] \setminus M(r) = \{x \in M \mid \tau(x) = r^2\}.$$

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From [4], we have

$$\int_{M\langle r\rangle} \sigma = \kappa,$$

where κ is a constant dependent only on the structure of M. We refer the reader to [13] and [14] for more details on p-Parabolic manifold.

Let $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic map defined on a p-Parabolic manifold M of dimension m, and let $\tilde{f} : M \longrightarrow C^{N+1}$ be a reduced representation of f. Then for some global meromorphic $(m-1,0)$-form B on M, we define the first B-derivative $\tilde{f}'_B$ of $\tilde{f}$ on local holomorphic coordinate chart (z, U_z) by

$$d\tilde{f} \wedge B = \tilde{f}'_B dz_1 \wedge \cdots \wedge dz_m,$$

and define inductively the kth B-derivative $\tilde{f}^{(k)}_B$ of $\tilde{f}$ by

$$d\tilde{f}^{(k-1)}_B \wedge B = \tilde{f}^{(k)}_B dz_1 \wedge \cdots \wedge dz_m$$

for $k = 1, \ldots N$. They are independent of the choice of the local holomorphic coordinate chart, and thus they are globally well defined. As a consequence, the kth preassociated map $\tilde{f}_k$ of f is defined by

$$\tilde{f}_k := \tilde{f} \wedge \tilde{f}'_B \wedge \cdots \wedge \tilde{f}^{(k)}_B : M \longrightarrow \wedge^{k+1} C^{N+1}$$

and the kth associated map f_k of f is defined by

$$f_k := [\tilde{f}_k] : M \longrightarrow \mathbb{P}(\wedge^{k+1} C^{N+1}) = \mathbb{P}^{n_k}(C), \quad n_k = \binom{N+1}{k+1} - 1$$

for $k = 1, \ldots, N$.

To establish the value distribution theory, we shall work on *admissible parabolic manifolds*, which satisfy the following assumptions:

$(\mathcal{A}_1)$: *(M, τ, ω) denotes a p-Parabolic manifold which possesses a globally defined meromorphic $(m-1)$-form B such that, for any linearly nondegenerate meromorphic map $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$, the kth associated map f_k is well defined for $k = 0, 1, \ldots N$, where we set $f_0 := f$.*

$(\mathcal{A}_2)$: *There exists a Hermitian holomorphic line bundle $(\mathfrak{L}, \mathfrak{h})$ that admits a holomorphic section μ such that, for some increasing function $\mathcal{Y}(\tau)$, we have*

$$m i_{m-1} |\mu|^2_{\mathfrak{h}} B \wedge \overline{B} \leq \mathcal{Y}(\tau)(dd^c\tau)^{p-1} \wedge \omega^{m-p},$$

where $i_{m-1} := (\frac{\sqrt{-1}}{2\pi})^{m-1}(m-1)!(-1)^{(m-1)(m-2)/2}$.

Remark 1.1. Let (M, τ) be a parabolic covering space of C^m with branching divisor β of π. Then it is an important class of admissible parabolic manifold (see [11]).

We set

$$\mathcal{T}_d := \{(i_0, \ldots, i_N) \in \mathbb{N}_0^{N+1} \mid i_0 + \cdots + i_N = d\}, \#\mathcal{T}_d = \binom{N+d}{N}.$$

Let Q be a homogeneous polynomial of degree d in $\mathbb{C}[x_0, \ldots, x_N]$ denote $\mathbf{x} = (x_0, \ldots, x_N)$, then we can write

$$Q(\mathbf{x}) = \sum_{I \in \mathcal{T}_d} a_I \mathbf{x}^I.$$

Let D be a hypersurface with degree d in $\mathbb{P}^N(\mathbb{C})$ which is define the homogeneous polynomial $Q \in \mathbb{C}[x_0, \ldots, x_N]$.In the case $d = 1$, we call D a hyperplane of $\mathbb{P}^N(\mathbb{C})$.

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Let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$, then the characteristic function of f, for a fixed $s > 0$ and any $r > s$ as

$$T_f(r,s) = \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} f^*\omega_{FS} \wedge (dd^c)^{p-1} \wedge \omega^{m-p}.$$

Let $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic map such that $f(M) \not\subset D$, then the Weil function of f with respect to D is defined by

$$\lambda_D(f) = \log \frac{\|\tilde{f}\|^d \cdot \|Q\|}{\mid Q(\tilde{f}) \mid},$$

where $\|\tilde{f}\| = \sqrt{\sum_{i=0}^n \mid \hat{f}_i \mid^2}$ for a reduced representation $\tilde{f} = (\hat{f}_0, \ldots, \hat{f}_N)$ on the local holomorphic coordinate chart (z, U_z) and $\|Q\| = \sqrt{\sum_I \mid a_I \mid^2}$. The proximity function and counting function of f with respect to D are defined respectively,

$$m_f(r, D) := \int_{M\langle r \rangle} \lambda_D(f)\sigma$$

and

$$N_f(r, s; D) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} \theta_f^D \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p},$$

where $\theta_f^D = div(Q(\tilde{f}))$on the local holomorphic coordinate chart (z, U_z). Let m be a positive integer, then the counting function with truncated level M is defined by

$$N_f^M(r, s; D) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} \theta_f^{M,D} \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p},$$

where $\theta_f^{M,D} = \min\{M, div(Q(\tilde{f}))\}$ on the local holomorphic coordinate chart (z, U_z).

From the Green–Jensen formula, the author in [4] derived the First Main Theorem as follows:

Theorem 1.2. [4] *Let $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a nonconstant meromorphic map defined on a p-Parabolic manifold M, and let D be a hypersurface of degree d such that $f(M) \not\subset D$. Then, for $r > s > 0$, we have*

$$dT_f(r, s) = N_f(r, s; D) + m_f(r, D) - m_f(s, D).$$

Then, the defect of f with respect to the hypersurface D is defined as

$$\delta_f(D) := \liminf_{r \to +\infty} \frac{m_f(r, D)}{dT_f(r, s)} = 1 - \limsup_{r \to +\infty} \frac{N_f(r, s; D)}{dT_f(r, s)}.$$

Accordingly, the defect of f with respect to the hypersurface D truncated to level M is defined by

$$\delta_f^M(D) := 1 - \limsup_{r \to +\infty} \frac{N_f^M(r, s; D)}{dT_f(r, s)}.$$

For each $0 \leq k \leq n - 1$ and a linearly non-degenerate meromorphic map on admissible parabolic manifold, define an important auxiliary function (see [4])

$$\Psi_k = \frac{m i_{m-1} f_k^* \omega_{FS}^k \wedge B \wedge \overline{B}}{(dd^c\tau)^p \wedge \omega^{m-p}} = \frac{\|\tilde{f}_{k-1}\|^2 \cdot \|\tilde{f}_{k+1}\|^2}{\|\tilde{f}_k\|^4} \cdot \frac{1}{A_p},$$

where ω_{FS}^k is the Fubini-Study metric on $\mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$, and $A_p,\ (1\le p\le m)$ is the pth symmetric polynomial of the matrix $(\tau_{a\bar{b}})$ with respect to the Kähler metric ω. Note that A_1 is the trace of $(\tau_{a\bar{b}})$, while A_m is the $\det(\tau_{a\bar{b}})(>0)$. We denote

$$Ric_p(r,s)=\int_s^r\frac{dt}{t^{2p-1}}\int_{M[t]}\theta^0_{A_p}\wedge(dd^c\tau)^{p-1}\wedge\omega^{m-p},$$

where $\theta^0_{A_p}$ is divisor zero of the holomorphic function A_p, and

$$m(\mathfrak{L};\mathfrak{r},\mathfrak{s})=\frac{1}{2}\int_{M\langle r\rangle}\log\frac{1}{|\mu|^2_{\mathfrak{h}}}\sigma-\frac{1}{2}\int_{M\langle s\rangle}\log\frac{1}{|\mu|^2_{\mathfrak{h}}}\sigma.$$

Definition 2. Let$V\subset\mathbb{P}^N(\mathbb{C})$ be a projective subvariety with dimension n. Let k be a positive integer and $D_1,\dots,D_k$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$. Let $l\ge n$ be a positive integer. We say that the hypersurfaces $D_1,\dots,D_k$ are in weak l-subgeneral position with respect to V if $k\le l+1$ such that either when $k=l+1$ we have $D_1\cap\cdots\cap D_{l+1}\cap V=\emptyset$ or when $k<l+1$; there exist hypersurfaces $S_1,\dots,S_{l+1-k}$ in $\mathbb{P}^N(\mathbb{C})$ such that $D_1\cap\cdots\cap D_k\cap S_1\cap\cdots\cap S_{l+1-k}\cap V=\emptyset$.

