Partitioning powers into sets of equal sum

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1 Introduction

Consider the following puzzle submitted by Dean Ballard to the Riddler column on the FiveThirtyEight website [15]:

> King Auric adored his most prized possession: a set of perfect spheres of solid gold. There was one of each size, with diameters of 1 centimeter, 2 centimeters, 3 centimeters, and so on. Their brilliant beauty brought joy to his heart. After many years, he felt the time had finally come to pass the golden spheres down to the next generation – his three children. He decided it was best to give each child precisely one-third of the total gold by weight, but he had a difficult time determining just how to do that. After some trial and error, he managed to divide his spheres into three groups of equal weight. He was further amused when he realized that his collection contained the minimum number of spheres needed for this division. How many golden spheres did King Auric have?

The puzzle translates into finding the smallest positive integer n such that $1^3, 2^3, \ldots, n^3$ can be partitioned into 3 sets where each set has the same sum. We can generalize to the following questions regarding partitioning the first n k-powers into m sets of equal sum.

Questions. *Let $k \geq 1$ and $m \geq 2$ be integers. For a positive integer n, consider the set of k-powers $S_{n,k} = \{1^k, 2^k, \ldots, n^k\}$.*

1. *Is there a positive integer n such that $S_{n,k}$ can be partitioned into m sets where the sum of the elements in each set is the same?*
2. *If such an n exists, what is the smallest possible n?*
3. *Can we classify all n for which a partition exists?*

These questions are closely related to a famous problem sometimes referred to as the Prouhet-Tarry-Escott (PTE) problem, but it goes all the way back to two letters between Euler and Goldbach [8, 9]. The PTE problem asks to find integers n, and a_{ij} for $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$[1] such that

$$\begin{aligned}
\sum_{j=1}^{n} a_{1j} &= \sum_{j=1}^{n} a_{2j} = \cdots = \sum_{j=1}^{n} a_{mj} \\
\sum_{j=1}^{n} a_{1j}^2 &= \sum_{j=1}^{n} a_{2j}^2 = \cdots = \sum_{j=1}^{n} a_{mj}^2 \\
&\vdots \\
\sum_{j=1}^{n} a_{1j}^k &= \sum_{j=1}^{n} a_{2j}^k = \cdots = \sum_{j=1}^{n} a_{mj}^k. \qquad (1)
\end{aligned}$$

[1] Most people use PTE to refer to the problem for $m = 2$, but we want to consider the more general case in this paper.

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The trivial PTE solutions are those for which there is an i and a $j \neq i$ for which the sets $\{a_{i1}, a_{i2}, \ldots, a_{in}\}$, $\{a_{j1}, a_{j2}, \ldots, a_{jn}\}$ are the same. Let $P(k,m)$ be the smallest positive integer n such that there is a nontrivial solution to (1). It has been shown that $P(k,m) \geq k+1$, and in fact $P(k,m) = k+1$ for $k \leq 10$ or $k = 12$ and $m = 2$, or for $k \in \{2,3,5\}$ for any m (see [3] for more details on the computational aspects of finding $P(k,m)$ and [13] for a very thorough review of all the literature on the problem). In general, it is hard to compute $P(k,m)$, but Prouhet [12] claimed the following theorem which yields $P(k,m) \leq m^k$.

Theorem A. *For each integer $i \in \{0,1,\ldots,m^{k+1}-1\}$, consider i written in base m. Let $s(i)$ be the sum of the base m digits of i. Let A_j be the set of integers i such that $s(i) \equiv j \bmod m$. Then for every $j, \ell \in \{0,1,2,\ldots,m-1\}$ and every $t \in \{0,1,\ldots,k\}$, we have*

$$\sum_{i \in A_j} i^t = \sum_{i \in A_\ell} i^t.$$

In other words, the sets A_i form a partition of $\{0,1,\ldots,m^{k+1}-1\}$ where the sums of like powers of each set are all equal. Lehmer [10] presented the first published proof of Theorem A in 1947. Since then, distinct proofs have been given by Wright [18] in 1948, Roberts [14] in 1965, and Nguyen [11] in 2016. Note that Theorem A certainly answers Question 1 above: If $n = m^{k+1}-1$, then $S_{n,k}$ can be partitioned into m pieces with equal sum.

The following proposition allows us to make some headway on our Questions 2 and 3.

Proposition 1. *Let $m \geq 2$ and $k \geq 1$ be positive integers. Let N, n be positive integers. Suppose we can partition the set $\{n, n-1, \ldots, n+1-N\}$ into m sets $A_{1,k}, A_{2,k}, \ldots, A_{m,k}$ such that for all integers t satisfying $0 \leq t \leq k$, and all $i, j \in \{1,2,\ldots,m\}$ we have*

$$\sum_{a \in A_{i,k}} a^t = \sum_{b \in A_{j,k}} b^t.$$

Then, we can partition $\{n, n-1, \ldots, n+1-mN\}$ into m sets $A_{1,k+1}, A_{2,k+1}, \ldots, A_{m,k+1}$ such that for all integers t satisfying $0 \leq t \leq k+1$, and all $i, j \in \{1,2,\ldots,m\}$ we have

$$\sum_{a \in A_{i,k}} a^t = \sum_{b \in A_{j,k}} b^t.$$

Proposition 1 gives another proof of Theorem A by starting with the sets $A_{1,0} = \{n\}, A_{2,0} = \{n-1\}, \ldots, A_{m,0} = \{n+1-m\}$. These m sets satisfy the conditions of the proposition for $k = 0$. We can iterate Proposition 1 k times to create m sets $A_{1,k}, A_{2,k}, \ldots, A_{m,k}$ that partition $\{n, n-1, \ldots, n+1-m^{k+1}\}$ into m sets satisfying (1). For $n = m^{k+1}-1$, the sets created with this recursive construction can be shown to coincide with those described in Theorem A.

