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A SZEGŐ LIMIT THEOREM RELATED TO THE HILBERT MATRIX

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ABSTRACT. The Szegő limit theorem by Fedele and Gebert for matrices of the type identity minus Hankel matrix is proved for the special case $\mathbb{1} - \frac{\beta}{\pi} H_{N,\alpha}$ where $H_{N,\alpha}$ is the $N \times N$-Hilbert matrix, $\alpha \geq \frac{1}{2}$, and $\beta \in \mathbb{C}$. The proof uses operator theoretic tools and a reduction to the classical Kac–Akhiezer theorem for the Carleman operator. Thereby, the validity of the theorem for this special Hankel matrix can be extended from $|\beta| < 1$ to $\beta \in \mathbb{C} \setminus]1, \infty[$. The bound on the correction term is improved to $O(1)$ instead of $o(\ln(N))$ for $\beta \in \mathbb{C} \setminus [1, \infty[$. The limit case $\beta = 1$ is derived directly from the asymptotics for general β.

1. Introduction

The Hilbert matrix appeared recently in the investigation of several problems such as Anderson's orthogonality catastrophe for Fermi gases [3], [7] and the spectral statistics of random matrices [4]. In particular, all those problems led to some sort of Szegő limit theorem for determinants. Subsequently, Fedele and Gebert [2] proved a Szegő limit theorem for $\det(\mathbb{1} - \frac{\beta}{\pi} H_N)$ with a general $N \times N$ Hankel matrix H_N and a parameter $\beta \in \mathbb{C}$, $|\beta| < 1$.

Here, we give an alternative proof for the special case when H_N is the Hilbert matrix. The proof uses operator theoretic methods. A key ingredient is Wouk's integral formula (3) for the operator logarithm instead of the usual Taylor series. Thereby, the restriction $|\beta| < 1$ can be replaced by the much weaker $\beta \notin [1, \infty[$ and the correction term is improved to $O(1)$ instead of $o(\ln(N))$ as in [2]. The limit case $\beta = 1$ is directly deduced from the asymptotics for general β's by use of a simple product formula, see (6), which eventually is a consequence of the third binomial formula.

To be more precise, we consider the general Hilbert matrix

$$H_{N,\alpha} = \Big(\frac{1}{j+k+\alpha}\Big)_{j,k=0,\dots,N-1}, \; N \in \mathbb{N}, \; \alpha > 0.$$

and obtain a Szegő limit theorem for $\det(\mathbb{1} - \frac{\beta}{\pi} H_{N,\alpha})$ with $\alpha \geq \frac{1}{2}$. The case $0 < \alpha < \frac{1}{2}$ is not treated herein since it would cause additional technical difficulties. The first main result of the paper is the following, see Theorem 4.5.

Theorem 1.1. *Let $N \in \mathbb{N}$, $\alpha \geq \frac{1}{2}$ and $\beta \in \mathbb{C} \setminus [1, \infty[$. Then, the Hilbert matrix $H_{N,\alpha}$ satisfies*

$$\det\big(\mathbb{1} - \frac{\beta}{\pi} H_{N,\alpha}\big) = \exp[\frac{1}{2} \ln(N) \gamma(\beta) + O(1)] \text{ as } N \to \infty$$

with the coefficient

$$\gamma(\beta) = \frac{1}{\pi^2} [\operatorname{arcosh}(-\beta)]^2 + \frac{1}{4}.$$

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Here, arcosh *is the principal branch on the cut plane* $\mathbb{C}\setminus]-\infty,-1[$.

Note that $\gamma(\beta)$ is given in a different but equivalent form in [2], see (38).

The special determinant in Theorem 1.1 appeared in the study of the free Fermi gas in a magnetic field [7]. The transition probability between the ground states of a system of N free fermions in an interval of length L with and without a magnetic field is given by a certain $N\times N$ determinant $\mathscr{D}_{N,L}$. In the so-called thermodynamic limit, $N,L\to\infty$ with the particle density $N/L=\rho>0$ kept fixed, this determinant satisfies

$$\ln(\mathscr{D}_{N,L}) = \ln(\det(\mathbb{1}-\beta K_N)) + O(1).$$

The $N\times N$ matrix K_N is explicitly given [7, p. 12] and does not depend on the magnetic field, which enters only through the parameter β. In order to determine the asymptotics of $\mathscr{D}_{N,L}$ we, thus, would have to prove a Szegő limit theorem related to K_N which is somewhat tricky due to the complicated structure of K_N. However, the asymptotically dominant part turns out to be given by the Hilbert matrix $H_{N,\alpha}$ with $\alpha=\frac{1}{2}$ (there is a slightly different notation in [7] concerning α). The $O(1)$ term in the asymptotic formula for $\mathscr{D}_{N,L}$ is due to so-called finite size effects caused by the electrons having been confined to an interval of finite length. The precise nature of these finite size effects is of physical interest in its own right. Therefore, it is desirable that the correction terms in the Szegő limit theorems are as small as possible, ideally $O(1)$.

The proof of Theorem 1.1 consists of two parts. In the first part, we relate the determinant of the Hilbert matrix $H_{N,\alpha}$ to the Fredholm determinant of an integral operator $G_{N,\alpha}$ on a Hilbert space, Lemma 3.1,

$$\det(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha}) = \det(\mathbb{1}-\frac{\beta}{\pi}G_{N,\alpha}).$$

The idea here is, essentially, to write the matrix entries of $H_{N,\alpha}$ as Laplace transforms

$$\frac{1}{j+\alpha} = \int_0^\infty e^{-(j+\alpha)x}\,dx,\ j\in\mathbb{N}_0,\ \alpha>0.$$

We then show, Proposition 3.5, that

$$\det(\mathbb{1}-\frac{\beta}{\pi}G_{N,\alpha}) = \det(\mathbb{1}-\frac{\beta}{\pi}P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}KP_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]})\Delta_N(\beta).$$

Here $P_{[a,b]}$ denotes the orthogonal projection corresponding to the characteristic function $\chi_{[a,b]}$ of the interval $[a,b]$ and K is the Carleman operator

$$(K\varphi)(x) = \int_0^\infty \frac{1}{x+y}\varphi(y)\,dy.$$

The so-called perturbation determinant $\Delta_N(\beta)$, cf. (9), can be shown to satisfy

$$\ln(\Delta_N(\beta)) = O(1) \text{ as } N\to\infty.$$

Here is where Wouk's integral formula (3) is used, see (10).

In the second part, we transform the Carleman operator K unitarily to a convolution operator K_0, Lemma 4.2. Since the projection $P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}$ has to be transformed accordingly N becomes $n_{\frac{\alpha}{2}}(N)$. Finally, we apply a general version of the classical Kac–Akhiezer theorem, Proposition 4.1, to K_0 thereby completing the proof.

If we had used the Taylor series of the logarithm as in [2] we would have to work with

$$\ln\bigl(\det(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha})\bigr)=-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\beta^n}{\pi^n}\operatorname{tr}(H_{N,\alpha}^n).$$

However, the infinite series restricts the result to those β for which the series converges, namely $|\beta|<1$.

The second main result concerns the limit case $\beta=1$, see Theorem 5.8.

Theorem 1.2. *Let $\alpha\geq\frac{1}{2}$. Then,*

$$\ln\bigl(\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\bigr)=\frac{1}{2}\ln(N)\gamma(1)+o(\ln(N))\ \textit{as}\ N\to\infty$$

with $\gamma(1)=-\frac{3}{4}$.

The key idea of the proof is to write, Lemma 2.1,

$$\frac{1}{\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})}=\prod_{m=0}^{\infty}\det\bigl(\mathbb{1}+(\frac{1}{\pi}H_{N,\alpha})^{2^m}\bigr)$$

and use, at least formally, the asymptotics of each factor from Corollary 4.6. The corollary itself follows easily from Theorem 4.5 with the aid of the roots of unity. This idea can be made rigorous yielding, however, only a lower bound for the desired asymptotics, Proposition 5.8. Fortunately, since $H_{N,\alpha}$ is a non-negative operator an upper bound, Proposition 5.1, follows immediately from

$$\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\leq\det(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha}),\ \beta<1,$$

and Theorem 4.5.

The limit case $\beta=1$ was (for a special α) also treated in [4, Thm. 1.4]. The method used therein relied on the explicit diagonalization of the infinite Hilbert matrix.

2. Determinants

For a trace class operator $A:\mathscr{H}\to\mathscr{H}$ on a separable Hilbert space $\mathscr{H}$ one can define the Fredholm determinant $\det(\mathbb{1}-A)$. One way to do this is via the trace

$$\ln(\det(\mathbb{1}-A))=\operatorname{tr}[\ln(\mathbb{1}-A)] \tag{1}$$

with the principal branch of the logarithm

$$\ln(1-z)=-z\int_0^1\frac{1}{1-rz},\ z\in\mathbb{C}\setminus[1,\infty[. \tag{2}$$

The operator logarithm on the right-hand side is given by Wouk's integral formula [15]

$$\ln(\mathbb{1}-A)=-\int_0^1 A(\mathbb{1}-rA)^{-1}\,dr \tag{3}$$

which is valid whenever the spectrum $\sigma(A)$ of A satisfies $\sigma(A)\cap[1,\infty[=\emptyset$. For alternative definitions and further properties see e.g. [13, XIII]. Standard estimates for trace class operators $A,B:\mathscr{H}\to\mathscr{H}$

are (see [13, Lemma 4, p. 323])

$$|\det(\mathbb{1}-A)| \le e^{\|A\|_1}, \tag{4}$$

$$|\det(\mathbb{1}-A)-\det(\mathbb{1}-B)| \le \|A-B\|_1 \exp\big[\|A\|_1+\|B\|_1+1\big]. \tag{5}$$

Another estimate, which is of special importance herein (see Section 5), is based upon the infinite product

$$\frac{1}{1-x} = \prod_{m=0}^{\infty}(1+x^{2^m}),\ x\in\mathbb{R},\ |x|<1, \tag{6}$$

more precisely on the version for determinant.