Definition 3. Let $l\ge n$ be a integer. We say that the hypersurfaces $D_1,\dots,D_q$ $(q\ge l+1)$ are in l-subgeneral position with respect to V if for any distinct indices $1\le j_1\le\cdots\le j_{l+1}\le q$, we have $D_{j_1}\cap\cdots\cap D_{j_{l+1}}\cap V=\emptyset$. If $l=n$ we said that $D_1,\dots,D_q\quad(q\ge l+1)$ are in general position in V.

Hence, if the hypersurfaces $D_1,\dots,D_q\quad(q\ge l+1)$ are in l-subgeneral position with respect to V, then for any set of hypersurfaces $\{D_s\}_{s\in S},\ S\subset\{1,\dots,q\},\ \#S\le l+1$ are in weak l-subgeneral position with respect to V.

As we known, In 2009, Ru[9] initially established a second main theorem for algebraically nondegenerate holomorphic maps from $\mathbb{C}$ into a projective subvariety $V\subset\mathbb{P}^n(\mathbb{C})$ with a family of hypersurfaces in general position w.r.t V. And then, in [8] Ru extended his result to the case of meromorphic mappings from a parabolic manifold. In 2019, Quang [7] initially proposed the replacing hypersurfaces method and using the method of Ru [9] to established the second main theorem for the case of family of hypersurfaces in N-subgeneral position w.r.t V.

Wong and Wong [14] introduced 'p-parabolic manifolds', and obtained certain First and Second Main Theorems. Q. Han [4] generalized their result for algebraically nondegenerate meromorphic maps over p-Parabolic manifolds intersecting with hypersurfaces in general position. Recently, Applying the method of Quang [7], Chen-Thin [1] proved the following the second main theorem for meromorphic mappings from a p-Parabolic manifold to V with a family of hypersurfaces in N-subgeneral position w.r.t V.

Theorem 1.3. [1] *Let $V\subset\mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension n. Let $f:M\longrightarrow V$ be an algebraically nondegenerate meromorphic mapping from a admissible p-Parabolic manifold M. Let $D_1,\dots,D_q$ be arbitrary hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are defined by homogeneous polynomials $Q_1,\dots,Q_q$ with degree $d_1,\dots,d_q$ respectively. Let $l\ge n$ be a integer. Then, for any $\varepsilon>0$ and $r>s>0$, we have*

$$\|\int_{M\langle r\rangle}\max_{K\subset\mathcal{K}}\sum_{j\in K}\frac{1}{d_j}\lambda_{D_j}(f)\sigma\le((l-n+1)(n+1)+\varepsilon)T_f(r.s)$$
$$+c(m(\mathfrak{L};\mathfrak{r},\mathfrak{s})+Ric_p(r,s)+\kappa\log^+\mathcal{Y}(r^2)+\kappa\log^+r)$$

where $\mathcal{K}$ *is the set of all subsets* $K \subset \{1,\dots,q\}$, $\#K \le l+1$, *such that the hypersurfaces* $\{D_j, j \in K\}$ *are in weak* l*-subgeneral position in* V *and* $c \gg 1$.

The notation "$\|$" means that the inequality holds for all $r \in [0,+\infty)$ except a set of finite Lebesgue measure.

Recently, Quang [6] considered the case of arbitrary families of hypersurfaces, not required to be in subgeneral position. To do so, he introduced a notion of distributive constant Δ of a family of hypersurface $\{D_i\}_{i=1}^q$ of $\mathbb{P}^N(\mathbb{C})$ in a subvariety $V \subset \mathbb{P}^N(\mathbb{C})$of dimension n, where $V \not\subset suppD_i (i = 1,\dots,q)$, as follows:

$$\Delta := \max_{\Gamma \subset \{1,\dots,q\}} \frac{\#\Gamma}{n - \dim(\bigcap_{j\in\Gamma} D_j) \cap V}$$

Here, $\dim \emptyset = -\infty$.

Remark 1.4. (see [6]) (1)If $D_1,\dots,D_q (q \ge n+1)$ are in general position with respect to V, then $\Delta = 1$.
(2)If $D_1,\dots,D_q (q \ge l+1)$ are in weak l-subgeneral position with respect to V, then we may see that for every subset $\{D_{i_1},\dots,D_{i_k}\} \quad (1 \le k \le l)$, one has

$$\dim(\bigcap_{j=1}^k D_{i_j}) \cap V \le \min\{n-1.l-k\}.$$

Hence $\Delta \le l-n+1$.

For more general, Quang [6] gave the following definition.

Definition 4. Let k be a number field and let V be a smooth projective subvariety of $\mathbb{P}^N(k)$ of dimension n. Let $D_0,\dots,D_l$ be l hypersurfaces in $\mathbb{P}^N(k)$. We say that the family $\{D_0,\dots,D_l\}$ is in $(t_1,t_2,\dots,t_n)$-subgeneral position with respect to V if for every $1 \le s \le n$ and t_s+1 hypersurfaces $D_{j_0},\dots,D_{j_{t_s}}$, we have

$$\dim(\bigcap_{i=0}^{t_s} D_{j_i}) \cap V(\bar{k}) \le n-s-1.$$

Remark 1.5. (see [6]) (1) If $\{D_0,\dots,D_l\}$ is in $(t_1,t_2,...,t_n)$-subgeneral position with respect to V, then its distributive constant in V satisfying

$$\Delta \le \max_{1\le k\le n} \frac{t_k}{n-(n-k)} = \max_{1\le k\le n} \frac{t_k}{k}.$$

(2)If $D_0,\dots,D_{q-1}$ $(q \ge l)$ are in l-subgeneral position with respect to V with index k (which is introduced by Q. Ji, Q. Yan and G. Yu [5] in 2019), then they are in $(1,2,\dots,k-1,l-n+k,l-n+k+1,\dots,l-1,l)$-subgeneral position with respect to V and hence $\Delta \le \frac{l-n+k}{k}$.

In this paper, we combine the method of Quang [7] with Ru [8] to prove the following general form of Second Main Theorem for meromorphic maps from p-Parabolic manifold into smooth projective variety intersecting with arbitrary families of hypersurfaces.

Main Theorem (I). *Let* $V \subset \mathbb{P}^N(\mathbb{C})$ *be a smooth complex projective variety of dimension* n. *Let* $D_1,\dots,D_q$ *be arbitrary hypersurfaces in* $\mathbb{P}^N(\mathbb{C})$ *with the distributive constant* Δ *in* V, $\deg D_j = d_j \quad (1 \le j \le q)$. *Let* $f : M \longrightarrow V$ *be an* *algebraically nondegenerate meromorphic mapping from a admissible p-Parabolic manifold M. Then, for any $\varepsilon > 0$ and $r > s > 0$, we have*

$$\|(q-\Delta(n+1)-\epsilon)T_f(r,s) \leq \sum_{j=1}^{q}\frac{1}{d_j}N(r,s;D_j)+c(m(\mathfrak{L};r,s)+Ric_p(r,s) \\ +\kappa\log^+\mathcal{Y}(r^2)+\kappa\log^+ r)$$

where $c \gg 1$.