Another important consequence of Proposition 1 is

Theorem 2. *Let $m \geq 2$ and $k \geq 1$ be positive integers. Let n be any integer. There exists a partition of $\{n, (n-1), \ldots, n+1-2m^k\}$ into m sets with equal sums of t-powers for every $t = 0,1,2,\ldots,k$.*

We prove Theorem 2 in §1. A similar argument for an equivalent result was found independently by Choudhry [7].

Theorem 2 has the following two important consequences:

- $S_{n,k}$ can be partitioned into m sets with equal sum when $n = 2m^k$.
- Whenever $S_{n,k}$ can be partitioned into m sets with the same sum, then $S_{n+2m^k,k}$ can also be partitioned.

The second of these allows us to classify all n such that $S_{n,k}$ can be partitioned into m sets of equal sum. Certainly if $S_{n,k}$ can be partitioned into m sets of equal sum, then $m \mid 1^k+2^k+\cdots+n^k$. Given an integer $r \geq 0$ such that $r < 2m^k$ and $m \mid 1^k+\cdots+r^k$, let n_r be the smallest positive integer such that $n_r \equiv r \bmod 2m^k$ and $S_{n_r,k}$ can be partitioned into m sets of equal sum. Let

$$R_m = \{r \in \mathbb{N}_0 \,|\, r < 2m^k,\ m \mid 1^k+2^k+\cdots+r^k\}.$$

Then the n for which $S_{n,k}$ can be partitioned into m sets of equal sum are

$$\{n_r + 2m^k\ell \,|\, r \in R_m, \ell \in \mathbb{N}_0\}.$$

Following this strategy, we classify all n for which $S_{n,3}$ can be partitioned into three sets of equal sum.

Theorem 3. *The set $\{1^3, 2^3, \ldots, n^3\}$ can be partitioned into three sets of equal sum if and only if $n = 23$ or $n \geq 26$ with $n \equiv 0, 2 \bmod 3$.*

We also include a few other examples of classification.

Theorem 4. *For positive integers n, k, let $S_{n,k} = \{1^k, 2^k, \ldots, n^k\}$.*

1. *The set $S_{n,3}$ can be partitioned into two sets of equal sum if and only if $n \geq 12$ and $n \equiv 0, 3 \bmod 4$.*
2. *The set $S_{n,2}$ can be partitioned into two sets of equal sum if and only if $n \geq 7$ and $n \equiv 0, 3 \bmod 4$.*
3. *The set $S_{n,2}$ can be partitioned into three sets of equal sum if and only if $n \geq 18$ and $n \equiv 0, 4, 8 \bmod 9$.*

In Theorems 3 and 4, the set $S_{n,k}$ can be partitioned into m sets with equal sum for all large enough n satisfying the necessary condition $m \mid 1^k + 2^k + \cdots + n^k$. In our final theorem, we prove this is true in general (that is, for all m and k).

Theorem 5. *Let $k \geq 1, m \geq 2$ be positive integers. There exists a constant $C_{k,m}$ depending on k and m such that $\{1^k, 2^k, \ldots n^k\}$ can be partitioned into m sets of equal sum whenever $n \geq C_{k,m}$ and $m \mid 1^k + 2^k + \cdots + n^k$.*

To get an idea of the proof, suppose $m = 2$. Let $T = \frac{1^k+2^k+\cdots+n^k}{2} \in \mathbb{N}$. It is enough to find a subset $A \subseteq S_{n,k} = \{1^k, 2^k, \ldots, n^k\}$ such that the sum of the elements of A is T, for then taking B to be the complement of A in $S_{n,k}$ yields our desired partition. To find A it is natural to try the following: Let r be the largest non-negative integer such that $n^k + (n-1)^k + \cdots + (n-r)^k < T$. Try to write $T - (n^k + (n-1)^k + \cdots + (n-r)^k)$ as a sum of distinct kth powers. If we find such a decomposition, appending the summands to $\{n^k, \ldots, (n-r)^k\}$ gives us A. We show that a close relative of this procedure always succeeds, and explain how to extend the method to $m > 2$, in §4. The linchpin of our proof of Theorem 5 is a deep theorem of Wright on Waring's problem (Proposition 7 below).

Finally, we remark that Boyd [4, 5], Berend and Golan [2], and Buhler, Golan, Pratt, and Wagon [6] have considered a variant of our classification problem in the spirit of PTE (for $m = 2$): Given a positive integer k, for which n can $\{1, 2, 3, \ldots, n\}$ can be partitioned into two sets A_1, A_2 such that $\sum_{a\in A_1} a^t = \sum_{a\in A_2} a^t$ for all $t = 0, 1, 2, \ldots, k-1$? They answer this question completely for $k = 1, 2, \ldots, 8$.