Lemma 2.1. *Let $A:\mathscr{H}\to\mathscr{H}$ be a trace class operator with $\|A\|<1$. Then,*

$$\frac{1}{\det(\mathbb{1}-A)} = \prod_{m=0}^{\infty}\det(\mathbb{1}+A^{2^m}) \tag{7}$$

where the infinite product converges absolutely. Furthermore,

$$\frac{1}{|\det(\mathbb{1}-A)|} \le \prod_{m=0}^{M}\det(\mathbb{1}+A^{2^m})\exp\Big[\sum_{m=M+1}^{\infty}\|A^{2^m}\|_1\Big],\ M\in\mathbb{N}_0. \tag{8}$$

Proof. We start off from the analogon of (6)

$$\frac{1}{\det(\mathbb{1}-A)} = \frac{1}{\det(\mathbb{1}-A^{2^M})}\prod_{m=0}^{M-1}\det(\mathbb{1}+A^{2^m}),\ M\in\mathbb{N}.$$

By (4), (5), and Hölder's inequality for the trace norm

$$\Big|\prod_{m=0}^{M-1}\det(\mathbb{1}+A^{2^m})\Big| \le \prod_{m=0}^{M-1}(1+\|A\|^{2^m-1}\|A\|_1)$$

and

$$|\det(\mathbb{1}-A^{2^M})-1| \le \|A\|^{2^M-1}\|A\|_1\exp\big[\|A^{2^M-1}\|\|A\|_1+1\big].$$

Using the assumption $\|A\|<1$ we deduce

$$\frac{1}{\det(\mathbb{1}-A)} = \lim_{M\to\infty}\frac{1}{\det(\mathbb{1}-A^{2^M})}\prod_{m=0}^{M-1}\det(\mathbb{1}+A^{2^m}) = \prod_{m=0}^{\infty}\det(\mathbb{1}+A^{2^m}).$$

This is (7). Finally, write

$$\frac{1}{\det(\mathbb{1}-A)} = \prod_{m=0}^{M}\det(\mathbb{1}+A^{2^m})\prod_{m=M+1}^{\infty}\det(\mathbb{1}+A^{2^m})$$

and apply (4) to the second factor. This shows (8). □

The determinants of two trace class operators A and B are related via the perturbation determinant Δ

$$\det(\mathbb{1}-A)=\det(\mathbb{1}-B)\Delta,\ \Delta:=\det(\mathbb{1}-(\mathbb{1}-A)^{-1}(B-A)) \tag{9}$$

as long as the operator $\mathbb{1} - A$ is invertible. Wouk's formula (3) yields

$$\ln(\Delta) = -\operatorname{tr}\Big[(B-A)\int_0^1 (\mathbb{1} - rA - (1-r)B)^{-1}\, dr\Big]. \tag{10}$$

3. Hilbert matrix and Carleman operator

The Hilbert matrix is

$$H_{N,\alpha} = \left(\frac{1}{j+k+\alpha}\right)_{j,k=0,\dots,N-1}, \ \alpha > 0. \tag{11}$$

It is well-known that as an operator $H_{N,\alpha} : \mathbb{C}^N \to \mathbb{C}^N$ it satisfies

$$0 \le H_{N,\alpha} \text{ for } \alpha > 0 \text{ and } H_{N,\alpha} < \pi\mathbb{1} \text{ for } \alpha \ge \frac{1}{2} \tag{12}$$

in the sense of quadratic forms. We will not treat the case $0 < \alpha < \frac{1}{2}$ and, thus, do not need the corresponding norm. With the aid of the Laplace transform

$$\frac{1}{j+\alpha} = \int_0^\infty e^{-jx} e^{-\alpha x}\, dx,\ j \in \mathbb{N}_0,\ \alpha > 0,$$

we obtain a Hankel integral operator with, essentially, the same spectrum as $H_{N,\alpha}$.

Lemma 3.1. *Let $\alpha > 0$ and $N \in \mathbb{N}$. Define the Hankel integral operator $G_{N,\alpha} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$,*

$$(G_{N,\alpha}\varphi)(x) = \int_0^\infty G_{N,\alpha}(x+y)\varphi(y)\, dy,\ x \in \mathbb{R}^+,$$

with kernel function

$$G_{N,\alpha}(x) := e^{-\frac{\alpha}{2}x} \sum_{j=0}^{N-1} e^{-jx} = e^{-\frac{\alpha}{2}x} \frac{e^{\frac{x}{2}}}{2\sinh(\frac{x}{2})}(1 - e^{-Nx}).$$

Then, $\sigma(H_{N,\alpha}) \setminus \{0\} = \sigma(G_{N,\alpha}) \setminus \{0\}$. In particular, $\|G_{N,\alpha}\| = \|H_{N,\alpha}\|$.

Proof. With the functions

$$e_j \in L^2(\mathbb{R}^+),\ e_j(x) = e^{-jx - \frac{\alpha}{2}x},\ j \in \mathbb{N}_0,$$

we define the operators $A : L^2(\mathbb{R}^+) \to \mathbb{C}^N$ and $B : \mathbb{C}^N \to L^2(\mathbb{R}^+)$, $c = (c_0, \dots, c_{N-1})$,

$$(A\varphi)_j = \int_0^\infty e_j(x)\varphi(x)\, dx,\ j = 0, \dots, N-1,\ (Bc)(x) = \sum_{j=0}^{N-1} c_j e_j(x),\ x \in \mathbb{R}^+.$$

It is easily checked that $H_{N,\alpha} = AB : \mathbb{C}^N \to \mathbb{C}^N$. On the other hand, $BA : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$,

$$(BA\varphi)(x) = \sum_{j=0}^{N-1} e_j(x) \int_0^\infty e_j(y)\varphi(y)\, dy = \int_0^\infty \varphi(y) \sum_{j=0}^{N-1} e_j(x)e_j(y)\, dy = (G_{N,\alpha}\varphi)(x)$$

since $e_j(x)e_j(y) = e_j(x+y)$. Now, $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ which completes the proof. □

We extract the asymptotically relevant part of the operator $G_{N,\alpha}$. This gives rise to orthogonal projections generated by characteristic functions. Throughout, we will use the notation

$$P_{[a,b]} : L^2(X) \to L^2(X), \ (P_{[a,b]}\varphi)(x) = \chi_{[a,b]}(x)\varphi(x) \tag{13}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$ and X may be $\mathbb{R}$ or $\mathbb{R}^+$.

Lemma 3.2. *Let $E_\alpha : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ be the integral operator with kernel function*

$$E_\alpha(x,y) = e^{-x(y+\frac{\alpha}{2})}. \tag{14}$$

Then E_α, $\alpha \geq 0$, is bounded with $\|E_\alpha\| \leq \sqrt{\pi}$. Moreover, $E_\alpha P_{[0,N]} E_\alpha$, $\alpha > 0$, is a trace class operator on $L^2(\mathbb{R}^+)$ with

$$\|E_\alpha P_{[0,N]} E_\alpha^*\|_1 = \frac{1}{2}\ln\Big(\frac{2N+\alpha}{\alpha}\Big). \tag{15}$$

The difference

$$D_N := G_{N,\alpha} - E_\alpha P_{[0,N]} E_\alpha^*$$

is trace class with $\|D_N\| \leq C_\alpha < \infty$ for all $N \in \mathbb{N}$.

Proof. We use a generalized version of the Schur test (see [5, Thm. 5.2]) with test functions $p(x) = q(x) = \frac{1}{\sqrt{x}}$. Then, by standard computations

$$\int_0^\infty e^{-x(y+\frac{\alpha}{2})} \frac{1}{\sqrt{y}}\, dy = \frac{\sqrt{\pi}}{\sqrt{x}} e^{-\frac{\alpha x}{2}} \leq \sqrt{\pi}\frac{1}{\sqrt{x}}, \ x > 0.$$

Likewise,

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-x(y+\frac{\alpha}{2})}\, dx \leq \int_0^\infty \frac{1}{\sqrt{x}} e^{-xy}\, dx = \sqrt{\pi}\frac{1}{\sqrt{y}}, \ y > 0.$$

This implies E_α is bounded with the given estimate for the norm.

In order to show the trace class property we start from the simple formula

$$1 - e^{-Nx} = x\int_0^N e^{-xt}\, dt$$

and rewrite the kernel function $G_{N,\alpha}$

$$G_{N,\alpha}(x) = e^{-\frac{\alpha}{2}x} \frac{e^{\frac{x}{2}}x}{2\sinh(\frac{x}{2})} \int_0^N e^{xt}\, dt = \int_0^N e^{-x(t+\frac{\alpha}{2})}\, dt + e^{-\frac{\alpha}{2}x}\Big[\frac{e^{\frac{x}{2}}x}{2\sinh(\frac{x}{2})} - 1\Big] \int_0^N e^{-xt}\, dt.$$

The first term gives rise to the Hankel operator $\tilde{G}_{N,\alpha}$ with kernel function

$$\tilde{G}_{N,\alpha}(x) = \int_0^N e^{-x(t+\frac{\alpha}{2})}\, dt.$$

We write this as follows (cf. (14))

$$\tilde{G}_{N,\alpha}(x+y) = \int_0^N e^{-(x+y)(t+\frac{\alpha}{2})}\, dt = \int_0^N e^{-x(t+\frac{\alpha}{2})} e^{-(t+\frac{\alpha}{2})y}\, dt = \int_0^N E_\alpha(x,t)E_\alpha(y,t)\, dt$$

which implies $\tilde{G}_{N,\alpha} = E_\alpha P_{[0,N]} E_\alpha^*$. Since, obviously, $E_\alpha P_{[0,N]} E_\alpha^* \geq 0$ we obtain

$$\|E_\alpha P_{[0,N]} E_\alpha^*\|_1 = \operatorname{tr}(E_\alpha P_{[0,N]} E_\alpha^*) = \int_0^\infty \int_0^N e^{-2x(y+\frac{\alpha}{2})}\,dy\,dx = \int_0^N \frac{1}{\alpha+2y}\,dy = \frac{1}{2}\ln\Big(\frac{\alpha+2N}{\alpha}\Big).$$