By remark1.5 (2), we get

Corollary 1.6. *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension n. Let $D_1,\ldots,D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are defined by homogeneous polynomials $D_1,\ldots,D_q$, $\deg D_j = d_j \quad (1 \leq j \leq q)$, which are located in l-general position with index k in V. Let $f : M \longrightarrow V$ be an algebraically nondegenerate meromorphic mapping from a admissible p-Parabolic manifold M. Then, for any $\varepsilon > 0$ and $r > s > 0$, we have*

$$\|\left(q-\frac{l-n+k}{k}(n+1)-\epsilon\right)T_f(r,s) \leq \sum_{j=1}^{q}\frac{1}{d_j}N(r,s;D_j)+c(m(\mathfrak{L};r,s)+Ric_p(r,s) \\ +\kappa\log^+\mathcal{Y}(r^2)+\kappa\log^+ r)$$

where $c \gg 1$.

From the above corollary, set $l = n$ *and* $k = 1$, we get again the result of Ru [8]. When $k = 1$, we have noticed that $D_1,\ldots,D_q$ are located in weak l- general position in V. Thus the above corollary is the generalization of theorem 1.3.

On the Second Main Theorem with truncated level, we get the result as follows:
Main Theorem (II). *Let $V \subseteq \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension n. Let $D_1,\ldots,D_q$ be arbitrary hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ with the distributive constant Δ in V, $\deg D_j = d_j \quad (1 \leq j \leq q)$. Let $f : M \longrightarrow V$ be an algebraically nondegenerate meromorphic mapping from a admissible p-Parabolic manifold M. Then, for any $\varepsilon > 0$ and $r > s > 0$, we have*

$$\|(q-\Delta(n+1)-\epsilon)T_f(r,s) \leq \sum_{j=1}^{q}\frac{1}{d_j}N^{M_0}(r,D_j)+c(m(\mathfrak{L};r,s)+Ric_p(r,s) \\ +\kappa\log^+\mathcal{Y}(r^2)+\kappa\log^+ r)$$

where $c \gg 1$ and $M_0 = \deg(V)^{n+1}e^n d^{n^2+n}\Delta^n(2n+4)^n(n+1)^n(q!)^n\epsilon^{-n}$.

Reference [4], when M is assumed to be either an affine algebraic variety or an algebraic vector bundle over an affine algebraic variety or its projectivization, it follows that

$$m(\mathfrak{L};r,s)+Ric_p(r,s)+\kappa\log^+\mathcal{Y}(r^2) = O(\log^+ r).$$

We naturally have a stronger estimate

$$\liminf_{r\to+\infty}\frac{m(\mathfrak{L};r,s)+Ric_p(r,s)+\kappa\log^+\mathcal{Y}(r^2)}{T_f(r,s)} = 0.$$

By the Main Theorem (II), we have

Corollary 1.7. *Let $f : M \longrightarrow V \subseteq \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate meromorphic map from M, either an affine algebraic variety or an algebraic vector* *bundle over an affine algebraic variety or its projectivization, to a smooth projective algebraic variety V with $\dim V = n-1$, and Let $D_1, \ldots, D_q$ be arbitrary hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ with the distributive constant Δ in V. Then, we have*

$$\sum_{i=1}^{q} \delta_f^{M_0}(D_j) \leq \Delta(n+1).$$

2. Some lemmas

Firstly, We recall the notion of Chow weights and Hilbert weights from [9].

Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a projective variety of dimension n and degree δ over $\mathbb{C}$. To X we associate up to a constant scalar, a unique polynomial

$$F_X(\mathbf{u}_0, \cdots, \mathbf{u}_N) = F_X(u_{00}, \cdots, u_{0N}; \cdots; u_{n0}, \cdots, u_{nN})$$

in $n+1$ blocks of variables $\mathbf{u}_i = (u_{i0}, \cdots, u_{iN})$, $i = 0, \ldots, n$, which is called the Chow form of X, with the following properties: F_X is irreducible in $\mathbb{C}[u_{00}, \ldots, u_{nN}]$, F_X is homogeneous of degree δ in each block $\mathbf{u}_i$, $i = 0, \ldots, n$, and $F_X(\mathbf{u}_0, \ldots, \mathbf{u}_n) = 0$ if and only if $X \cap H_{\mathbf{u}_0} \cap \cdots \cap H_{\mathbf{u}_n} \neq \emptyset$. where $H_{\mathbf{u}_i}$, $i = 0, \ldots, n$, are the hyperplanes given by

$$\mathbf{u}_{i0}x_0 + \cdots + \mathbf{u}_{iN}x_N = 0.$$

Let F_X be the Chow form associated to X. Let $\mathbf{c} = (c_0, \cdots, c_N)$ be a tuple of real numbers. Let t be an auxiliary variable. We consider the decomposition

$$\begin{aligned} &F_X(t^{c_0}u_{00}, \ldots, t^{c_N}u_{0N}; \ldots; t^{c_0}u_{n0}, \ldots, t^{c_N}u_{nN}) \\ =&t^{e_0}G_0(\mathbf{u}_0, \ldots, \mathbf{u}_n) + \cdots + t^{e_r}G_r(\mathbf{u}_0, \ldots, \mathbf{u}_n). \end{aligned}$$

with $G_0, \ldots, G_r \in \mathbb{C}[u_{00}, \ldots, u_{0N}; \ldots; u_{n0}, \ldots, u_{nN}]$ and $e_0 > e_1 > \cdots > e_r$. The Chow weight of X with respect to $\mathbf{c}$ is defined by

$$e_X(\mathbf{c}) := e_0.$$

For each subset $J = \{j_0, \ldots, j_n\}$ of $\{0, \ldots, N\}$ with $j_0 < j_1 < \cdots < j_n$, we define the bracket

$$[J] = [J](\mathbf{u}_0, \ldots, \mathbf{u}_N) := \det(u_{ij_t}), i, t = 0, \ldots, n,$$

where $\mathbf{u}_i = (u_{i0}, \ldots, u_{iN})$ denotes the blocks of $N+1$ variables. Let $J_1, \ldots, J_\beta$ with $\beta = \binom{N+1}{n+1}$ be all subsets of $\{0, \ldots, N\}$ of cardinality $n+1$. Then the Chow form F_X of X can be written as a homogeneous polynomial of degree δ in $[J_1], \ldots, [J_\beta]$. We may see that for $\mathbf{c} = (c_0, \cdots, c_N) \in \mathbb{R}^{N+1}$ and for any J among $J_1, \ldots, J_\beta$,

$$\begin{aligned} &[J](t^{c_0}u_{00}, \ldots, t^{c_N}u_{0N}; \ldots; t^{c_0}u_{n0}, \ldots, t^{c_N}u_{nN}) \\ =&t^{\sum_{j \in J} c_j}[J](u_{00}, \ldots, u_{0N}; \ldots; u_{n0}, \ldots, u_{nN}). \end{aligned}$$

For $\mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1}$, we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \ldots x_N^{a_N}$. Let $I = I_X$ be the prime ideal in $\mathbb{C}[x_0, \ldots, x_N]$ deffning X. Let $\mathbb{C}[x_0, \ldots, x_N]_u$ denote the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ of degreee u(including 0). Put $I_u := \mathbf{C}[x_0, \ldots, x_N]_u \cap I$ and define the Hilbert function H_X of X, for $u = 1, 2, \ldots,$

$$H_X(u) := \dim\left(\frac{\mathbb{C}[x_0, \ldots, x_N]_u}{I_u}\right).$$

By the usual theory of Hilbert polynomials,

$$H_X(u) = \delta \cdot \frac{u^N}{N!} + O(u^{N-1}).$$

The u-th Hilbert weight $S_X(u, \mathbf{c})$ of X with respect to the tuple $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{n+1}$ is defined by

$$S_X(u, \mathbf{c}) := \max \left(\sum_{i=1}^{H_X(u)} \mathbf{a}_i \cdot \mathbf{c} \right),$$

where the maximum is take over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{H_X(u)}}$ whose residue classes modulo I form a basis of $\frac{\mathbb{C}[x_0,\dots,x_N]_u}{I_u}$.