2 Constructive upper bounds: Proofs of Proposition 1 and Theorem 2

Let's illustrate the idea behind the proof of Proposition 1 and Theorem 2 by showing how it works for a particular m and k. Let's take $m = k = 3$. We will show that we can partition 54 consecutive integers into 3 sets where the sum of the cubes of the elements in each set are equal. First, note that given six consecutive integers $n-5, n-4, \ldots, n$, then $A = \{n, n-5\}, B = \{n-1, n-4\}, C = \{n-2, n-3\}$ satisfy that the sum of the terms in each set is $2n - 5$ for all three sets. If we now consider the sum of the squares of each set and view as a polynomial in n, then we get $2n^2 - 10n + 25, 2n^2 - 10n + 17, 2n^2 - 10n + 13$, respectively. Therefore, they only differ in the constant term. We can now consider $n-6, n-7, \ldots, n-11$ and distribute by rotation, i.e., $n-6, n-11$ go to B, $n-7, n-10$ go to C, and $n-8, n-9$ go to A. We then rotate $n-12, \ldots, n-17$ once more. At the end we get $A = \{n, n-5, n-8, n-9, n-13, n-16\}$, $B = \{n-1, n-4, n-6, n-11, n-14, n-15\}$, and $C = \{n-2, n-3, n-7, n-10, n-12, n-17\}$. Note

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that we get

$$\sum_{a\in A} a^2 = 2n^2 - 10n + 25 + 2(n-6)^2 - 10(n-6) + 13 + 2(n-12)^2 - 10(n-12) + 17$$
$$\sum_{b\in B} b^2 = 2n^2 - 10n + 17 + 2(n-6)^2 - 10(n-6) + 25 + 2(n-12)^2 - 10(n-12) + 13$$
$$\sum_{c\in C} c^2 = 2n^2 - 10n + 13 + 2(n-6)^2 - 10(n-6) + 17 + 2(n-12)^2 - 10(n-12) + 25.$$

By rotating three times we forced the constant terms to be equal. Therefore these 18 consecutive numbers could be split into three sets of 6 each, such that the sum of the squares of the elements of each set are all equal. Now, by construction, if we consider the sum of cubes as a polynomial in n, the result will be a cubic in n where the coefficients of n^3, n^2, n match, with the constant term the only different coefficient. We can then use the same trick of rotating three times to get a construction of 54 consecutive integers such that the sum of the cubes are equal. The sets are

$$A = \{n-\alpha \,|\, \alpha \in \{0, 5, 8, 9, 13, 16, 20, 21, 25, 28, 30, 35, 37, 40, 42, 47, 50, 51\}\}$$
$$B = \{n-\alpha \,|\, \alpha \in \{1, 4, 6, 11, 14, 15, 18, 23, 26, 27, 31, 34, 38, 39, 43, 46, 48, 53\}\}$$
$$C = \{n-\alpha \,|\, \alpha \in \{2, 3, 7, 10, 12, 17, 19, 22, 24, 29, 32, 33, 36, 41, 44, 45, 49, 52\}\}.$$

Proof of Proposition 1. We view the elements of the sets $A_{i,k}$ as polynomials in n of the form $n-\alpha$, where α is an integer. Then for each $i \in \{1, 2, \dots, m\}$, we may expand

$$\sum_{n-\alpha\in A_{i,k}} (n-\alpha)^{k+1}$$

as a polynomial in n of degree $k+1$, i.e.,

$$\sum_{n-\alpha\in A_{i,k}} (n-\alpha)^{k+1} = c_{i,k+1}n^{k+1} + c_{i,k}n^k + \cdots + c_{i,1}n + c_{i,0}.$$

Note that for any $\ell \in \{0, 1, 2, \dots, k+1\}$

$$\begin{aligned} c_{i,\ell} &= (-1)^{k+1-\ell}\binom{k+1}{\ell}\sum_{n-\alpha\in A_{i,k}} \alpha^{k+1-\ell} = (-1)^{k+1-\ell}\binom{k+1}{\ell}\sum_{a\in A_{i,k}} (n-a)^{k+1-\ell} \\ &= (-1)^{k+1-\ell}\binom{k+1}{\ell}\sum_{a\in A_{i,k}}\sum_{h=0}^{k+1-\ell}\binom{k+1-\ell}{h} n^{k+1-\ell-h}(-1)^h a^h \\ &= (-1)^{k+1-\ell}\binom{k+1}{\ell}\sum_{h=0}^{k+1-\ell}\binom{k+1-\ell}{h} n^{k+1-\ell-h}(-1)^h \sum_{a\in A_{i,k}} a^h. \end{aligned}$$

Now for any $j \in \{1, 2, \dots, m\}$ and any $h < k+1$, the inner sum can be replaced:

$$\sum_{a\in A_{i,k}} a^h = \sum_{a\in A_{j,k}} a^h.$$

When $\ell > 0$, we have $h < k+1$. Therefore

$$c_{i,\ell} = (-1)^{k+1-\ell}\binom{k+1}{\ell}\sum_{h=0}^{k+1-\ell}\binom{k+1-\ell}{h} n^{k+1-\ell-h}(-1)^h \sum_{a\in A_{j,k}} a^h = c_{j,\ell}.$$

Therefore $c_{i,\ell} = c_{j,\ell}$ for any $\ell > 0$. We can now relabel $c_{i,\ell}$ as c_ℓ for $\ell > 0$ and d_i for $\ell = 0$. In other words,

$$\sum_{n-\alpha\in A_{i,k}} (n-\alpha)^{k+1} = c_{k+1}n^{k+1} + c_kn^k + \cdots + c_1n + d_i. \tag{2}$$

For $0 \le q \le m-1$ define the sets $A_{i,k}^{(q)}$ as

$$A_{i,k}^{(q)} = \{a - qN \mid a \in A_{i,k}\}.$$

Now we will define $A_{i,k+1}$ in terms of these:

$$A_{i,k+1} = A_{i,k} \cup A_{i-1,k}^{(1)} \cup \cdots \cup A_{1,k}^{(i-1)} \cup A_{m,k}^{(i)} \cup \cdots \cup A_{i+1,k}^{(m-1)}.$$