The remaining difference is the Hankel operator D_N with kernel function

$$D_N(x) := e^{-\frac{\alpha}{2}x}\Big[\frac{e^{\frac{x}{2}}x}{2\sinh(\frac{x}{2})} - 1\Big]\int_0^N e^{-xt}\,dt = \Big[\frac{e^{\frac{x}{2}}x}{2\sinh(\frac{x}{2})} - 1\Big]\int_{\frac{\alpha}{2}}^{N+\frac{\alpha}{2}} e^{-xt}\,dt.$$

In order to show that D_N is in the trace class we use Howland's criterion [6, Thm. 2.1], which also gives a bound on the trace norm. To this end, we need the derivative

$$D_N'(x) = \Big\{\frac{1-e^{-x}-xe^{-x}}{(1-e^{-x})^2} - \Big[\frac{x}{1-e^{-x}} - 1\Big]x\Big\}\int_{\frac{\alpha}{2}}^{N+\frac{\alpha}{2}} e^{-xt}\,dt.$$

Via the elementary estimates

$$0 \leq \frac{x}{1-e^{-x}} - 1 \leq x,\ 0 \leq \frac{1-e^{-x}-xe^{-x}}{(1-e^{-x})^2} \leq 1 \text{ for } x \geq 0,$$

we obtain

$$|D_N'(x)| \leq (1+x^2)\int_{\frac{\alpha}{2}}^{N+\frac{\alpha}{2}} e^{-xt}\,dt \leq \Big(x+\frac{1}{x}\Big)e^{-\frac{\alpha}{2}x}.$$

Then, Howland's criterion shows that D_N is in the trace class with

$$\|D_N\|_1 \leq \int_0^\infty x^{\frac{1}{2}}\Big[\int_x^\infty |D_N'(y)|^2\,dy\Big]^{\frac{1}{2}}dx \leq \int_0^\infty x^{\frac{1}{2}}\Big[\int_x^\infty \Big(y^2+2+\frac{1}{y^2}\Big)e^{-\alpha y}\,dy\Big]^{\frac{1}{2}}dx =: C_\alpha.$$

Elementary estimates show that $C_\alpha < \infty$ for $\alpha > 0$. Note that C_α does not depend on N. □

We relate $E_\alpha P_{[0,N]} E_\alpha^*$ to the Carleman operator $K : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$,

$$(K\varphi)(x) = \int_0^\infty \frac{1}{x+y}\varphi(y)\,dy,\ x \in \mathbb{R}^+. \tag{16}$$

It is well-known that K is self-adjoint and satisfies (see [12, Theorem 8.14] for the operator norm)

$$0 \leq K \leq \pi. \tag{17}$$

We define the translation operator

$$T_\alpha : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+),\ (T_\alpha\varphi)(x) = \begin{cases} \varphi(x-\frac{\alpha}{2}) & \text{for } x \geq \frac{\alpha}{2}, \\ 0 & \text{for } 0 \leq x < \frac{\alpha}{2}. \end{cases} \tag{18}$$

Its pseudo inverse is given by

$$T_\alpha^+ : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+),\ (T_\alpha^+\varphi)(x) = \varphi\Big(x+\frac{\alpha}{2}\Big),\ x \geq 0.$$

That is to say,

$$P_{[\frac{\alpha}{2},\infty[}T_\alpha T_\alpha^+ = P_{[\frac{\alpha}{2},\infty[}. \tag{19}$$

We move the α from the integral operator to the projection.

Lemma 3.3. *Let $\alpha > 0$ and $N \in \mathbb{N}$. The operator $E_\alpha P_{[0,N]} E_\alpha^*$ and the Carleman operator K, cf. (14) and (16), satisfy*

$$\sigma(E_\alpha P_{[0,N]} E_\alpha^*) \setminus \{0\} = \sigma(P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} K P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}) \setminus \{0\}. \tag{20}$$

Proof. We know that

$$\sigma(E_\alpha P_{[0,N]} E_\alpha^*) \setminus \{0\} = \sigma(E_\alpha^* E_\alpha P_{[0,N]}) \setminus \{0\}.$$

The product $E_\alpha^* E_\alpha$ is a quasi-Carleman operator

$$(E_\alpha^* E_\alpha)(x,y) = \int_0^\infty e^{-(x+\frac{\alpha}{2})t} e^{-t(y+\frac{\alpha}{2})}\, dt = \frac{1}{x+y+\alpha}.$$

By using T_α (cf. (18))

$$\begin{aligned}
(E_\alpha^* E_\alpha P_{[0,N]} \varphi)(x) &= \int_0^N \frac{1}{x+y+\alpha} \varphi(y)\, dy \\
&= \int_{\frac{\alpha}{2}}^{N+\frac{\alpha}{2}} \frac{1}{x+y+\frac{\alpha}{2}} \varphi(y-\frac{\alpha}{2})\, dy \\
&= \int_0^\infty \frac{1}{x+y+\frac{\alpha}{2}} \chi_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} (T_\alpha \varphi)(y)\, dy \\
&= (T_\alpha^+ K P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} T_\alpha \varphi)(x).
\end{aligned}$$

In operator form this reads

$$E_\alpha^* E_\alpha P_{[0,N]} = T_\alpha^+ K P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} T_\alpha$$

which implies

$$\sigma(E_\alpha^* E_\alpha P_{[0,N]}) \setminus \{0\} = \sigma(K P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} T_\alpha T_\alpha^+) \setminus \{0\} = \sigma(K P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}) \setminus \{0\}.$$

Here we used (19). This implies (20). □

In order to use the perturbation determinant (9) we need a certain inverse.

Lemma 3.4. *Let $\alpha \geq \frac{1}{2}$. Furthermore, let $\beta \in \mathbb{C} \setminus [1,\infty[$, $s \in [0,1]$, and $N \in \mathbb{N}$. Then, the operator $\mathbb{1} - \beta A_{N,\alpha}(s)$,*

$$A_{N,\alpha}(s) := \frac{1}{\pi}\big((1-s) E_\alpha P_{[0,N]} E_\alpha^* + s G_{N,\alpha}\big),$$

is invertible with

$$\|(\mathbb{1} - \beta A_{N,\alpha}(s))^{-1}\| \leq \begin{cases} 1 & \text{for } \mathrm{Re}(\beta) \leq 0, \\ \frac{1}{1-\mathrm{Re}(\beta)} & \text{for } 0 < \mathrm{Re}(\beta) < 1, \\ \frac{|\beta|}{|\mathrm{Im}(\beta)|} & \text{for } \mathrm{Im}(\beta) \neq 0. \end{cases}$$

Proof. We use the Lax–Milgram theorem to show invertibility of $\mathbb{1} - \beta A_{N,\alpha}(s)$ and to prove the estimates for the norm of the inverse. Note that Lemmas 3.2 and 3.1 along with (12) imply $0 \leq A_{N,\alpha}(s) \leq \mathbb{1}$ in the sense of quadratic forms. Furthermore,

$$\mathrm{Re}(\mathbb{1} - \beta A_{N,\alpha}(s)) = \mathbb{1} - \mathrm{Re}(\beta) A_{N,\alpha}(s).$$

Hence, for $\mathrm{Re}(\beta) \le 0$

$$\mathrm{Re}(\mathbb{1} - \beta A_{N,\alpha}(s)) \ge \mathbb{1}$$

and for $0 < \mathrm{Re}(\beta) < 1$

$$\mathrm{Re}(\mathbb{1} - \beta A_{N,\alpha}(s)) \ge (1 - \mathrm{Re}(\beta))\mathbb{1} \text{ with } 1 - \mathrm{Re}(\beta) > 0,$$

which yield the first two cases. In the third case, surely $\beta \ne 0$. Hence,

$$\mathbb{1} - \beta A_{N,\alpha}(s) = \beta(\frac{1}{\beta}\mathbb{1} - A_{N,\alpha}(s))$$

and

$$\mathrm{Im}(\frac{1}{\beta}\mathbb{1} - A_{N,\alpha}(s)) = -\frac{\beta}{|\beta|^2}\mathbb{1}.$$

This implies that the inverse exists and is bounded with

$$\|(\mathbb{1} - \beta A_{N,\alpha}(s))^{-1}\| = \frac{1}{|\beta|}\|(\frac{1}{\beta}\mathbb{1} - A_{N,\alpha}(s))^{-1}\| \le \frac{1}{|\beta|}\frac{|\beta|^2}{|\mathrm{Im}(\beta)|}.$$

This completes the proof. □

The asymptotics of the determinant under study is given by the corresponding determinant of the Carleman operator.

Proposition 3.5. *Let $\alpha > 0$ and $N \in \mathbb{N}$. The operator $P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}KP_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, cf. (16), is in the trace class. Furthermore, if $\alpha \ge \frac{1}{2}$ and $\beta \in \mathbb{C} \setminus [1,\infty[$,*

$$\det(\mathbb{1} - \frac{\beta}{\pi}H_{N,\alpha}) = \det(\mathbb{1} - \frac{\beta}{\pi}P_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]}KP_{[\frac{\alpha}{2},N+\frac{\alpha}{2}]})\Delta_N(\beta)$$

where the perturbation determinant can be bounded as

$$\exp[-C(\beta)|\beta|\|D_N\|_1] \le |\Delta_N(\beta)| \le \exp[C(\beta)|\beta|\|D_N\|_1]$$

with $0 \le C(\beta) < \infty$ independent of N, cf. Lemmas 3.4 and 3.2.