According to Mumford,

$$S_X(u, \mathbf{c}) = e_X(\mathbf{c}). \frac{u^{N+1}}{(N+1)!} + O(u^N),$$

this implies that

$$\lim_{u \to \infty} \frac{1}{uH_X(u)} \cdot S_X(u, \mathbf{c}) = \frac{1}{(n+1)\delta} \cdot e_X(\mathbf{c}).$$

We call $\frac{1}{uH_X(u)} \cdot S_X(u, \mathbf{c})$ the u-th normalized Hilbert weight and $\frac{1}{(N+1)\delta} \cdot e_X(\mathbf{c})$ the normalized Chow weight of X with respect to $\mathbf{c}$.

The following lemmas are due to J. Evertse and R. Ferretti.

Lemma 2.1. *(Theorem 4.1[2]) Let $X \subset \mathbb{P}^N(\mathbb{C})$ be an algebraic variety of dimension n and degree δ. let $u > \delta$ be an integer and let $\mathbf{c} = (c_0, \cdots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$. Then*

$$\frac{1}{uH_X(u)} S_X(u, \mathbf{c}) \geq \frac{1}{(n+1)\delta} e_X(\mathbf{c}) - \frac{(2n+1)\delta}{u} \cdot \left(\max_{i=0,\dots,N} c_i \right).$$

Lemma 2.2. *(see [3], [9],) Let Y be a subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ of dimension n and degree δ. Let $\mathbf{c} = (c_1, \dots, c_q)$ be a tuple of positive reals. Let $\{i_0, \cdots, i_n\}$ be a subset of $\{1, \dots, q\}$ such that*

$$Y \cap \{y_{i_0} = \cdots = y_{i_n} = 0\} = \emptyset.$$

Then

$$e_Y(\mathbf{c}) \geq (c_{i_0} + \cdots + c_{i_n})\delta.$$

The following general form of the second main theorem is due to Han [4].

Lemma 2.3. [4] *Let $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic map defined on a p-Parabolic manifold M satisfying the general condition $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and let $\{H_j\}_{j=1}^q$ be q arbitrary hyperplanes. Then, for $r > s > 0$, we have*

$$\begin{aligned} \| \int_{M\langle r \rangle} \max_{K \subset \mathcal{K}} \sum_{j \in K} \frac{1}{d_j} \lambda_{H_j}(f) \sigma &\leq (N+1) T_f(r.s) - N_{Ramf}(r, s) \\ &+ \frac{N(N+1)}{2} (m(\mathfrak{L}; \mathfrak{r}, \mathfrak{s}) + Ric_p(r, s) + \kappa \log^+ T_f(r, s)) \\ &+ \frac{\kappa N(N+1)}{2} (\log^+ m(\mathfrak{L}; r, s) + \log^+ Ric_p(r, s) + \log^+ \mathcal{Y}(r^2) + \kappa \log^+ r). \end{aligned}$$

where maximum is taken over all subsets K of $\{1, \dots, q\}$ such that the generating linear forms of the hyperplanes in each set are linearly independent and $N_{Ramf}(r, s)$ is the counting function of the ramification divisor $div \tilde{f}_N$.

Lemma 2.4. *([11] Lemma 13.3) Let M be a p-Parabolic manifold of dimensional m. Let $f : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping which is linearly nondegenerate over $\mathbb{C}$. Let $\{H_j\}_{j=1}^q$ be a family of hyperplanes of $\mathbb{P}^N(\mathbb{C})$ in general position. We have*

$$\sum_{j=1}^{q}\left(\theta_f^{H_j} - \theta_f^{N,H_j}\right) \leq div\tilde{f}_N.$$

The following two lemmas are the important key lemmas of [6] to deal with the case of arbitrary families of hypersurfaces.

Lemma 2.5. *([6] Lemma 3.1) Let $t_0, t_1, \ldots, t_n$ be $n+1$ integers such that $t_0 < t_1 < \cdots < t_n$, and let $\Delta = \max_{1\leq s\leq n} \frac{t_s - t_0}{s}$. Then for every n real numbers $a_0, a_2, \ldots a_{n-1}$ with $a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq 1$, we have*

$$a_0^{t_1-t_0} a_1^{t_2-t_1} \cdots a_{n-1}^{t_n-t_{n-1}} \leq (a_0 a_1 \cdots a_{n-1})^{\Delta}.$$

Lemma 2.6. *([6] Lemma 3.2) Let V be a projective subvariety of $\mathbb{P}^N(\mathbb{C})$ of dimension n. Let $D_0, \ldots, D_l$ be l hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of the same degree $d \geq 1$, such that $\bigcap_{i=0}^{l} D_i \cap V = \emptyset$ and*

$$\dim\left(\bigcap_{i=0}^{s} D_i \cap V\right) = n - u, \quad t_{u-1} \leq s < t_u, \quad 1 \leq u \leq n,$$

where $t_0, t_1, \ldots, t_n$ integers with $0 = t_0 < t_1 < \cdots < t_n = l$. Then there exist $n+1$ hypersurfaces $P_0, \ldots, P_n$ in $\mathbb{P}^N(\mathbb{C})$ of the forms

$$P_u = \sum_{j=1}^{t_u} c_{uj} D_j, \quad c_{uj} \in \mathbf{C}, \quad u = 0, \ldots, n$$

such that $\bigcap_{u=0}^{n} P_u \cap V = \emptyset$.

3. The proof of main theorem (I)

Proof. By the First Main theorem, it is suffice to consider the case where $\Delta < \frac{q}{n+1}$. Note that $\Delta \geq 1$, hence $q > n+1$. If there exists $i \in \{1, \ldots, q\}$ such that $\bigcap_{j=1, j\neq i} D \cap V \neq \emptyset$, then

$$\Delta \geq \frac{q-1}{n} > \frac{q}{n+1}.$$

This is a contradiction. Therefore, $\bigcap_{j=1, j\neq i} D_j \cap V = \emptyset$ for all $i \in \{1, \ldots, q\}$. Firstly, we will prove the theorem for the case where all hypersurfaces $D_j (1 \leq j \leq q)$ are of the same degree d. Let Q_j, $1 \leq j \leq q$, be homogeneous polynomials in$\mathbb{C}[x_0, \ldots, x_N]$ of degree d_j which is defined by D_j. We denote by $\mathcal{I}$ the set of all permutations of the set $\{1, \ldots, q\}$. Denote by n_0 the cardinality of $\mathcal{I}$, $n_0 = q!$, and we write $\mathcal{I} = \{I_1, \ldots, I_{n_0}\}$, where $I_i = (I_i(0), \ldots, I_i(q-1)) \in \mathbb{N}^q$ and $I_1 < I_2 < \cdots < I_{n_0}$ in the lexicographic order.

For each $I_i \in \mathcal{I}$, since $\bigcap_{j=1, j\neq i} D_j \cap V = \emptyset$, there exist $n+1$ integers $t_{i,0}, t_{i,1}, \ldots, t_{i,n}$ with $0 = t_{i,0} < \cdots < t_{i,n} = l_i$, where $l_i \leq q-2$ such that $\bigcap_{j=0}^{l_i} D_{I_i(j)} \cap V = \emptyset$ and

$$\dim\left(\bigcap_{j=0}^{s} D_{I_i(j)}\right) \cap V = n - u, \quad \forall\, t_{i,u-1} \leq s < t_{i,u}, \quad 1 \leq u \leq n.$$

Then $\Delta > \frac{t_{i,u}-t_{i,0}}{u}$ for all $1 \leq u \leq n$. Denote by $P_{i,0}, \ldots, P_{i,n}$ the hypersurfaces obtained in Lemma 2.6 with respect to the hypersurfaces $D_{I_i(0)}, \ldots, D_{I_i(l_i)}$. We may choose a positive constant $B \geq 1$, commonly for all $I_i \in \mathcal{I}$, such that

$$| P_{i,j}(\mathbf{x}) | \leq B \max_{1 \leq s \leq t_{i,j}} | Q_{I_i(j)}(\mathbf{x}) |,$$

for all $0 \leq j \leq n$ and $\mathbf{x} = (x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$.