From (2), for any $j \in \{1, 2, \ldots, m\}$, we get

$$\begin{aligned}\sum_{n-\alpha \in A_{i,k+1}} (n-\alpha)^{k+1} &= \sum_{\ell=1}^{k+1} c_\ell \sum_{q=0}^{m-1} (n-qN)^\ell + \sum_{r=1}^{m} d_r \\ &= \sum_{n-\alpha \in A_{j,k+1}} (n-\alpha)^{k+1}.\end{aligned}$$

For $t \in \{0, 1, \ldots k\}$ and $q \in \{0, 1, \ldots, m-1\}$, we have

$$\begin{aligned}\sum_{a \in A_{i,k}} (a-qN)^t &= \sum_{a \in A_{i,k}} \sum_{\ell=0}^{t} \binom{t}{\ell} a^\ell (-qN)^{t-\ell} = \sum_{\ell=0}^{t} \binom{t}{\ell} (-qN)^{t-\ell} \sum_{a \in A_{i,k}} a^\ell \\ &= \sum_{\ell=0}^{t} \binom{t}{\ell} (-qN)^{t-\ell} \sum_{a \in A_{j,k}} a^\ell = \sum_{a \in A_{j,k}} (a-qN)^t.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{a \in A_{i,k+1}} a^t &= \sum_{a \in A_{i,k}} a^t + \sum_{a \in A_{i-1,k}} (a-N)^t + \cdots + \sum_{a \in A_{i+1,k}} (a-(m-1)N)^t \\ &= \sum_{a \in A_{j,k}} a^t + \sum_{a \in A_{j-1,k}} (a-N)^t + \cdots + \sum_{a \in A_{j+1,k}} (a-(m-1)N)^t = \sum_{a \in A_{j,k+1}} a^t.\end{aligned}$$

Note that the size of $A_{i,k+1}$ is m times the size of $A_{i,k}$, which is what we wanted to prove. □

Proof of Theorem 2. Consider the partition $A_1 = \{n, n+1-2m\}, A_2 = \{n-1, n+2-2m\}, A_3 = \{n-2, n+3-2m\}, \ldots, A_m = \{n-(m-1), n-m\}$. Then for $k=1$, we have $N = 2m$. By applying Proposition 1 $k-1$ times, we get a partition of $\{n, (n-1), \ldots, (n-2m^k)\}$ into m sets, where the t-th powers of the elements of each set have the same sum for $0 \le t \le k$. □

3 Solving the classification problem for some $m, k \in \{2, 3\}$: Proofs of Theorems 3 and 4

To showcase the usefulness of Theorem 2, we will classify all n partitioning $S_{n,k}$ into m sets of equal sum for some values of m, k. Since the original puzzle that inspired this work was the case $k=3, m=3$, we will first prove Theorem 3, which classifies all n such that $\{1^3, 2^3, \ldots, n^3\}$ can be partitioned into three sets of equal sum.

Proof of Theorem 3. We know

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{3}$$

Hence, for $S_{n,3}$ to be partitioned into three sets, we need $n \equiv 0, 2 \bmod 3$. From sequence A330212 in OEIS [1] we know that the smallest n for which there is such a partition is $n = 23$. To search for solutions for $n = 24$, we searched among all subsets $S \subseteq \{1, 2, \ldots, 24\}$ whether

$$\sum_{i \in S} i^3 = \frac{n^2(n+1)^2}{12} = 30000.$$

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There are 163 subsets with such a property, but no two of them are disjoint. Therefore, there are no solutions for $n = 24$. Tables 1, 2 show there are partitions for $26 \le n \le 75$ with $n \equiv 0, 2 \bmod 3$, and there is a partition for $n = 78$. Since all residues modulo 54 that are 0 or 2 modulo 3 are covered, by applying Proposition 1, we get that any $n > 78$ with $n \equiv 0, 2 \bmod 3$ has a partition. $\square$