Proof. The trace class property follows immediately from Lemmas 3.3 and 3.2. We apply the formula (9) for the perturbation determinant to the operator (cf. Lemma 3.2)

$$G_{N,\alpha} = E_\alpha P_{[0,N]}E_\alpha^* + D_N$$

thereby obtaining

$$\det(\mathbb{1} - \frac{\beta}{\pi}G_{N,\alpha}) = \det(\mathbb{1} - \frac{\beta}{\pi}E_\alpha P_{[0,N]}E_\alpha^*)\Delta_N(\beta).$$

Using the formula (10) for the perturbation determinant we write this as

$$\Delta_N(\beta) = \exp\left[-\frac{\beta}{\pi}\int_0^1 \mathrm{tr}\left\{\left[\mathbb{1} - A_{N,\alpha}(s)\right]^{-1}D_N\right\}ds\right]$$

with $A_{N,\alpha}(s)$ from Lemma 3.4. Finally, we bound the trace by the trace norm and use Lemma 3.4 to estimate the norm of the inverse. This completes the proof. □

4. Szegő limit theorem

In order to handle the complex parameter β we formulate an abstract Szegő theorem for normal operators based upon [10] and [1].

Proposition 4.1. *Let $A : \mathscr{H} \to \mathscr{H}$ be a bounded normal operator with*

$$\mathrm{Re}(\lambda) \geq m, \ \mathrm{Im}(\lambda) \in [y_0 - h, y_0 + h] \text{ for all } \lambda \in \sigma(A) \tag{21}$$

where $m, y_0 \in \mathbb{R}$ and $0 \leq h < \frac{\pi}{2}$. Furthermore, let $P : \mathscr{H} \to \mathscr{H}$ be an orthonormal projection such that PAP is trace class. Then, the determinant of the operator $Pe^A P : \mathrm{ran}(P) \to \mathrm{ran}(P)$ satisfies

$$\det(Pe^A P) = \exp[\mathrm{tr}(PAP) + \rho(A)] \tag{22}$$

where the correction term $\rho(A) \in \mathbb{C}$ satisfies

$$|\rho(A)| \leq \frac{1}{2} \frac{e^{|m|}}{\cos(h)} e^{\|A\|} \|PA(\mathbb{1} - P)\|_2 \|(\mathbb{1} - P)AP\|_2. \tag{23}$$

Proof. From (19) in [10] follows

$$|\rho(A)| \leq e^{\|A\|} \|PA(\mathbb{1} - P)\|_2 \|(\mathbb{1} - P)AP\|_2 \int_0^1 t \|(Pe^{tA}P)^{-1}P\| \, dt.$$

From (15) and (16) in [1] we infer

$$\|(Pe^{tA}P)^{-1}P\| \leq \frac{e^{|m|}}{\cos(h)}, \ 0 \leq t \leq 1,$$

which proves the statement. □

In the special case when $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a convolution operator with even kernel function $A(x) = A(-x)$ and $P = P_{[-n,n]}$ the Hilbert–Schmidt norm appearing in Proposition 4.1 can be written after some simple calculations as

$$\begin{aligned} \|P_{[-n,n]}A(\mathbb{1} - P_{[-n,n]})\|_2^2 &= \int_{|x| \leq n} \int_{|y| \geq n} |A(x-y)|^2 \, dy \, dx \\ &= 2\int_0^n x|A(x)|^2 \, dx + 2n \int_n^\infty |A(x)|^2 \, dx + 2\int_0^n \int_n^\infty |A(x+y)|^2 \, dy \, dx. \end{aligned} \tag{24}$$

In order to apply the abstract result in Proposition 4.1 to our case, we have to write the operator at hand as $\mathbb{1} - \frac{\beta}{\pi} K = e^A$. In other words we need a logarithm which is no problem here since the Carleman operator K can be diagonalized explicitly by means of the Mellin transform. For our purposes it is more convenient to stop halfway and transform it into a convolution operator.

Lemma 4.2. *The operator $W_a : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$, $a \in \mathbb{R}$,*

$$(W_a \varphi)(s) = \sqrt{2} e^{s+a} \varphi(e^{2s+2a}), \ s \in \mathbb{R}, \ \varphi \in L^2(\mathbb{R}^+) \tag{25}$$

is unitary. It transforms the Carleman operator K into a convolution operator

$$W_a K W_a^* = K_0, \ K_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \ K_0(x-y) = \frac{1}{\cosh(x-y)} \tag{26}$$

and the projection with $a = \frac{1}{4}(\ln(N+\frac{\alpha}{2}) + \ln(\frac{\alpha}{2}))$

$$W_a P_{[\frac{\alpha}{2}, N+\frac{\alpha}{2}]} W_a^* = P_{[-n_{\frac{\alpha}{2}}(N), n_{\frac{\alpha}{2}}(N)]}, \ n_{\frac{\alpha}{2}}(N) = \frac{1}{4}\ln(\frac{N+\frac{\alpha}{2}}{\frac{\alpha}{2}}). \tag{27}$$

Proof. Cf. [12, Ch. 10, Thm. 2.1] and also [16]. We will use the substitution

$$x = e^{2s+2a}, \ dx = 2e^{2s+2a}\, ds.$$

The unitarity follows from, $\varphi \in L^2(\mathbb{R}^+)$,

$$\|W_a\varphi\|^2 = 2\int_{\mathbb{R}} |\varphi(e^{2s+2a})|^2 e^{2s+2a}\, ds = \int_0^\infty |\varphi(x)|^2\, dx = \|\varphi\|^2$$

and the analogous calculation for W_a^*. For the Carleman operator we obtain

$$\begin{aligned}
(W_a K\varphi)(s) &= \sqrt{2}e^{s+a}\int_0^\infty \frac{1}{e^{2s+2a}+y}\varphi(y)\, dy \\
&= \sqrt{2}e^{s+a}\int_{\mathbb{R}} \frac{2e^{2t+2a}}{e^{2s+2a}+e^{2t+2a}}\varphi(e^{2t+2a})\, dt \\
&= \sqrt{2}\int_{\mathbb{R}} \frac{2}{e^{s-t}+e^{t-s}} e^{t+a}\varphi(e^{2t+2a})\, dt \\
&= \int_{\mathbb{R}} \frac{1}{\cosh(s-t)}(W_a\varphi)(t)\, dt \\
&= (K_0 W_a \varphi)(s)
\end{aligned}$$

which reads in operator form

$$W_a K = K_0 W_a.$$

This yields (26). Finally,

$$\chi_{[\frac{\alpha}{2}, \mathrm{N}+\frac{\alpha}{2}]}(e^{2s+2a}) = \begin{cases} 1 & \text{for } \frac{\alpha}{2} \le e^{2s+2a} \le N+\frac{\alpha}{2}, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1 & \text{for } \frac{1}{2}\ln(\frac{\alpha}{2}) - a \le s \le \frac{1}{2}\ln(N+\frac{\alpha}{2}) - a, \\ 0 & \text{otherwise}. \end{cases}$$

The special a yields the formula (27) for the projection. □

Via the Fourier transform

$$(\mathscr{F}\varphi)(\omega) := \hat{\varphi}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\omega x}\varphi(x)\, dx, \ \varphi \in L^2(\mathbb{R}), \tag{28}$$

the convolution operator K_0 can be transformed into a multiplication operator

$$\mathscr{F}K_0\varphi = \sqrt{2\pi}\hat{K}_0\hat{\varphi}, \ \hat{K}_0(\omega) = \sqrt{\frac{\pi}{2}}\frac{1}{\cosh(\frac{\omega\pi}{2})}. \tag{29}$$

Thereby, we can construct the logarithm needed for the Szegő theorem.

Lemma 4.3. *Let* $\beta \in \mathbb{C}\setminus[1,\infty[$ *and let the convolution operator* $A_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *be given by its kernel function*

$$A_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x}\ln\big(1 - \frac{\beta}{\cosh(\frac{\omega\pi}{2})}\big)\, d\omega, \ \hat{A}_0(\omega) = \frac{1}{\sqrt{2\pi}}\ln\big(1 - \frac{\beta}{\cosh(\frac{\omega\pi}{2})}\big). \tag{30}$$

Then,

$$e^{A_0} = \mathbb{1} - \frac{\beta}{\pi} K_0. \tag{31}$$

Furthermore, the spectrum $\sigma(A_0)$ of A_0 satisfies

$$\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A_0)\} = [m, M] \textit{ with } M = \max\{0, \ln|1-\beta|\},$$
$$m = \begin{cases} \ln|1 - \frac{\mathrm{Re}(\beta)}{\beta}| & \textit{if } 0 \le \mathrm{Re}(\beta) \le |\beta|^2, \\ \min\{0, \ln|1-\beta|\} & \textit{otherwise}, \end{cases} \tag{32}$$

and

$$\{\mathrm{Im}(\lambda) \mid \lambda \in \sigma(A_0)\} = [y_0 - h, y_0 + h],\ y_0 = \frac{1}{2}a(\beta),\ h = \frac{1}{2}|a(\beta)| < \frac{\pi}{2},$$
$$a(\beta) = -\,\mathrm{sign}(\mathrm{Im}(\beta))\left[\frac{\pi}{2} - \arctan\left(\frac{1-\mathrm{Re}(\beta)}{|\mathrm{Im}(\beta)|}\right)\right]. \tag{33}$$

Proof. To get all the π's right note that (31) is, via the Fourier transform (cf. (28)), equivalent to

$$\exp(\sqrt{2\pi}\hat{A}_0(\omega)) = 1 - \frac{\beta}{\pi}\sqrt{2\pi}\hat{K}_0(\omega).$$

Solving for $\hat{A}_0(\omega)$ and using (29) for $\hat{K}_0(\omega)$ as well as the inverse Fourier transform prove (30).