Consider a reduced representation $\tilde{f} = (f_0, \ldots, f_n) : M \longrightarrow \mathbb{C}^{N+1}$ of f. Fix an element $I_i \in \mathcal{I}$. Denote by $S(i)$ the set of all points $z \in M \setminus \left(\bigcup_{i=1}^q Q_i(\tilde{f})^{-1}(\{0\}) \cup I_f \right)$,where I_f is indeterminacy of f, such that

$$| Q_{I_i(0)}(\tilde{f})(z) | \leq | Q_{I_i(1)}(\tilde{f})(z) | \leq \cdots \leq | Q_{I_i(q-1)}(\tilde{f})(z) | .$$

Since $\bigcap_{j=0}^{l_i} D_{I_i(j)} \cap V = \emptyset$, there exists a positive constant A, which is chosen common for all I_i, such that

$$\|\tilde{f}(z)\|^d \leq \max_{0 \leq j \leq l_i} | Q_{I_i(j)}(\tilde{f})(z) |, \quad z \in S(i).$$

Therefore, for $z \in S(i)$, By Lemma 2.5, we have

$$\begin{aligned}
\prod_{i=1}^q \frac{\|\tilde{f}(z)\|^d}{| Q_i(\tilde{f})(z) |} &\leq A^{q-l_j} \prod_{j=0}^{l_j-1} \frac{\|\tilde{f}(z)\|^d}{| Q_{I_i(j)}(\tilde{f})(z) |} \leq A^{q-l_j} \prod_{j=0}^{n-1} \left(\frac{\|\tilde{f}(z)\|^d}{| Q_{I_i(t_j)}(\tilde{f})(z) |} \right)^{t_{i.j+1}-t_{i,j}} \\
&\leq A^{q-l_j} \prod_{j=0}^{n-1} \left(\frac{\|\tilde{f}(z)\|^d}{| Q_{I_i(t_j)}(\tilde{f})(z) |} \right)^{\Delta} \\
&\leq A^{q-l_j} B^{n\Delta} \prod_{j=0}^{n-1} \left(\frac{\|\tilde{f}(z)\|^d}{| P_{i,j}(\tilde{f})(z) |} \right)^{\Delta}
\end{aligned}$$

Since the number of hypersurfaces in the proof is finite, we may choose a positive constant c such that for all $1 \leq j \leq q$ and all $\mathbf{x} = (x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$, we have

$$Q_j(\mathbf{x}) \leq c \|\mathbf{x}\|^d.$$

Thus $| P_{i,n}(\tilde{f})(z) | \leq B \max_{1 \leq s \leq t_{i,n}} | D_{I_i(n)}(\mathbf{x}) | \leq Bc \|\tilde{f}(z)\|^d$. It yields that

$$\prod_{i=1}^q \frac{\|\tilde{f}(z)\|^d}{| Q_i(\tilde{f})(z) |} \leq A^{q-l_j} B^{(n+1)\Delta} c^{\Delta} \prod_{j=0}^n \left(\frac{\|\tilde{f}(z)\|^d}{| P_{i,j}(\tilde{f})(z) |} \right)^{\Delta}. \tag{1}$$

Consider the mapping Φ from V into $\mathbb{P}^{l-1}(\mathbb{C}) \quad (l = n_0(n+1))$, which maps a point $\mathbf{x} = (x_0 : \cdots : x_N) \in V$ to the point $\Phi(\mathbf{x}) \in \mathbb{P}^{l-1}(\mathbb{C})$ given by

$$\Phi(\mathbf{x}) = (P_{1,0}(x) : \cdots : P_{1,n}(x) : P_{2,0}(x) \cdots : P_{2,n}(x) : \cdots : P_{n_0,0}(x) : \cdots : P_{n_0,n}(x)),$$

where $x = (x_0, \ldots, x_N)$. Let $Y = \Phi(V)$. Since $\bigcap_{j=0}^n P_{1,j} \cap V = \emptyset$, Φ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{l-1}(\mathbb{C})$ with $\dim Y = n$ and $\delta := degY \leq d^n \deg V$ (see,[10]). For $\mathbf{a} = (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l$ and $\mathbf{y} = (y_1, \ldots, y_l)$, we denote $\mathbf{y}^{\mathbf{a}} := y_1^{a_1} \cdots y_l^{a_l}$. Let u be a positive integer and set

$$n_u := H_Y(u) - 1, \quad \xi_u := \binom{l+u-1}{u} - 1.$$

Follow from [9], consider the Veronese embedding

$$\Phi_u : \mathbb{P}^{l-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{\xi_u}(\mathbb{C}) : [\mathbf{y}] \longrightarrow [\mathbf{y}^{\mathbf{a}_0} : \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}].$$

where $\mathbf{y}^{\mathbf{a}_0}, \ldots, \mathbf{y}^{\mathbf{a}_{\xi_u}}$ are the monomials of degree u in $y_1, \ldots, y_l$ in some order. Denote by Y_u the smallest linear subvariety of $\mathbb{P}^{\xi_u}(\mathbb{C})$ containing $\Phi_u(Y)$. Then, clearly, a linear form $\sum_{i=0}^{\xi_u} \gamma_i z_i$ vanishes identically on Y_u if and only if $\sum_{i=0}^{\xi_u} \gamma_i \mathbf{y}^{\mathbf{a}_i}$, as a polynomial of degree u, vanishes identically on Y. In other words, there is an isomorphism

$$\mathbb{C}[y_1, \ldots, y_l]_u / \mathfrak{I}_u(Y) \simeq (Y_u)^\vee : \ \mathbf{y}_i^{\mathbf{a}} \to z_i.$$

where $\mathfrak{I}(Y)$ is the prime ideal in $\mathbb{C}[y_1, \ldots, y_l]$ define Y, $\mathbb{C}[y_1, \ldots, y_l]_u$ is the vector space of homogeneous polynomials in$\mathbb{C}[y_1, \ldots, y_l]$ of degree u (including 0), $(Y_u)^\vee$ is the vector space of linear forms in $\mathbb{C}[z_0, \ldots, z_{\xi_u}]$ modulo the linear forms vanishing identically on Y_u. Hence Y_u is an n_u-dimensional linear subspace of $\mathbb{P}^{\xi_u}(\mathbb{C})$. Thus, there are linear forms $L_0, \ldots, L_{\xi_u} \in \mathbb{C}[w_0, \ldots, w_{n_u}]$ such that the map

$$\Psi_u : \mathbf{w} \in \mathbb{P}^{n_u}(\mathbb{C}) \longrightarrow [L_0(\mathbf{w}) : \cdots : L_{\xi_u}(\mathbf{w})] \in Y_u$$

is a linear isomorphism from $\mathbb{P}^{n_u}(\mathbb{C})$ to Y_u. Therefore, $\Psi_u^{-1} \circ \Phi_u : \ Y \longrightarrow \mathbb{P}^{n_u}(\mathbb{C})$ is an injective map such that

$$\Psi_u^{-1} \circ \Phi_u(\mathbf{y}) = [\mathbb{L}_0([\mathbf{y}^{\mathbf{a}_0} : \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}]) : \cdots : \mathbb{L}_{n_u}([\mathbf{y}^{\mathbf{a}_0} : \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}])]$$

for all $\mathbf{y} \in Y$, where $\mathbb{L}_0, \ldots, \mathbb{L}_{n_u}$ are linear forms independent in $\mathbb{P}^{\xi_u}(\mathbb{C})$. Then $\{\mathbb{L}_0([\mathbf{y}^{\mathbf{a}_0} \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}]), \ldots, \mathbb{L}_{n_u}([\mathbf{y}^{\mathbf{a}_0} : \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}])\}$ is a base of $\mathbb{C}[y_1, \ldots, y_l]_u / \mathfrak{I}_u(Y)$. Denote $\phi_i = \mathbb{L}_0([\mathbf{y}^{\mathbf{a}_0} \cdots : \mathbf{y}^{\mathbf{a}_{\xi_u}}]), \ i = 0, \ldots, n_u$. We consider $F = \Psi_u^{-1} \circ \Phi_u \circ \Phi \circ f :$ $M \longrightarrow \mathbb{P}^{n_u}(\mathbb{C})$ with the following reduced representation

$$\tilde{F} = (\phi_0(\Phi \circ \tilde{f}), \ldots, \phi_{n_u}(\Phi \circ \tilde{f}))$$

on each local chart (z, U_z). Furthermore, F is linearly nondegenerate, since f is algebraically nondegenerate.