n	Partition
23	$\{3,6,10,13,18,19,21\},\{1,4,7,8,12,16,20,22\},\{2,5,9,11,14,15,17,23\}$
26	$\{4,14,19,24,26\},\{2,3,5,11,15,16,18,22,25\},\{1,6,7,8,9,10,12,13,17,20,21,23\}$
27	$\{11,12,21,25,27\},\{7,13,14,15,17,18,22,26\},\{1,2,3,4,5,6,8,9,10,16,19,20,23,24\}$
29	$\{7,12,14,19,24,25,28\},\{2,6,8,17,20,23,26,27\},\{1,3,4,5,9,10,11,13,15,16,18,21,22,29\}$
30	$\{4,7,8,16,19,25,26,30\},\{3,5,11,14,17,20,21,23,24,27\},\{1,2,6,9,10,12,13,15,18,22,28,29\}$
32	$\{16,22,25,31,32\},\{2,4,5,8,11,12,17,18,19,20,23,29,30\},$ $\{1,3,6,7,9,10,13,14,15,21,24,26,27,28\}$
33	$\{4,13,16,21,24,26,28,33\},\{1,3,6,7,10,18,20,25,27,29,31\},$ $\{2,5,8,9,11,12,14,15,17,19,22,23,30,32\}$
35	$\{7,17,24,25,28,32,35\},\{11,18,19,20,22,29,33,34\},$ $\{1,2,3,4,5,6,8,9,10,12,13,14,15,16,21,23,26,27,30,31\}$
36	$\{5,7,10,14,22,29,31,33,35\},\{1,6,12,15,16,17,18,20,24,26,27,28,36\},$ $\{2,3,4,8,9,11,13,19,21,23,25,30,32,34\}$
38	$\{5,17,21,24,29,32,35,38\},\{1,6,9,10,12,13,14,15,20,31,33,36,37\},$ $\{2,3,4,7,8,11,16,18,19,22,23,25,26,27,28,30,34\}$
39	$\{6,22,25,27,36,37,39\},\{2,3,4,5,8,11,12,13,16,23,26,29,30,32,33,35\},$ $\{1,7,9,10,14,15,17,18,19,20,21,24,28,31,34,38\}$
41	$\{2,5,18,26,27,32,35,39,41\},\{10,13,20,23,24,28,31,34,38,40\},$ $\{1,3,4,6,7,8,9,11,12,14,15,16,17,19,21,22,25,29,30,33,36,37\}$
42	$\{2,8,9,20,24,26,35,38,39,42\},\{3,4,6,7,11,12,14,15,19,21,25,36,37,40,41\},$ $\{1,5,10,13,16,17,18,22,23,27,28,29,30,31,32,33,34\}$
44	$\{1,2,9,20,28,31,36,38,43,44\},\{4,5,8,10,11,13,15,22,25,26,27,29,32,39,40,42\},$ $\{3,6,7,12,14,16,17,18,19,21,23,24,30,33,34,35,37,41\}$
45	$\{5,10,11,19,24,27,28,29,32,34,36,40,44\},\{4,9,14,15,22,23,26,30,31,37,39,41,42\},$ $\{1,2,3,6,7,8,12,13,16,17,18,20,21,25,33,35,38,43,45\}$
47	$\{16,24,25,28,34,38,43,45,47\},\{5,7,10,20,21,31,35,36,37,40,42,46\},$ $\{1,2,3,4,6,8,9,11,12,13,14,15,17,18,19,22,23,26,27,29,30,32,33,39,41,44\}$
48	$\{8,10,12,19,22,24,25,27,32,36,37,39,45,48\},\{2,3,5,6,9,11,18,20,23,29,40,41,42,46,47\},$ $\{1,4,7,13,14,15,16,17,21,26,28,30,31,33,34,35,38,43,44\}$
50	$\{6,12,16,28,33,36,41,42,43,45,49\},\{1,2,9,18,19,20,22,25,27,29,30,31,34,35,37,39,46,47\},$ $\{3,4,5,7,8,10,11,13,14,15,17,21,23,24,26,32,38,40,44,48,50\}$
51	$\{2,8,11,15,20,26,32,37,42,44,45,47,49\},\{3,14,16,22,25,28,30,31,34,35,38,43,50,51\},$ $\{1,4,5,6,7,9,10,12,13,17,18,19,21,23,24,27,29,33,36,39,40,41,46,48\}$
53	$\{4,6,13,17,18,21,30,32,46,47,49,51,53\},\{3,8,24,25,27,31,36,38,42,44,45,48,52\},$ $\{1,2,5,7,9,10,11,12,14,15,16,19,20,22,23,26,28,29,33,34,35,37,39,40,41,43,50\}$
54	$\{17,22,38,39,47,48,49,51,52\},\{4,5,18,24,26,33,36,40,42,43,45,53,54\},$ $\{1,2,3,6,7,8,9,10,11,12,13,14,15,16,19,20,21,23,25,27,28,29,30,31,32,34,35,37,41,44,46,50\}$

Table 1: Partitions of $\{1, 2, \ldots, n\}$ into three sets whose cubes have equal sum.

Remark 6. Finding a partition of $S_{n,3}$ into three sets of equal sum is not trivial in a computer, as there are an enormous amount of partitions to consider. Indeed, there are $\left\{ {n \atop 3} \right\} = \frac{3^{n-1}-2^n+1}{2}$ partitions to consider. Because of this, our strategy was not to try all partitions in some order, but to pick a random subset of $S = \{1^3, 2^3, \ldots, n^3\}$ and check if the sum of the elements of the subset is $N = (1^3 + 2^3 + \cdots + n^3)/3$. The way we chose the random subset was as follows: we chose $i^3 \in S$ and subtracted it from N, we then continued to choose distinct cubes, until the difference between N and the sum of these cubes was smaller than the