The spectrum of A_0 is given up to factor through the numerical range of the function $\hat{A}_0$

$$\sigma(A_0) = \{\ln(1 - \frac{\beta}{\cosh(\frac{\omega\pi}{2})}) \mid \omega \in \mathbb{R}\} \cup \{0\} = \{\ln(1 - s\beta) \mid 0 \le s \le 1\}.$$

Using the the principal branch of the logarithm as in (2) yields

$$\ln(1 - s\beta) = -\beta \int_0^s \frac{1}{1-\beta t}\,dt = -\int_0^s \frac{\beta - |\beta|^2 t}{|1-\beta t|^2}\,dt.$$

The imaginary part is

$$\mathrm{Im}(\ln(1-s\beta)) = -\,\mathrm{Im}(\beta)\int_0^s \frac{1}{|1-\beta t|^2}\,dt.$$

The integral vanishes at $s = 0$ and attains its maximal value at $s = 1$. For $\mathrm{Im}(\beta) \ne 0$ we obtain after some standard substitutions

$$\mathrm{Im}(\ln(1-\beta)) = -\,\mathrm{Im}(\beta)\int_1^\infty \frac{1}{|t-\beta|^2}\,dt = -\,\mathrm{sign}(\mathrm{Im}(\beta))\int_{\frac{1-\mathrm{Re}(\beta)}{|\mathrm{Im}(\beta)|}}^\infty \frac{1}{t^2+1}\,dt$$

and for the remaining case

$$\mathrm{Im}(\ln(1 - \frac{\beta}{s})) = 0 \text{ for } \mathrm{Im}(\beta) = 0.$$

this implies (33). The bound $h \le \frac{\pi}{2}$ is obvious. Since $h = \frac{\pi}{2}$ would require $1 - \mathrm{Re}(\beta) < 0$ and $\mathrm{Im}(\beta) = 0$ this cannot occur due to the assumptions on β.

The real part is

$$\mathrm{Re}(\ln(1-s\beta)) = -\int_0^s \frac{\mathrm{Re}(\beta) - |\beta|^2 t}{|1-\beta t|^2}\,dt = \ln|1-s\beta| =: f(s).$$

For those β's satisfying

$$0 \le \mathrm{Re}(\beta) \le |\beta|^2$$

the function f has a single local extremum at $s_- \in [0,1]$, which is a minimum with

$$f(s_-) = \ln\left|1 - \frac{\mathrm{Re}(\beta)}{\beta}\right| = \ln\left(\frac{|\mathrm{Im}(\beta)|}{|\beta|}\right) \le 0.$$

For any other β the extremal values are given by $f(0) = 0$ and $f(1) = \ln|1-\beta|$. This proves (32). $\square$

We apply Proposition 4.1 to the operator K_0.

Proposition 4.4. *Let $\beta \in \mathbb{C} \setminus [1,\infty[$ and $n \ge 0$. Then for K_0 from (26),*

$$\det\left(\mathbb{1} - \frac{\beta}{\pi} P_{[-n,n]} K_0 P_{[-n,n]}\right) = \exp[2n\gamma(\beta) + \rho_n]. \tag{34}$$

Here

$$\gamma(\beta) = \frac{1}{\pi^2}\int_0^\infty \ln\left(1 - \frac{\beta}{\cosh(\omega)}\right) d\omega = \frac{1}{\pi^2}\left[\operatorname{arcosh}(-\beta)\right]^2 + \frac{1}{4} \tag{35}$$

and the correction term $\rho_n \in \mathbb{C}$ satisfies (cf. (30))

$$|\rho_n| \le \frac{3}{4\pi} \frac{e^{|m|}}{\cos(h)} \left(\|\hat{A}_0\|_1 + \|\hat{A}_0''\|_1\right)^2$$

with m from (32), $0 \le h < \frac{\pi}{2}$ from (33), and $\hat{A}_0$ from (30). $\|\cdot\|_1$ denotes the $L^1(\mathbb{R})$-Norm.

Proof. We check the conditions of Proposition 4.1. The second part of (21) follows immediately from (33) since $0 \le h < \frac{\pi}{2}$ for $\beta \in \mathbb{C} \setminus [1,\infty[$. For the real part the only critical cases in (32) are $\beta = 1$ and $\frac{\mathrm{Re}(\beta)}{\beta} = 1$, which is equivalent to $\beta = 1$. Since $\beta \notin]1,\infty]$ this cannot occur. Hence, there is an $m \in \mathbb{R}$ with $|m| < \infty$ such that the first part in (21) holds true.

In order to bound the correction ρ_n term we use (24). Since A_0 is the Fourier transform of an L^1-function $\hat{A}_0$ that is arbitrarily often differentiable and vanishes at infinity appropriately, cf. (30), a simple integration by parts shows

$$|A_0(x)| \le \frac{1}{\sqrt{2\pi}} \frac{1}{1+x^2} \left[\|\hat{A}_0\|_1 + \|\hat{A}_0''\|_1\right], \; x \in \mathbb{R}.$$

For $\beta \notin]1,\infty]$ the L^1-norms are finite which follows most conveniently from the representation

$$\hat{A}_0(\omega) = -\frac{\beta}{\sqrt{2\pi}} \int_0^1 \frac{1}{\cosh(\frac{\omega\pi}{2}) - t\beta}\, dt$$

and the analogous formula for $\hat{A}_0''(\omega)$. The integrals in (24) become in our case

$$2\int_0^n \frac{x}{(1+x^2)^2}\,dx \le 1,\; 2n\int_n^\infty \frac{1}{(1+x^2)^2}\,dx \le \int_n^\infty \frac{2x}{(1+x^2)^2}\,dx = \frac{1}{1+n^2},$$
$$2\int_0^n \int_n^\infty \frac{1}{(1+(x+y)^2)^2}\,dy\,dx \le 2n\int_n^\infty \frac{1}{(1+y^2)^2}\,dy \le \frac{1}{1+n^2}.$$

Thereby,

$$\|P_{[-n,n]}A_0(\mathbb{1}-P_{[-n,n]})\|\cdot\|(\mathbb{1}-P_{[-n,n]})A_0P_{[-n,n]}\|\leq\frac{3}{2\pi}[\|\hat{A}_0\|_1+\|\hat{A}_0''\|_1]$$

Finally, the leading term in (22) is

$$\operatorname{tr}(P_{[-n,n]}A_0P_{[-n,n]})=2nA_0(0)$$

Now,

$$A_0(0)=\frac{1}{2\pi}\int_{\mathbb{R}}\ln\big(1-\frac{\beta}{\cosh(\frac{\omega\pi}{2})}\big)\,d\omega=\frac{2}{\pi^2}\int_0^\infty\ln\big(1-\frac{\beta}{\cosh(\omega)}\big)\,d\omega=\frac{1}{\pi^2}[\operatorname{arcosh}(-\beta)]^2+\frac{1}{4}$$

where we evaluated the integral via Lemma A.2. This completes the proof. □

We summarize our findings by formulating the main result, the Szegő limit theorem for the Hilbert matrix.

Theorem 4.5. *Let $\alpha\geq\frac{1}{2}$ and $N\in\mathbb{N}$. Then, the Hilbert matrix $H_{N,\alpha}$, see (11), satisfies for all $\beta\in\mathbb{C}\setminus[1,\infty[$*

$$\det\big(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha}\big)=\exp[\frac{1}{2}\ln(N)\gamma(\beta)+O(1)]\ \text{as } N\to\infty \tag{36}$$

with the coefficient

$$\gamma(\beta)=\frac{1}{\pi^2}[\operatorname{arcosh}(-\beta)]^2+\frac{1}{4}. \tag{37}$$

Proof. From Proposition 3.5 we know

$$\ln\big(\det(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha})\big)=\ln\big(\det(\mathbb{1}-\frac{\beta}{\pi}P_{[-\frac{\alpha}{2},N+\frac{\alpha}{2}]}KP_{[-\frac{\alpha}{2},N+\frac{\alpha}{2}]})\big)+O(1)$$

with the Carleman operator K from (16). From Proposition 4.4 we infer

$$\det(\mathbb{1}-\frac{\beta}{\pi}P_{[-\frac{\alpha}{2},N+\frac{\alpha}{2}]}KP_{[-\frac{\alpha}{2},N+\frac{\alpha}{2}]})=\det(\mathbb{1}-P_{[-n_{\frac{\alpha}{2}}(N),n_{\frac{\alpha}{2}}(N)]}K_0P_{[-n_{\frac{\alpha}{2}}(N),n_{\frac{\alpha}{2}}(N)]}).$$

The Szegő theorem for K_0, Proposition 4.4, is

$$\ln\big(\det(\mathbb{1}-P_{[-n_{\frac{\alpha}{2}}(N),n_{\frac{\alpha}{2}}(N)]}K_0P_{[-n_{\frac{\alpha}{2}}(N),n_{\frac{\alpha}{2}}(N)]})\big)=2n_{\frac{\alpha}{2}}(N)\gamma(\beta)+O(1).$$

Combining these formulae and noting

$$n_{\frac{\alpha}{2}}(N)=\frac{1}{4}\ln\Big(\frac{N+\frac{\alpha}{2}}{\frac{\alpha}{2}}\Big)=\frac{1}{4}\ln(N)+O(1)\text{ as } N\to\infty$$

prove the theorem. □

Though the result in [2] looks a bit different from ours it is actually the same. For,

$$\operatorname{arcosh}(x)=i\arccos(x),\ \arccos(x)=\frac{\pi}{2}-\arcsin(x),\ x\in[-1,1].$$

These imply

$$\frac{1}{\pi^2}(\operatorname{arcosh}(-\beta))^2+\frac{1}{4}=-\frac{1}{\pi^2}(\frac{\pi}{2}-\arcsin(-\beta))^2+\frac{1}{4}=-\frac{1}{\pi^2}(\arcsin(\beta)^2+\pi\arcsin(\beta))$$

which yields the asymptotic formula from [2, (1.5)]

$$\ln\big(\det(\mathbb{1}-\frac{\beta}{\pi}H_{N,\alpha})\big) \sim -\frac{1}{2\pi^2}\big([\arcsin(\beta)]^2+\pi\arcsin(\beta)\big)\ln(N) \text{ as } N\to\infty. \tag{38}$$

By a simple argument based upon the roots of unity we extend our Szegő theorem to even powers of the Hilbert matrix. This will be used for the limit case $\beta=1$, which is not covered by Theorem 4.5.