Now, for every fixed $i \in \{1, \ldots, n_0\}$ and a point $z \in S(i)$, we define

$$\mathbf{c} = (c_{1,0,z}, \ldots, c_{1,n,z}, c_{2,0,z}, \ldots, c_{2,n,z}, c_{n_0,0,z}, \ldots, c_{n_0,n,z}) \in \mathbb{Z}^l$$

where

$$c_{i,j,z} := \log \frac{\|\tilde{f}(z)\|^d \|P_{i,j}\|}{\mid P_{i,j}(\tilde{f})(z) \mid} \ for\ i = 1, \ldots, n_0 \ and\ j = 0, \ldots, n.$$

Then $c_{i,j,z} \geq 0$ for all i and j. By the definition of the Hilbert weight, there are $\mathbf{a}_{1,z}, \ldots, \mathbf{a}_{H_Y(u),z} \in \mathbb{N}^l$ with

$$\mathbf{a}_{i,z} = (a_{i,1,0,z}, \ldots, a_{i,1,n,z}, \ldots, a_{i,n_0,0,z}, \ldots, a_{i,n_0,n,z}),$$

where $a_{i,j,s,z} \in \{1, \ldots, \xi_u\}$, such that the residue classes modulo $(I_Y)_u$ of $\mathbf{y}^{\mathbf{a}_{1,z}}, \ldots, \mathbf{y}^{\mathbf{a}_{H_Y(u),z}}$ form a basic of $\mathbb{C}[y_1, \ldots, y_l]_u / \mathfrak{I}_u(Y)$ and

$$S_Y(u, \mathbf{c}_z) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_{i,z} \cdot \mathbf{c}_z.$$

Since $\mathbf{y}^{\mathbf{a}_{i,z}}, 1 \leq i \leq H_Y(u)$ are basis of $\mathbb{C}[y_1, \ldots, y_l]_u / \mathfrak{I}_u(Y)$, then there exist $H_Y(u)$ independent linear forms $\mathcal{L}_z = \{L_{j,z}, 1 \leq j \leq H_Y(u)\}$ such that

$$\mathbf{y}^{\mathbf{a}_{j,z}} = L_{j,z}(\phi_0, \ldots, \phi_{n_u}), \ 1 \leq j \leq H_Y(u).$$

We denote $\mathcal{L} = \cup_z \mathcal{L}_z$, then $\mathcal{L}$ is finite since $\#\mathcal{L} \leq \binom{\xi_u+1}{n_u+1}$. We have

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$$\log \prod_{i=1}^{H_Y(u)} \mid L_{i,z}(\tilde{F}(z)) \mid = \log \prod_{i=1}^{H_Y(u)} \prod_{\substack{1\le t\le n_0\\ 0\le j\le n}} \mid P_{t,j}(\tilde{f}(z)) \mid^{a_{i,j,z}}$$
$$= -S_Y(u, \mathbf{c}_z) + duH_Y(u)\log\|\tilde{f}(z)\| + O(uH_Y(u)).$$

It implies that

$$\log \prod_{i=1}^{H_Y(u)} \frac{\|\tilde{F}(z)\|\|L_{i,z}\|}{\mid L_{i,z}(\tilde{F}(z)) \mid} = S_Y(u, \mathbf{c}_z) - duH_Y(u)\log\|\tilde{f}(z)\|$$
$$+ H_Y(u)\log\|\tilde{F}(z)\| + O(uH_Y(u)).$$

Thus

$$S_Y(u, \mathbf{c}_z) \le \max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{\|\tilde{F}(z)\|\|L\|}{\mid L(\tilde{F}(z)) \mid} + duH_Y(u)\log\|\tilde{f}(z)\|$$
$$- H_Y(u)\log\|\tilde{F}(z)\| + O(uH_Y(u)). \tag{2}$$

where the maximum is taken over all subsets $\mathcal{J} \subset \mathcal{L}$ with $\#\mathcal{J} = H_Y(u)$ and $\{L \mid L \in \mathcal{J}\}$ is linearly independent. From Lemma 2.1, we have

$$\frac{1}{uH_Y(u)} S_Y(u, \mathbf{c}_z) \ge \frac{1}{(n+1)\delta} e_Y(\mathbf{c}_z) - \frac{(2n+1)\delta}{u} \max_{\substack{1\le i\le n_0\\ 0\le j\le n}} c_{i,j,z}. \tag{3}$$

Combining (2) and (3), we get

(4)
$$\frac{1}{(n+1)\delta} e_Y(\mathbf{c}_z) \le \frac{1}{uH_Y(u)} \left(\max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{\|\tilde{F}(z)\|\|L\|}{\mid L(\tilde{F}(z)) \mid} - H_Y(u)\log\|\tilde{F}(z)\| \right)$$
$$+ d\log\|\tilde{f}(z)\| + \frac{(2n+1)\delta}{u} \sum_{\substack{1\le i\le n_0\\ 0\le j\le n}} \log \frac{\|\tilde{f}(z)\|^d\|P_{i,j}\|}{\mid P_{i,j}(\tilde{f})(z) \mid} + O(\frac{1}{u}).$$

Since $\{P_{i,0} = \cdots = P_{i,n} = 0\} \cap V = \emptyset$ for $1 \le i \le n_0$, by Lemma 2, we get

$$e_Y(\mathbf{c}_z) \ge (c_{i,0,z} + \cdots + c_{i,n,z}) \cdot \delta = \left(\sum_{0\le j\le n} \log \frac{\|\tilde{f}(z)\|^d\|P_{i,j}\|}{\mid P_{i,j}(\tilde{f})(z) \mid} \right) \cdot \delta. \tag{5}$$

From (1),(4) and (5),we obtain

(6)
$$\frac{1}{\Delta} \log \prod_{i=1}^{q} \frac{\|\tilde{f}(z)\|^d}{\mid Q_i(\tilde{f})(z) \mid} \le \frac{n+1}{uH_Y(u)} \left(\max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{\|\tilde{F}(z)\|\|L\|}{\mid L(\tilde{F}(z)) \mid} - H_Y(u)\log\|\tilde{F}(z)\| \right)$$
$$+ d(n+1)\log\|\tilde{f}(z)\| + \frac{(2n+1)(n+1)\delta}{u} \sum_{\substack{1\le i\le n_0\\ 0\le j\le n}} \log \frac{\|\tilde{f}(z)\|^d\|P_{i,j}\|}{\mid P_{i,j}(\tilde{f})(z) \mid} + O(\frac{1}{u}).$$

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By Lemma 2.3, for any $\epsilon' > 0,\ r > s > 0$ large enough, we have

(7)

$$\| \int_{M\langle r\rangle} \max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{\|\tilde{F}(z)\|\|L\|}{\mid L(\tilde{F}(z))\mid}\sigma \leq (H_Y(u)+\epsilon')T_F(r.s) - N_{RamF}(r,s)$$
$$+ (\frac{H_Y(u)(H_Y(u)-1)}{2} + \epsilon')(m(\mathfrak{L};r,s) + Ric_p(r,s) + \kappa\log^+\mathcal{Y}(r^2) + \kappa\log^+ r).$$

where maximum is taken over all subsets $\mathcal{J}\subset\mathcal{L}$ with $\#\mathcal{J} = H_Y(u)$ and $\{L \mid L\in\mathcal{J}\}$ are linearly independent.