n	Partition
56	$\{1,4,14,15,28,34,36,37,39,42,49,50,51,56\}$, $\{3,5,8,11,16,23,26,27,30,31,32,38,43,44,45,47,52,53\}$, $\{2,6,7,9,10,12,13,17,18,19,20,21,22,24,25,29,33,35,40,41,46,48,54,55\}$
57	$\{7,31,40,52,53,55,56,57\}$, $\{2,3,12,13,17,18,20,22,23,25,26,27,29,32,36,38,39,41,42,43,47,48,54\}$, $\{1,4,5,6,8,9,10,11,14,15,16,19,21,24,28,30,33,34,35,37,44,45,46,49,50,51\}$
59	$\{1,2,6,9,10,18,23,27,37,38,43,46,55,56,57,58\}$, $\{3,4,5,11,13,15,17,20,30,33,34,35,36,44,48,50,53,54,59\}$, $\{7,8,12,14,16,19,21,22,24,25,26,28,29,31,32,39,40,41,42,45,47,49,51,52\}$
60	$\{14,16,23,26,30,31,32,35,38,40,42,43,44,47,50,56,57\}$, $\{1,2,3,5,7,11,13,15,19,21,29,33,36,37,45,49,53,55,58,60\}$, $\{4,6,8,9,10,12,17,18,20,22,24,25,27,28,34,39,41,46,48,51,52,54,59\}$
62	$\{2,4,7,9,11,21,30,31,48,50,51,57,58,60,62\}$, $\{1,3,8,10,15,18,19,22,29,33,39,42,47,49,52,53,54,56,59\}$, $\{5,6,12,13,14,16,17,20,23,24,25,26,27,28,32,34,35,36,37,38,40,41,43,44,45,46,55,61\}$
63	$\{1,4,5,8,15,23,24,29,35,42,46,51,53,55,58,60,61\}$, $\{7,10,11,12,14,17,18,21,28,33,34,37,38,39,40,44,47,48,49,52,59,62\}$, $\{2,3,6,9,13,16,19,20,22,25,26,27,30,31,32,36,41,43,45,50,54,56,57,63\}$
65	$\{1,7,8,18,22,25,27,32,38,40,48,50,52,55,63,64,65\}$, $\{2,4,6,11,14,15,16,19,21,28,30,34,37,39,41,43,49,56,58,59,61,62\}$, $\{3,5,9,10,12,13,17,20,23,24,26,29,31,33,35,36,42,44,45,46,47,51,53,54,57,60\}$
66	$\{3,7,9,13,17,18,23,31,34,41,46,49,53,55,58,61,62,65\}$, $\{2,4,6,8,14,20,24,29,32,36,37,38,40,47,48,54,59,60,63,66\}$, $\{1,5,10,11,12,15,16,19,21,22,25,26,27,28,30,33,35,39,42,43,44,45,50,51,52,56,57,64\}$
68	$\{1,2,6,7,13,33,36,37,41,48,54,55,57,58,62,64,68\}$, $\{3,5,11,16,17,20,22,23,25,26,27,32,38,42,47,49,50,51,52,59,60,63,66\}$, $\{4,8,9,10,12,14,15,18,19,21,24,28,29,30,31,34,35,39,40,43,44,45,46,53,56,61,65,67\}$
69	$\{1,6,7,28,33,38,39,40,55,57,60,63,65,67,68\}$, $\{3,4,9,14,15,20,21,22,24,25,30,31,32,35,45,49,58,61,62,64,66,69\}$, $\{2,5,8,10,11,12,13,16,17,18,19,23,26,27,29,34,36,37,41,42,43,44,46,$ $47,48,50,51,52,53,54,56,59\}$
71	$\{2,8,14,18,20,22,31,35,37,39,43,46,47,51,53,55,57,61,62,66,67\}$, $\{4,5,9,10,11,13,17,24,26,28,30,33,38,41,44,49,52,56,58,63,64,70,71\}$, $\{1,3,6,7,12,15,16,19,21,23,25,27,29,32,34,36,40,42,45,48,50,54,59,60,65,68,69\}$
72	$\{1,2,3,5,10,17,25,26,34,38,46,49,52,54,55,56,59,61,62,68,69\}$, $\{4,7,12,14,15,20,23,24,28,29,33,40,41,48,50,51,53,57,58,60,66,67,70\}$, $\{6,8,9,11,13,16,18,19,21,22,27,30,31,32,35,36,37,39,42,43,44,45,47,63,64,65,71,72\}$
74	$\{1,2,4,8,9,18,21,22,27,29,46,47,50,52,56,57,64,67,70,72,73\}$, $\{5,6,10,11,12,13,14,17,32,33,34,43,44,48,49,51,53,62,63,65,66,68,74\}$, $\{3,7,15,16,19,20,23,24,25,26,28,30,31,35,36,37,38,39,40,41,42,45,54,55,58,59,60,61,69,71\}$
75	$\{1,2,3,6,7,12,13,17,25,26,40,43,48,57,59,60,64,71,72,73,75\}$, $\{4,9,19,20,23,24,28,34,37,38,46,47,49,50,51,54,55,56,61,63,65,69,70\}$, $\{5,8,10,11,14,15,16,18,21,22,27,29,30,31,32,33,35,36,39,41,42,44,45,52,53,58,62,66,67,68,74\}$
78	$\{1,6,11,12,20,31,40,42,43,45,46,54,62,67,69,72,74,76,78\}$, $\{2,3,4,5,7,9,22,33,37,38,48,49,51,52,55,59,63,64,71,73,75,77\}$, $\{8,10,13,14,15,16,17,18,19,21,23,24,25,26,27,28,29,30,32,34,35,36,39,41,44,47,$ $50,53,56,57,58,60,61,65,66,68,70\}$

Table 2: Partitions of $\{1,2,\ldots,n\}$ into three sets whose cubes have equal sum.

smallest cube left in S. We repeated this process multiple times until we found two disjoint subsets whose cubes added to N. Finding the partitions for all 36 cases with this strategy took a few hours. Stan Wagon has informed us that the techniques from [6] can be adapted to speed up these calculations.

Proof of Theorem 4. 1. From (3) we see that for $S_{n,3}$ to be partitioned into two sets of equal sum we need $2 \mid \left(\frac{n(n+1)}{2}\right)^2$, therefore $n \equiv 0, 3 \bmod 4$. The smallest n for which $S_{n,3}$ can be partitioned in two sets of equal sum is 12 (see sequence A330212 in OEIS [1]). From Corollary 2 we know that we can partition $2(2)^3 = 16$ consecutive cubes into two sets of equal sums. Therefore, all positive $n \equiv 12 \bmod 16$ work, as do all positive $n \equiv 0 \bmod 16$. In Table 3 we show for $12 \leq n \leq 27$ with $n \equiv 0, 3 \bmod 4$ that there is a partition of $S_{n,3}$ into two sets of equal sum. Therefore, for every $n \geq 12$ with $n \equiv 0, 3 \bmod 4$ we can partition $S_{n,3}$ into two sets of equal sum.

n	Partition
12	$\{1, 2, 4, 8, 9, 12\}, \{3, 5, 6, 7, 10, 11\}$
15	$\{1, 2, 4, 7, 8, 11, 13, 14\}, \{3, 5, 6, 9, 10, 12, 15\}$
16	$\{1, 4, 6, 7, 10, 11, 13, 16\}, \{2, 3, 5, 8, 9, 12, 14, 15\}$
19	$\{1, 2, 5, 6, 8, 9, 11, 14, 15, 16, 17\}, \{3, 4, 7, 10, 12, 13, 18, 19\}$
20	$\{1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 16, 17, 20\}, \{8, 10, 12, 14, 15, 18, 19\}$
23	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 14, 15, 16, 19, 20, 21\}, \{10, 11, 13, 17, 18, 22, 23\}$
24	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 21, 22, 23\}, \{13, 15, 17, 18, 19, 20, 24\}$
27	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 21, 25, 26, 27\}, \{12, 15, 16, 17, 18, 19, 20, 22, 23, 24\}$

Table 3: Partitions of $\{1, 2, \ldots, n\}$ into two sets whose cubes have equal sum.