Corollary 4.6. *Let $m\in\mathbb{N}$ and $\alpha\geq\frac{1}{2}$. Then, the Hilbert matrix $H_{N,\alpha}$ satisfies*

$$\det\big(\mathbb{1}+\frac{1}{\pi^{2m}}H^{2m}_{N,\alpha}\big)=\exp[\frac{1}{2}\ln(N)\gamma_{2m}+O(1)] \text{ as } N\to\infty \tag{39}$$

where

$$\gamma_{2m}=\frac{2}{\pi^2}\int_0^\infty \ln\big(1+\frac{1}{\cosh(\omega)^{2m}}\big)\,d\omega. \tag{40}$$

Proof. Let us define

$$\eta_k=\frac{2k-1}{2m},\ k=1,\dots,m,$$

whereby we can factorize the determinant into

$$\det(\mathbb{1}+\frac{1}{\pi^{2m}}H^{2m}_{N,\alpha})=\prod_{k=1}^m\det(\mathbb{1}+\frac{1}{\pi}e^{i\pi\eta_k}H_{N,\alpha})\prod_{k=1}^m\det(\mathbb{1}+\frac{1}{\pi}e^{-i\pi\eta_k}H_{N,\alpha}).$$

Note that $e^{\pm i\eta_k}\neq -1$. Therefore, we may apply Theorem 4.5 to each factor in the product which yields for the leading term in the asymptotics

$$\gamma_{2m}=\frac{2}{\pi^2}\Big\{\sum_{k=1}^m\int_0^\infty\ln\big(1-\frac{e^{i\pi\eta_k}}{\cosh(\omega)}\big)\,d\omega+\sum_{k=1}^m\int_0^\infty\ln\big(1-\frac{e^{-i\pi\eta_k}}{\cosh(\omega)}\big)\,d\omega\Big\}.$$

Here we used the integral representation (35) for the coefficients. In order to rewrite this we note that for the principal branch of the logarithm

$$\ln(z)+\ln(\bar{z})=\ln(z\bar{z})=2\ln(|z|) \text{ for all } z\in\mathbb{C}\setminus\{0\}$$

which implies

$$\gamma_{2m}=\frac{2}{\pi^2}\int_0^\infty\ln\Big[\prod_{k=-m}^m(1-\frac{e^{i\pi\eta_k}}{\cosh(\omega)})\Big]\,d\omega$$

and thus (40). Since the product is finite the sum of the $O(1)$ terms in (36) is still $O(1)$ which shows (39). □

5. Limit case

We treat the limit case $\beta=1$, which was not covered by Theorem 4.5, by showing that it is the limit, hence the name, of the asymptotics for admissible β. More precisely, we provide an upper and lower bound for the asymptotics. The upper bound is straightforward.

Proposition 5.1. *Let $\alpha\geq\frac{1}{2}$ and $N\in\mathbb{N}$. Then,*

$$\limsup_{N\to\infty}\frac{2}{\ln(N)}\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\big)\leq\gamma(1). \tag{41}$$

Proof. Let $\beta < 1$. Since $H_{N,\alpha} \geq 0$,

$$\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha}) \leq \det(\mathbb{1} - \frac{\beta}{\pi} H_{N,\alpha}). \tag{42}$$

We already know the asymptotics for these β's from Theorem 4.5

$$\limsup_{N\to\infty} \frac{2}{\ln(N)} \ln\big(\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha})\big) \leq \liminf_{N\to\infty} \frac{2}{\ln(N)} \ln\big(\det(\mathbb{1} - \frac{\beta}{\pi} H_{N,\alpha})\big) = \gamma(\beta).$$

Since this is valid for all $\beta < 1$ and, moreover, $\gamma(\beta) \to \gamma(1)$ as $\beta \to 1$ we obtain (41) $\square$

For the lower bound we employ Lemma 2.1. To this end, we need estimates for $\operatorname{tr}(H^m_{N,\alpha})$. The method parallels that of Section 3 in that we replace the Hilbert matrix by the Carleman operator. For an intermediate step we need the so-called 'odd' Hilbert matrix

$$H_- : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0),\; H_- = (h_{j+k})_{j,k\in\mathbb{N}_0},\; h_j = \begin{cases} \frac{1}{j+1} & \text{for } j \text{ even,} \\ 0 & \text{for } j \text{ odd.} \end{cases} \tag{43}$$

It is more convenient here to work with the projection operator

$$P_N : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0),\; (P_N c)_j = \begin{cases} c_j & \text{for } 0 \leq j \leq N-1, \\ 0 & \text{for } j \geq N+1 \end{cases} \tag{44}$$

instead of the finite odd Hilbert matrix.

Lemma 5.2. *Let $\alpha \geq \frac{1}{2}$. Then, for all $m, N \in \mathbb{N}$*

$$\operatorname{tr}[H^m_{N,\alpha}] \leq 2^m \operatorname{tr}[(P_{2N} H_-)^m] \leq 2^m \operatorname{tr}[P_{2N} H^m_-]. \tag{45}$$

Proof. We start with the odd Hilbert matrix

$$\begin{aligned}
\operatorname{tr}[(P_{2N}H_-)^m] &= \sum_{j_1,\dots,j_m=0}^{2N-1} \prod_{l=1}^{m} h_{j_l+j_{l+1}} \\
&= \sum_{k_1,\dots,k_m=0}^{N-1} \prod_{l=1}^{m} \frac{1}{2k_l + 2k_{l+1} + 1} + \sum_{k_1,\dots,k_m=0}^{N-1} \prod_{l=1}^{m} \frac{1}{2k_l + 1 + 2k_{l+1} + 1 + 1} \\
&\geq \frac{1}{2^m} \sum_{k_1,\dots,k_m=0}^{N-1} \prod_{l=1}^{m} \frac{1}{k_l + k_{l+1} + \frac{1}{2}} \\
&\geq \frac{1}{2^m} \sum_{k_1,\dots,k_m=0}^{N-1} \prod_{l=1}^{m} \frac{1}{k_l + k_{l+1} + \alpha} \\
&= \frac{1}{2^m} \operatorname{tr}[H^m_{N,\alpha}].
\end{aligned}$$

Here we used that $h_{j_l+j_{l+1}} \neq 0$ only if $j_l + j_{l+1}$ is even which is the case when either all of the j_l are even or all are odd. This yields the first inequality in (45). The second inequality follows from P_{2N} being an orthogonal projection and $H^*_{N,\alpha} = H_{N,\alpha}$. $\square$

With the aid of the orthonormal Laguerre functions l_j, $j \in \mathbb{N}_0$, we define the unitary operator

$$U : L^2(\mathbb{R}^+) \to \ell^2(\mathbb{N}_0),\ (U\varphi)_j = \int_0^\infty l_j(x)\varphi(x)\,dx,\ j \in \mathbb{N}_0. \tag{46}$$

This transforms P_N into the projection with Christoffel–Darboux kernel

$$P_N = U\Pi_N U^*,\ \Pi_N(x,y) := \sum_{k=0}^{N} l_k(x)l_k(y) \tag{47}$$

and the odd Hilbert matrix into the Carleman operator [12, pp. 54, 55]

$$2H_- = UKU^*,\ 2^m \operatorname{tr}(P_N H_-^m) = \operatorname{tr}(\Pi_N K^m). \tag{48}$$

The kernel function of the Carleman operator has a critical behavior at $x = 0$ and $x = \infty$, cf. (16). Therefore, we use an appropriate cut-off.

Lemma 5.3. *Let $0 \le \delta \le L$. Then, for all $m, N \in \mathbb{N}$*

$$2^m \operatorname{tr}[P_N H_-^m] \le 2\operatorname{tr}[P_{[\delta,L]}K^m] + (1+\pi^m)\operatorname{tr}[P_{[\delta,L]}^{\perp}\Pi_N],\ P_{[\delta,L]}^{\perp} := \mathbb{1} - P_{[\delta,L]}. \tag{49}$$

Proof. We use (48) and decompose the trace

$$\operatorname{tr}(\Pi_N K^m) = \operatorname{tr}[P_{[\delta,L]}\Pi_N P_{[\delta,L]}K^m] + 2\operatorname{Re}\operatorname{tr}[P_{[\delta,L]}^{\perp}\Pi_N P_{[\delta,L]}K^m] + \operatorname{tr}[P_{[\delta,L]}^{\perp}\Pi_N P_{[\delta,L]}^{\perp}K^m].$$

Since all operators involved are non-negative we can bound the traces through the operator norm

$$\begin{aligned}
\operatorname{tr}(\Pi_N K^m) &\le \|P_{[\delta,L]}\Pi_N P_{[\delta,L]}\| \operatorname{tr}[P_{[\delta,L]}K^m] + 2\big(\operatorname{tr}(P_{[\delta,L]}^{\perp}\Pi_N\big)^{\frac12}\big(\operatorname{tr}(P_{[\delta,L]}K^m)\big)^{\frac12} + \operatorname{tr}(P_{[\delta,L]}^{\perp}\Pi_N)\|K\|^m \\
&\le \operatorname{tr}[P_{[\delta,L]}K^m] + 2\big(\operatorname{tr}(P_{[\delta,L]}^{\perp}\Pi_N\big)^{\frac12}\big(\operatorname{tr}(P_{[\delta,L]}K^m)\big)^{\frac12} + \operatorname{tr}(P_{[\delta,L]}^{\perp}\Pi_N)\pi^m \\
&\le 2\operatorname{tr}[P_{[\delta,L]}K^m] + (1+\pi^m)\operatorname{tr}(P_{[\delta,L]}^{\perp}\Pi_N).
\end{aligned}$$

Here we used the Cauchy–Schwarz inequality for the trace and (17). This proves the lemma. □

The trace of the Carleman operator can be expressed as a simple integral.