However,In order to take integration over $M\langle r\rangle$, we now encounter a problem, that is, the functions $\log\|\tilde{F}(z)\|$ and $\log\|\tilde{f}(z)\|$ are usually not globally defined. Hence, we use the concept of 'reduced representation sections' of F and f (see [11]) to avoid this difficulty. We only do this for F in detail, as the case for f is similar (ref. [4]).

Set $\{\tilde{F}_\alpha, U_\alpha\}$ to be a system of local reduced representations of $\tilde{F}$ such that, on $U_\alpha\cap U_\beta\neq\emptyset$, we have

$$\tilde{F}_\alpha = h_{\alpha\beta}\tilde{F}_\beta$$

for a non-vanishing holomorphic function $h_{\alpha\beta}: U_\alpha\cap U_\beta\to\mathbb{C}^*$. Then, $\{h_{\alpha\beta}\}$ forms a basic cocycle so that there exists a holomorphic line bundle $\mathbb{H}_F$ on M, with a holomorphic frame atlas $\{s_F^\alpha, U_\alpha\}$ such that, on $U_\alpha\cap U_\beta\neq\emptyset$, we have

$$s_F^\alpha = h_{\beta\alpha}s_F^\beta,$$

which is called the hyperplane section bundle of F. Now, define a holomorphic section

$$\tilde{F}_\alpha^\star(z) := (z, \tilde{F}_\alpha(z)) \in \mathit{\Gamma}(U_\alpha, M\times\mathbb{C}^{n_u+1}).$$

Hence, there is a global holomorphic section $\chi\in\mathit{\Gamma}(M,(M\times\mathbb{C}^{n_u+1})\otimes\mathcal{H}_F)$, called the standard reduced representation section of F, such that $\chi\mid_{U_\alpha} = \tilde{F}_\alpha^\star\otimes s_\alpha^F$.

Set ζ_1 to be the standard Hermitian metric along the fibres of the trivial bundle $M\times\mathbb{C}^{n_u+1}$ and $\wp_1$ to be a Hermitian metric along the fibres of $\mathcal{H}_F$. Then, we can apply our Green–Jensen formula to the function $\log\|\chi\|_{\zeta_1\otimes\wp_1}$to get

$$(8)\ \ T_F(r,s) - T_{\mathcal{H}_F}(r,s) = \int_{M\langle r\rangle}\log\|F\|_{\zeta_1}\otimes\|s^F\|_{\wp_1}\sigma - \int_{M\langle s\rangle}\log\|F\|_{\zeta_1}\otimes\|s^F\|_{\wp_1}\sigma$$

where $T_{\mathcal{H}_F}(r,s)$ is defined via the pull-back of the first Chern form on $(\mathcal{H}_F,\wp_1)$. Analogously,

$$(9)\ \ T_f(r,s) - T_{\mathcal{H}_f}(r,s) = \int_{M\langle r\rangle}\log\|f\|_{\zeta_2}\otimes\|s^f\|_{\wp_2}\sigma - \int_{M\langle s\rangle}\log\|f\|_{\zeta_2}\otimes\|s^f\|_{\wp_2}\sigma.$$

The construction of F leads to

$$(\|F\|_{\zeta_1})\mid_{U_\alpha} = (\|f\|_{\zeta_2})\mid_{U_\alpha}.$$

Thus $T_{\mathcal{H}_F}(r,s) = duT_{\mathcal{H}_f}(r,s)$. Combining with (8) and (9), yields

$$T_F(r,s) = duT_f(r,s).$$

Taking integral of (6) and combining it with (7), we have

$$\tag{10} \begin{aligned} \|\frac{1}{d}\sum_{j=1}^{q} m_f(r,D_j) \leq{}& \Delta(n+1)T_f(r,s) - \frac{\Delta(n+1)}{udH_Y(u)}N_{RamF}(r,s) + \epsilon'\frac{\Delta(n+1)}{H_Y(u)}T_f(r,s) \\ &+ \frac{\Delta(n+1)}{udH_Y(u)}\left(\frac{H_Y(u)(H_Y(u)-1)}{2} + \epsilon'\right)(m(\mathfrak{L};r,s) + Ric_p(r,s) + \kappa\log^+\mathcal{Y}(r^2) \\ &+ \kappa\log^+ r) + \frac{\Delta(2n+1)(n+1)\delta}{ud}\sum_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} m_f(r,P_{i,j}) + O(1). \end{aligned}$$

Using the First Main Theorem, for r large enough, we assume $T_f(r,s)\geq 1$, then

$$\begin{aligned} \sum_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} m_f(r,P_{i,j}) \leq{}& d\left((n+1)n_0T_f(r,s) + \frac{1}{d}\sum_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} m_f(s,P_{i,j})\right) \\ \leq{}& d\left((n+1)n_0 + \frac{1}{d}\sum_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} m_f(s,P_{i,j})\right)T_f(r,s) \end{aligned}$$

Now we choose $u\geq u_0$ large enough and ϵ', such that

$$\tag{11} \begin{aligned} &\frac{\Delta(2n+1)(n+1)\delta}{u_0}\left((n+1)n_0 + \frac{1}{d}\sum_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} m_f(s,P_{i,j})\right) < \frac{\epsilon}{4}, \\ &\epsilon'\frac{\Delta(n+1)}{H_Y(u_0)} < \frac{\epsilon}{4}. \end{aligned}$$

Denote $c = \max\{1, \frac{\Delta(n+1)}{udH_Y(u_0)}\left(\frac{H_Y(u_0)(H_Y(u_0)-1)}{2} + \epsilon'\right)\}$. Using First Main Theorem and combining (10) and (11), notice $N_{RamF}(r,s)\geq 0$, then

$$\tag{12} \begin{aligned} \|(q-\Delta(n+1)-\epsilon)T_f(r,s) \leq{}& \sum_{j=1}^{q}\frac{1}{d}N(r,s;D_j) + c(m(\mathfrak{L};r,s) + Ric_p(r,s) \\ &+ \kappa\log^+\mathcal{Y}(r^2) + \kappa\log^+ r). \end{aligned}$$

Now, for the general case where $D_i(1\leq i\leq q)$ is of the degree d_i, then all $D^{\frac{d}{d_i}}$ are of the same degree $d(1\leq i\leq q)$, where d is the l.c.m of $d_j, j=1,\ldots,q$. Applying the above result, the theorem is proved.

4. The proof of main theorem (II)

Proof. We can replace $D_i(1\leq i\leq q)$ by $D^{\frac{d}{d_i}}$ if necessary, where d is the l.c.m of $d_j, j=1,\ldots,q$, we may assume that $D_1,\ldots,D_q$ have the same degree of d.