2. For $S_{n,2}$ to be partitioned into two sets, we need $2 \mid 1^2 + 2^2 + \cdots n^2 = n(n+1)(2n+1)/6$. Therefore, we need $n \equiv 0, 3 \bmod 4$. From Proposition 1, we know we can partition any 8 consecutive integers into two sets where the sum of their squares are equal. Therefore, we need only check $n \equiv 0, 3, 4, 7 \bmod 8$. Table 4 shows partitions for $n = 7, 8, 10, 12$. One can also check that it is not possible for $n = 3, 4$.

n	Partition
7	$\{1, 2, 4, 7\}, \{3, 5, 6\}$
8	$\{1, 4, 6, 7\}, \{2, 3, 5, 8\}$
10	$\{1, 3, 4, 5, 9, 11\}, \{2, 6, 7, 8, 10\}$
12	$\{1, 2, 3, 4, 5, 6, 7, 8, 11\}, \{9, 10, 12\}$

Table 4: Partitions of $\{1, 2, \ldots, n\}$ into two sets whose squares have equal sum.

3. From sequence A330431 in [1], we know 13 is the least n with such a property. Since $3 \mid n(n+1)(2n+1)/6$, we need $n \equiv 0, 4, 8 \bmod 9$. From Proposition 1, we know that any list of 18 consecutive integers can be partitioned into three sets such that the sum of squares in each set is the same. Therefore, we need only check $n \equiv 0, 4, 8, 9, 13, 17 \bmod 18$. In Table 5, we show that if $13 \leq n < 31$ and $n \equiv 0, 4, 8, 9, 13, 17 \bmod 18$, then there is a partition. The proof follows.

□

4 The classification problem for large enough n: Proof of Theorem 5

The proof uses work of Wright on Waring's problem with "proportionality conditions".

n	Partition
13	$\{2,10,13\},\{4,7,8,12\},\{1,3,5,6,9,11\}$
17	$\{1,2,13,14,15\},\{6,7,8,9,10,11,12\},\{3,4,5,16,17\}$
18	$\{1,6,8,11,15,16\},\{2,5,9,10,13,18\},\{3,4,7,12,14,17\}$
22	$\{1,2,3,7,19,20,21\},\{4,8,11,16,18,22\},\{5,6,9,10,12,13,14,15,17\}$
26	$\{1,4,19,22,23,26\},\{5,20,21,24,25\},\{2,3,6,7,8,9,10,11,12,13,14,15,16,17,18\}$
27	$\{1,3,5,21,23,24,27\},\{2,11,20,22,25,26\},\{4,6,7,8,9,10,12,13,14,15,16,17,18,19\}$

Table 5: Partitions of $\{1,2,\dots,n\}$ into three sets whose squares have equal sum.

Proposition 7. *For each fixed positive integer k there is an $s_0 = s_0(k)$ for which the following holds. Fix a positive integer $s \geq s_0$, and fix positive real numbers $\lambda_1,\dots,\lambda_s$ with $\lambda_1+\cdots+\lambda_s = 1$. If n is sufficiently large, there are positive integers $m_1,\dots,m_s$ with*

$$m_1^k + \cdots + m_s^k = n.$$

Furthermore, one can choose $m_1,\dots,m_s$ such that each $m_i^k = (\lambda_i + o(1))n$, as $n \to \infty$.

Proposition 7 is proved for $k \geq 3$ in [17] (see that paper's Theorem 1). For $k = 2$, stronger results are proved in [16]. Proposition 7 is trivial when $k = 1$ (in that case one can take $s_0 = 1$ and each $m_i = \lambda_i n + O_s(1)$).

Corollary 8. *For each fixed positive integer k there is an $s_0 = s_0(k)$ for which the following holds. Fix a positive integer $s \geq s_0$. For all large enough n, there are distinct positive integers $m_1,\dots,m_s$ with $m_1^k + \cdots + m_s^k = n$ and each $m_i^k \in (\frac{n}{2s}, \frac{3n}{2s})$.*

Proof. This follows immediately from Proposition 7, choosing $\lambda_1,\dots,\lambda_s$ as s distinct real numbers that sum to 1 from the interval $(\frac{1}{2s}, \frac{3}{2s})$. □

The next lemma is a consequence of the the "integral test" in calculus. Below we write $f(n) \sim g(n)$ to mean that $f(n)/g(n) \to 1$ as $n \to \infty$.

Lemma 9. *Fix a positive integer k, and fix real numbers α,β with $0 < \alpha < \beta$. As $n \to \infty$,*

$$\sum_{\alpha n < a < \beta n} a^k \sim \frac{(\beta^{k+1} - \alpha^{k+1})}{k+1} n^{k+1}.$$

In particular, $1^k + 2^k + \cdots + n^k \sim \frac{n^{k+1}}{k+1}$.