Lemma 5.4. *Let $\delta > 0$ and $N \ge 0$. Then, for all $m \in \mathbb{N}$*

$$\operatorname{tr}[P_{[\delta,N+\delta]}K^m] = 2n_\delta(N)\pi^{m-2}\int_{\mathbb{R}} \frac{1}{[\cosh(\omega)]^m}\,d\omega,\ n_\delta(N) = \frac14 \ln\Big(\frac{N+\delta}{\delta}\Big).$$

Proof. From Lemma 4.2 we immediately infer

$$\operatorname{tr}[P_{[\delta,N+\delta]}K^m] = \operatorname{tr}[P_{[-n_\delta(N),n_\delta(N)]}K_0^m].$$

Via the diagonalization $\mathscr{F}K_0\mathscr{F}^* = \sqrt{2\pi}\hat{K}_0$, see (28) and (29), we obtain

$$\operatorname{tr}[P_{[-n_\delta(N),n_\delta(N)]}K_0^m] = (2\pi)^{\frac{m}{2}}\operatorname{tr}[P_{-n_\delta(N),n_\delta(N)]}\mathscr{F}^*\hat{K}_0^m\mathscr{F}] = (2\pi)^{\frac{m}{2}}\operatorname{tr}[\mathscr{F}P_{[-n_\delta(N),n_\delta(N)]}\mathscr{F}^*\hat{K}_0^m].$$

Now,

$$\mathscr{F}P_{[-n_\delta(N),n_\delta(N)]}\mathscr{F}^*(x,y) = \frac{1}{2\pi}\int_{-n}^{n} e^{-i\omega(x-y)}\,d\omega$$

and thus

$$\operatorname{tr}[P_{[-n_\delta(N),n_\delta(N)]}K_0^m] = \frac{1}{2\pi}2n_\delta(N)(2\pi)^{\frac{m}{2}}\int_{\mathbb{R}}\hat{K}_0(\omega)^m\,d\omega.$$

This implies

$$\operatorname{tr}[P_{[-n_\delta(N),n_\delta(N)]}K_0^m] = 2n_\delta(N)(2\pi)^{\frac{m-2}{2}} \int_{\mathbb{R}} \left[\sqrt{\frac{\pi}{2}}\frac{1}{\cosh(\frac{\pi\omega}{2})}\right]^m d\omega = 2n_\delta(N)\pi^{m-2}\int_{\mathbb{R}}\frac{1}{[\cosh(\omega)]^m}\,d\omega$$

which proves the lemma. □

In order to bound the traces of the projection operator in (49) we need pointwise estimates for the Laguerre polynomials. The first one is Szegő's inequality, [14, (7.21.3)],

$$|L_n(x)| \le e^{\frac{x}{2}},\ x\ge 0,\ n\in\mathbb{N}_0. \tag{50}$$

The second one is the less known Lewandowski–Szynal inequality [8, Corollary 1], which bounds the Laguerre polynomial via the incomplete Gamma function

$$|L_n(x)| \le \frac{e^x}{n!}\int_x^\infty t^n e^{-t}\,dt,\ x\ge 0,\ n\in\mathbb{N}_0. \tag{51}$$

We will also need the simple formula

$$\sum_{k=0}^{n}\frac{1}{k!}x^k = \frac{e^x}{n!}\int_x^\infty t^n e^{-t}\,dt, \tag{52}$$

whereby one could replace the integral in (51) by the partial sum of the exponential function e^x. In particular, (51) is better for large x than (50) but does not converge to (50) for large n and fixed x because of the different exponents.

Lemma 5.5. *Let $\delta\ge 0$. Furthermore, let $N\in\mathbb{N}$ and $L>0$ such that $\frac{N}{L}<\frac{1}{2}$. Then,*

$$\operatorname{tr}[P^{\perp}_{[\delta,L]}\Pi_N] \le \delta(N+1) + \frac{4}{\frac{1}{2}-\frac{N}{L}}\frac{1}{N!}e^{-\frac{L}{2}}L^N.$$

Proof. First note that $P^{\perp}_{[\delta,L]} = P_{[0,\delta]} + P_{[L,\infty[}$. Using Szegő's inequality (50) we obtain

$$\operatorname{tr}[P_{[0,\delta]}\Pi_N] = \int_0^\delta \sum_{n=0}^N l_n(x)^2\,dx \le \delta(N+1).$$

The remaining trace is a bit more difficult. To simplify the calculations, we apply Szegő's inequality to one factor in

$$0\le \Pi_N(x,x) = \sum_{n=0}^N l_n(x)^2 \le \sum_{n=0}^N |l_n(x)|,\ x\ge 0$$

and then use the Lewandowski–Szynal inequality (51), $x\ge 0$,

$$0\le \Pi_N(x,x) \le \sum_{n=0}^N \frac{e^{\frac{x}{2}}}{n!}\int_x^\infty t^n e^{-t}\,dt = e^{\frac{x}{2}}\int_x^\infty e^{-t}\sum_{n=0}^N\frac{t^n}{n!}\,dt = \frac{1}{N!}e^{\frac{x}{2}}\int_x^\infty\int_t^\infty s^N e^{-s}\,ds\,dt.$$

In the last step we used (52). Furthermore,

$$\begin{aligned} N!\,\mathrm{tr}[P_{[L,\infty[}\Pi_N] &= \int_L^\infty e^{\frac{x}{2}} \int_x^\infty s^N e^{-s}(s-x)\,ds\,dx \\ &= e^{-\frac{L}{2}} \int_0^\infty e^{\frac{x}{2}} \int_x^\infty (s+L)^N e^{-s}(s-x)\,ds\,dx \\ &= e^{-\frac{L}{2}} L^N \int_0^\infty (1+\frac{s}{L})^N e^{-\frac{s}{2}} \int_0^s e^{-\frac{x}{2}} x\,dx\,ds \\ &\leq e^{-\frac{L}{2}} L^N \int_0^\infty e^{\frac{N}{L}s} e^{-\frac{s}{2}} \int_0^s e^{-\frac{x}{2}} x\,dx\,ds. \end{aligned}$$

For simplicity we bound the x-integral by 4

$$N!\,\mathrm{tr}[P_{[L,\infty[}\Pi_N] \leq 4e^{-\frac{L}{2}} L^N \int_0^\infty e^{\frac{N}{L}s} e^{-\frac{s}{2}}\,ds = 4e^{-\frac{L}{2}} L^N \frac{1}{\frac{1}{2}-\frac{N}{L}}.$$

This completes the proof. □

We combine the preceding estimates to obtain a bound on the trace of the Hilbert matrix.

Lemma 5.6. *Let $\alpha \geq \frac{1}{2}$ and $N, m \in \mathbb{N}$ with $m \geq 5$. Then,*

$$\frac{1}{\pi^{2m}}\,\mathrm{tr}(H_{N,\alpha}^{2m}) \leq C\Big\{\frac{1}{2^{\frac{m}{2}}}\big[\ln(m)+\ln(N)\big] + \frac{1}{m^2} + \frac{1}{(2N)!}(mN)^{2N} e^{-\frac{1}{2}mN}\Big\} \tag{53}$$

with some explicitely given constant $0 \leq C < \infty$.

Proof. Lemmas 5.2 and 5.3 imply

$$\frac{1}{\pi^{2m}}\,\mathrm{tr}[H_{N,\alpha}^{2m}] \leq \frac{2}{\pi^{2m}}\,\mathrm{tr}(P_{[\delta,L]}K^{2^m}) + 2\,\mathrm{tr}\big(P_{[\delta,L]}^{\perp}\Pi_{2N}\big). \tag{54}$$

We let δ and L depend on m and N in an appropriate way

$$\delta := \frac{1}{(2N+1)m^2},\ L := mN,$$

and bound the first term in (54) with the aid of Lemma 5.4 and (62)

$$\frac{2}{\pi^{2m}}\,\mathrm{tr}(P_{[\delta,L]}K^{2^m}) \leq \frac{4}{\pi^2}\frac{n_\delta(L-\delta)}{\sqrt{2^{m-1}-1}} \leq \frac{2}{\pi^2}\frac{1}{2^{\frac{m}{2}}}\ln(m^3N(2N+1)). \tag{55}$$

For the second term follows via Lemma 5.5 ($m \geq 5$)

$$\begin{aligned} 2\,\mathrm{tr}\big(P_{[\delta,L]}^{\perp}\Pi_{2N}\big) &\leq 2\Big(\delta(2N+1) + \frac{4}{\frac{1}{2}-\frac{2N}{L}}\frac{1}{(2N)!}L^{2N}e^{-\frac{L}{2}}\Big) \\ &= 2\Big(\frac{1}{m^2} + \frac{4}{\frac{1}{2}-\frac{2}{m}}\frac{1}{(2N)!}(mL)^{2N}e^{-\frac{1}{2}mN}\Big). \end{aligned} \tag{56}$$

Via some elementary estimates, (55) and (56) imply (53). □

Now, everything is at hand to prove the complement of Proposition 5.1.