From (10), we need estimate the quantity $N_{RamF}(r,s)$. Without loss of generality, we may assume that $z\in S(1)$, where $I_1=(1,\ldots,q)$ and moreover

$$\theta_f^{D_1} \geq \theta_f^{D_2} \geq \cdots \geq \theta_f^{D_q}$$

where $\theta_f^{D_j}(z) = div(Q_j(\tilde{f}))(z),\ j = 1,\dots,q$. Since $\bigcap_{j=1}^{l_1+1} D_j \cap V = \emptyset$, then $div(Q_j(\tilde{f}))(z) = 0$ for $j \geq l_1 + 1$. Set

$$c_{i,j} = \max\{0, div(P_{i,j}(\tilde{f}))(z) - n_u\}$$

and

$$\mathbf{c} = (c_{1,0},\dots,c_{1,n},\dots,c_{n_0,0},\dots,c_{n_0,n}) \in \mathbb{Z}^l_{\geq 0}.$$

Then there are

$$\mathbf{a}_i = (a_{i,1,0},\dots,a_{i,1,n},\dots,a_{i,n_0,0},\dots,a_{i,n_0,n}) \in \{1,\dots,\xi_u\}.$$

such that $\mathbf{y}^{\mathbf{a}_1},\dots,\mathbf{y}^{\mathbf{a}_{H_Y(u)}}$ is a basic of $\mathbb{C}[y_1,\dots,y_l]_u/\mathfrak{I}_u(Y)$ and

$$S_Y(u,\mathbf{c}) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \cdot \mathbf{c}.$$

Similarly as above, we write $\mathbf{y}^{\mathbf{a}_i} = L_i(\phi_1, \dot{\phi}_{H_Y(u)})$, where $L_1,\dots,L_{H_Y(u)}$ are independent linear forms in variables $y_{i,j}(1 \leq i \leq n_0,\ 0 \leq j \leq n)$. For any divisor ν on M, we denote ν^u by a divisor such that $\nu^u(z) = \min u, \nu(z)$. Then we see

$$\begin{aligned} div(L_i(\tilde{F}))(z) - div^{n_u}(L_i(\tilde{F}))(z) &\geq \sum_{\substack{1\leq j\leq n_0\\ 0\leq s\leq n}} a_{i,j,s}(div(P_{j,s}(\tilde{f})) - div^{n_u}(P_{j,s}(\tilde{f}))) \\ &= \sum_{\substack{1\leq j\leq n_0\\ 0\leq s\leq n}} a_{i,j,s}\max\{0, div(P_{j,s}(\tilde{f}))(z) - n_u\} = \mathbf{a}_i \cdot \mathbf{c}. \end{aligned}$$

Using Lemma 2.4, we get

$$S_Y(u,\mathbf{c}) \leq \sum_{i=1}^{H_Y(u)} div(L_i(\tilde{F}))(z) - div^{n_u}(L_i(\tilde{F}))(z) \leq div\tilde{F}_{n_u}(z).$$

Since $\bigcap_{j=0}^{n} P_{1,j} \cap V = \emptyset$, then by Lemma 2.2, we have

$$e_Y(\mathbf{c}) \geq \delta \cdot \sum_{j=0}^{n} c_{1,j} = \delta \cdot \sum_{j=0}^{n} \max\{0, div(P_{1,j}(\tilde{f}))(z) - n_u\}.$$

On the other hand, by Lemma2.1, we obtain

$$\begin{aligned} S_Y(u,\mathbf{c}) \geq& \frac{uH_Y(u)}{(n+1)\delta} e_Y(\mathbf{c}) - (2n+1)\delta H_Y(u) \max_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} c_{i,j} \\ \geq& \frac{uH_Y(u)}{n+1} \sum_{j=0}^{n} \max\{0, div(P_{1,j}(\tilde{f}))(z) - n_u\} \\ &- (2n+1)\delta H_Y(u) \max_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} div(P_{i,j}(\tilde{f}))(z). \end{aligned}$$

Thus

$$\begin{aligned} div\tilde{F}_{n_u}(z) \geq& \frac{uH_Y(u)}{n+1} \sum_{j=0}^{n} \max\{0, div(P_{1,j}(\tilde{f}))(z) - n_u\} \\ &- (2n+1)\delta H_Y(u) \max_{\substack{1\leq i\leq n_0\\ 0\leq j\leq n}} div(P_{i,j}(\tilde{f}))(z). \end{aligned} \tag{13}$$

Since $div(P_{1,j}(\tilde{f}))(z) \geq div(Q_{I_1(t_{1,j})}(\tilde{f}))(z)$ for all $0 \leq j \leq n$ and $I_1(t_{1,j}) = t_{1,j} + 1, P_{1,0} = D_1$, therefore

$$\Delta\sum_{j=0}^{n}\max\{0, div(P_{1,j}(\tilde{f}))(z) - n_u\} \geq \Delta\sum_{j=0}^{n}\max\{0, div(Q_{I_1(t_{1,j})}(\tilde{f}))(z) - n_u\}$$
$$\geq \sum_{j=0}^{n}(t_{1,j+1} - t_{1,j})\max\{0, div(Q_{I_1(t_{1,j})}(\tilde{f}))(z) - n_u\}$$
$$\geq \sum_{i=0}^{l_1}\max\{0, div(Q_{I_1(j)}(\tilde{f}))(z) - n_u\} = \sum_{i=1}^{q}\max\{0, div(Q_j(\tilde{f}))(z) - n_u\}.$$

Combining this inequality and (13), we have

$$\begin{aligned} div\tilde{F}_{n_u}(z) \geq &\frac{uH_Y(u)}{(n+1)\Delta}\sum_{i=1}^{q}\max\{0, div(Q_j(\tilde{f}))(z) - n_u\} \\ &- (2n+1)\delta H_Y(u)\max_{\substack{1\leq i\leq n_0\\0\leq j\leq n}} div(P_{i,j}(\tilde{f}))(z) \\ \geq &\frac{uH_Y(u)}{(n+1)\Delta}\sum_{i=1}^{q}\Big(div(Q_j(\tilde{f}))(z) - \min\{n_u, div(Q_j(\tilde{f}))(z)\}\Big) \\ &- (2n+1)\delta H_Y(u)\max_{\substack{1\leq i\leq n_0\\0\leq j\leq n}} div(P_{i,j}(\tilde{f}))(z). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\Delta(n+1)}{udH_Y(u)}N_{RamF}(r,s) \geq &\sum_{j=1}^{q}\frac{1}{d}\left[N(r,s;D_j) - N^{n_u}(r,s;D_j)\right] \\ &- \frac{\Delta(n+1)(2n+1)\delta}{ud}\max_{\substack{1\leq i\leq n_0\\0\leq j\leq n}} N(r,s;P_{i,j}). \end{aligned} \tag{14}$$

By (10), (14) and the First Main Theorem, we get

$$\begin{aligned} \| \quad &(q - \Delta(n+1))T_f(r,s) \\ &\leq \sum_{j=1}^{q}\frac{1}{d}N^{n_u}(r,s;D_j) + \left(\epsilon'\frac{\Delta(n+1)}{H_Y(u)} + \frac{\Delta(2n+1)(n+1)l\delta}{u}\right)T_f(r,s) \\ &\quad + \frac{\Delta(n+1)}{udH_Y(u)}\left(\frac{H_Y(u)(H_Y(u)-1)}{2} + \epsilon'\right)(m(\mathfrak{L};r,s) + Ric_p(r,s) + \kappa\log^+\mathcal{Y}(r^2) \\ &\quad + \kappa\log^+ r) + O(1). \end{aligned}$$

We now choose u is the smallest integer such that

$$u > \Delta(2n+1)(n+1)l\delta\epsilon^{-1}$$

and

$$\epsilon' = \frac{H_Y(u)}{\Delta(n+1)}\left(\epsilon - \frac{\Delta(2n+1)(n+1)l\delta}{u}\right) > 0.$$

Hence

$$\begin{aligned}\| \quad (q-\Delta(n+1)-\epsilon)T_f(r,s) \leq & \sum_{j=1}^{q}\frac{1}{d}N^{n_u}(r,s;D_j)\\ & + c(m(\mathfrak{L};r,s)+Ric_p(r,s)+\kappa\log^+\mathcal{Y}(r^2)+\kappa\log^+ r).\end{aligned}$$

where $c \geq \{1, \frac{\Delta(n+1)}{udH_Y(u)}\left(\frac{H_Y(u)(H_Y(u)-1)}{2}+\epsilon'\right)\}$ and

$$\begin{aligned} n_u = & \begin{pmatrix} H_Y(u)-1 \leq \delta(n+u \\ n) \leq d^n \deg(V)e^n(1+\frac{u}{n})^n \end{pmatrix}\\ \leq & d^n\deg(V)e^n(\Delta(2n+4)l\delta\epsilon^{-1})^n\\ \leq & \deg(V)^{n+1}e^n d^{n^2+n}\Delta^n(2n+4)^n l^n\epsilon^{-n} = M_0.\end{aligned}$$

The theorem is proved.

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