Proof (sketch). The sum is (essentially) a Riemann sum for the integral $\int_{\alpha n}^{\beta n} t^k \, dt = \frac{(\beta^{k+1}-\alpha^{k+1})}{k+1} n^{k+1}$. Drawing the graph, we see that the difference between the sum and the integral is bounded by a constant multiple of n^k; the result follows. □

Proof of Theorem 5. Put $T = \frac{1}{m}(1^k+2^k+\cdots+n^k)$. For all large n, we construct disjoint sets $A_1,\dots,A_{m-1} \subseteq \{1,2,\dots,n\}$ with $\sum_{a\in A_i} a^k = T$ for each $i = 1,\dots,m-1$. Setting $A_m = \{1,2,\dots,n\} \setminus (A_1 \cup \cdots \cup A_{m-1})$, we then have that $A_1,\dots,A_m$ form a partition $\{1,2,\dots,n\}$ into m sets with equal sums of kth powers.

Let $s_0(k)$ be as in Corollary 8. We fix $s_1 \geq s_0(k)$ satisfying

$$\left(\frac{3}{2s_1}\right)^{1/k} < \frac{1}{2}\left(\frac{1}{m}\right)^{\frac{1}{k+1}}$$

and fix positive integers $s_2,\dots,s_{k-1}$ with

$$s_{i+1} > 3s_i \qquad \text{for } 0 < i < k-1.$$

In what follows, n is always assumed sufficiently large. All asymptotic notation refers to behavior as $n \to \infty$.

To construct A_1, let r_1 be the largest nonnegative integer with $n^k + \cdots + (n-r_1)^k \le T$. From Lemma 9,

$$n - r_1 \sim c_1 n, \quad \text{where} \quad c_1 = (1-1/m)^{\frac{1}{k+1}}.$$

Put

$$T_1 = T - (n^k + (n-1)^k + \cdots + (n-(r_1-1))^k).$$

Then

$$(n-r_1)^k \le T_1 < (n-r_1)^k + (n-(r_1+1))^k < 2(n-r_1)^k.$$

By Corollary 8, we can write $T_1 = a_{1,1}^k + \cdots + a_{1,s_1}^k$, where the $a_{1,j}$ are distinct and

$$\frac{1}{2s_1}(n-r_1)^k < a_{1,j}^k < \frac{3}{2s_1}(n-r_1)^k. \tag{4}$$

In particular, since $1 - \frac{1}{m} \ge \frac{1}{m}$, each

$$a_{1,j} < \left(\frac{3}{2s_1}\right)^{1/k}(n-r_1) < \frac{1}{2}\left(\frac{1}{m}\right)^{\frac{1}{k+1}}(n-r_1) < \frac{1}{2}c_1 n < n - r_1; \tag{5}$$

in particular, each $a_{1,j}$ is smaller than each of $n-1, \ldots, n-(r_1-1)$. We take $A_1 = \{n, n-1, \ldots, n-(r_1-1)\} \cup \{a_{1,1}, \ldots, a_{1,s_1}\}$.

If $m = 2$, we are done. Otherwise, we proceed to construct A_2 as follows. Choose the largest positive integer r_2 with $(n-r_1)^k + \cdots + (n-r_2)^k \le T$. Then $n - r_2 \sim c_2 n$, for $c_2 = (1-2/m)^{1/(k+1)}$. Put

$$T_2 = T - ((n-r_1)^k + \cdots + (n-(r_2-1))^k).$$

Arguing as above, we see we can represent $T_2 = a_{2,1}^k + \cdots + a_{2,s_2}^k$, where the $a_{2,j}$ are distinct and

$$\frac{1}{2s_2}(n-r_2)^k < a_{2,j}^k < \frac{3}{2s_2}(n-r_2)^k.$$

Since $s_2 > 3s_1$,

$$\frac{3}{2s_2}(n-r_2)^k < \frac{3}{6s_1}(n-r_2)^k = \frac{1}{2s_1}(n-r_2)^k < \frac{1}{2s_1}(n-r_1)^k.$$

Comparing with (4), we see each $a_{2,j}$ is smaller than each $a_{1,j'}$. Furthermore, since $s_2 > s_1$ and $1 - \frac{2}{m} \ge \frac{1}{m}$, a calculation analogous to (5) shows that each

$$a_{2,j} < n - r_2.$$

We take $A_2 = \{n - r_1, n-(r_1+1), \ldots, n-(r_2-1)\} \cup \{a_{2,1}, \ldots, a_{2,s_2}\}$.

If $m > 3$, we continue in the obvious way to construct $A_3, \ldots, A_{m-1}$.

It remains to argue that $A_1, \ldots, A_{m-1}$ are disjoint. Each A_i consists of terms $n - t_i$ (for t_i in a certain interval) together with terms $a_{i,j}$. The t_i come from disjoint intervals, as i varies. Moreover, when $i < i'$, our construction guarantees that each $a_{i',j}$ is smaller than each $a_{i,j'}$. So it is enough to argue that no $n - t_i$ is equal to any $a_{i',j}$. Each $n - t_i \ge n - r_{m-1}$, and $n - r_{m-1} \sim c_{m-1} n$, where $c_{m-1} = (1-(m-1)/m)^{1/(k+1)} = (1/m)^{1/(k+1)}$. On the other hand, each

$$a_{i',j} < \left(\frac{3}{2s_{i'}}\right)^{1/k}(n-r_{i'}) < \left(\frac{3}{2s_1}\right)^{1/k}(n-r_{i'}) < \frac{1}{2}\left(\frac{1}{m}\right)^{1/(k+1)} n < n - r_{m-1}.$$

Hence, $n - t_i$ is strictly larger than $a_{i',j}$. □

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