Proposition 5.7. *Let $\alpha \geq \frac{1}{2}$. Then,*

$$-\liminf_{N\to\infty} \frac{2}{\ln(N)} \ln\big(\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha})\big) \leq \frac{3}{4}. \tag{57}$$

Proof. Since $H_{N,\alpha}^{2^m}$ is a non-negative operator the trace norm in (8) equals the trace

$$-\ln\big(\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha})\big) \leq \sum_{m=0}^{M} \ln\big(\det(\mathbb{1} + (\frac{1}{\pi} H_{N,\alpha})^{2^m})\big) + \sum_{m=M+1}^{\infty} \frac{1}{\pi^{2^m}} \operatorname{tr}(H_{N,\alpha}^{2^m}). \tag{58}$$

We bound the traces via Lemma 5.6 (with $M \geq 4$)

$$\sum_{m=M+1}^{\infty} \frac{1}{\pi^{2^m}} \operatorname{tr}(H_{N,\alpha}^{2^m}) \leq C_1 \sum_{m=M+1}^{\infty} \Big\{ \frac{1}{2^{\frac{m}{2}}} \big[\ln(m) + \ln(N)\big] + \frac{1}{m^2} \Big\} + C_1 \frac{N^{2N}}{(2N)!} \sum_{m=M+1}^{\infty} m^{2N} e^{-\frac{1}{2}mN}.$$

For the first sum

$$\liminf_{N\to\infty} \frac{1}{\ln(N)} \sum_{m=M+1}^{\infty} \Big\{ \frac{1}{2^{\frac{m}{2}}} \big[\ln(m) + \ln(N)\big] + \frac{1}{m^2} \Big\} = \sum_{m=M+1}^{\infty} \frac{1}{2^{\frac{m}{2}}}. \tag{59}$$

The second series requires a bit more reasoning. For sufficiently large $M \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{(2N)!} N^{2N} \sum_{m=M+1}^{\infty} m^{2N} e^{-\frac{1}{2}mN} &\leq \frac{1}{(2N)!} N^{2N} \int_M^{\infty} t^{2N} e^{-\frac{1}{2}Nt}\, dt \\
&= \frac{1}{(2N)!} \frac{2}{N} e^{-\frac{1}{2}MN} (MN)^{2N} \int_0^{\infty} \big(1 + \frac{2t}{MN}\big)^{2N} e^{-t}\, dt \\
&\leq \frac{1}{(2N)!} \frac{2}{N} e^{-\frac{1}{2}MN} (MN)^{2N} \int_0^{\infty} e^{\frac{4t}{M}} e^{-t}\, dt \\
&\leq C_2 \frac{1}{N^{\frac{3}{2}}} \big(\frac{e}{2}\big)^{2N} e^{-\frac{1}{2}MN} M^{2N} \\
&\leq C_2 \frac{1}{N^{\frac{3}{2}}} \exp\big[(2 - 2\ln(2) - \frac{1}{2}M + 2\ln(M))N\big]
\end{aligned}$$

with some constant $C_2 \geq 0$. In the next to last step we used the lower bound from Stirling's formula. For M large enough, the argument of the exponential function becomes negative which shows that the expression coverges to zero as $N \to \infty$ even without the factor $\ln(N)$. Now, divide (58) by $\frac{1}{2}\ln(N)$ and use Corollary 4.6 and (59) to deduce

$$-\liminf_{N\to\infty} \frac{2}{\ln(N)} \ln\big(\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha})\big) \leq \sum_{m=0}^{M} \gamma_{2^m} + C_3 \sum_{m=M+1}^{\infty} \frac{1}{2^{\frac{m}{2}}} \tag{60}$$

with $C_3 \geq 0$ to adjust for a different factor in (59). Since (60) is true for all (sufficiently large) $M \in \mathbb{N}$ we may perform the limit $M \to \infty$

$$-\liminf_{N\to\infty} \frac{2}{\ln(N)} \ln\big(\det(\mathbb{1} - \frac{1}{\pi} H_{N,\alpha})\big) \leq \sum_{m=0}^{\infty} \gamma_{2^m}.$$

We evaluate the infinite sum by using the explicit form of the γ_k's in (40)

$$\sum_{m=0}^{\infty}\gamma_{2^m}=\frac{2}{\pi^2}\sum_{m=0}^{\infty}\int_0^{\infty}\ln\big(1+\frac{1}{[\cosh(\omega)]^{2^m}}\big)\,d\omega=\frac{2}{\pi^2}\int_0^{\infty}\ln\Big(\prod_{m=0}^{\infty}\big(1+\frac{1}{[\cosh(\omega)]^{2^m}}\big)\Big)\,d\omega.$$

Interchanging summation and integration can be justified via Lebesgue's convergence theorem. With (6) we obtain

$$\sum_{m=0}^{\infty}\gamma_{2^m}=\frac{2}{\pi^2}\int_0^{\infty}\ln\Big(\frac{1}{1-\frac{1}{\cosh(\omega)}}\Big)\,d\omega=-\frac{2}{\pi^2}\int_0^{\infty}\ln\Big(1-\frac{1}{\cosh(\omega)}\Big)\,d\omega=\frac{3}{4}.$$

In the last step we used Lemma A.3. This yields (57). □

We combine the lower and upper bound.

Theorem 5.8. *Let $\alpha\ge\frac{1}{2}$. Then,*

$$\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\big)=\frac{1}{2}\ln(N)\gamma(1)+o(\ln(N))\ \text{as}\ N\to\infty$$

with $\gamma(1)=-\frac{3}{4}$.

Proof. From Propositions 5.1 and 5.7 we obtain

$$-\frac{3}{4}\le\liminf_{N\to\infty}\frac{2}{\ln(N)}\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\big)\le\limsup_{N\to\infty}\frac{2}{\ln(N)}\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,\alpha})\big)\le\gamma(1)=-\frac{3}{4},$$

cf. (37). This proves the statement. □

6. Limit case for $\alpha=1$

For the special Hilbert matrix with $\alpha=1$, cf. (11), there is an alternative way to prove the trace estimates (Lemmas 5.2, 5.3, 5.5, 5.6) used in Proposition 5.8 to bound the limit inferior. Starting point is a simple estimate for the hyperbolic sine.

Lemma 6.1. *Let $0\le\delta\le\frac{1}{3}$. Then, the hyperbolic sine satisfies the estimate*

$$\frac{y}{\sinh(y)}\le 2^{\delta}e^{-\delta y},\ \ y>0.$$

Proof. We use Lazarevic's inequality [9, 3.6.9]

$$\cosh(y)\le\left[\frac{\sinh(y)}{y}\right]^p,\ y\ne0,\ p\ge3.$$

For the proof note that $\sinh(y)/y\ge1$ whence one only has to consider the case $p=3$. Using $\cosh(y)\ge e^y/2$ yields the claimed inequality with $\delta=1/p$. □

We replace the Hilbert matrix by the Carleman operator.

Lemma 6.2. *Let $N,m\in\mathbb{N}$ and $0<\delta\le\frac{1}{3}$. Then,*

$$0\le\operatorname{tr}[H_{N,1}^m]\le2^{m\delta}\operatorname{tr}[P_{[\delta,N+\delta]}K^m]$$

with K the Carleman operator (16).

Proof. From Lemma 3.1 follows

$$\mathrm{tr}[H^m_{N,1}] = \mathrm{tr}[G^m_{N,1}],\ m \in \mathbb{N}.$$

Recall the kernel function (see Lemma 3.1 and the proof of Lemma 3.2)

$$G_{N,1}(x) = \frac{x}{2\sinh(\frac{x}{2})} \int_0^N e^{-sx}\,ds.$$

With the aid of Lemma 6.1

$$0 \le G_{N,1}(x+y) \le 2^\delta e^{-\delta(x+y)} \int_0^N e^{-s(x+y)}\,ds = 2^\delta \int_0^N e^{-(s+\delta)(x+y)}\,ds = 2^\delta (E_{2\delta} P_{[0,N]} E^*_{2\delta})(x,y)$$

where $E_{2\delta}$ is from (14) with $\alpha = 2\delta$. Since $\delta > 0$ we may take the trace, Lemma 3.2

$$0 \le \mathrm{tr}[H^m_{N,1}] = \mathrm{tr}[G^m_{N,1}] \le 2^{m\delta}\,\mathrm{tr}[(E_{2\delta} P_{[0,N]} E^*_{2\delta})^m]$$

where we used that the kernel functions are (pointwise) non-negative. Via Lemma 3.3

$$\mathrm{tr}[(E_{2\delta} P_{[0,N]} E^*_{2\delta})^m] = \mathrm{tr}[(P_{[\delta,N+\delta]} K P_{[\delta,N+\delta]})^m] \le \mathrm{tr}[P_{[\delta,N+\delta]} K^m]$$

In the last step we used $0 \le P_{[\delta,N+\delta]} \le \mathbb{1}$ in the sense of quadratic forms. □

We replace the Carleman operator K by the convolution operator K_0.

Lemma 6.3. *Let* $0 < \delta \le \frac{1}{3}$. *With the convolution operator* K_0 *from Lemma 4.2*

$$\mathrm{tr}[P_{[\delta,N+\delta]} K^m] = \mathrm{tr}[P_{[-n,n]} K^m_0],\ n_\delta(N) = \frac{1}{4} \ln \frac{N+\delta}{\delta}.$$

Proof. See Lemma 4.2. □

Using the diagonalization of the convolution operator K_0, see (29), we express the trace as a simple integral.

Lemma 6.4. *Let* $m \in \mathbb{N}$ *and* $n \ge 0$. *Then,*

$$\mathrm{tr}[P_{[-n,n]} K^m_0] = 2n\pi^{m-2} \int_{\mathbb{R}} \frac{1}{[\cosh(\omega)]^m}\,d\omega.$$

Proof. Via the diagonalization $\mathscr{F} K_0 \mathscr{F}^* = \sqrt{2\pi}\hat{K}_0$, see (28) and (29), we obtain

$$\mathrm{tr}[P_{[-n,n]} K^m_0] = (2\pi)^{\frac{m}{2}}\,\mathrm{tr}[P_{-n,n]} \mathscr{F}^* \hat{K}^m_0 \mathscr{F}] = (2\pi)^{\frac{m}{2}}\,\mathrm{tr}[\mathscr{F} P_{[-n,n]} \mathscr{F}^* \hat{K}^m_0].$$

Now,

$$\mathscr{F} P_{[-n,n]} \mathscr{F}^*(x,y) = \frac{1}{2\pi} \int_{-n}^n e^{-i\omega(x-y)}\,d\omega$$

and thus

$$\mathrm{tr}[P_{[-n,n]} K^m_0] = \frac{1}{2\pi} 2n (2\pi)^{\frac{m}{2}} \int_{\mathbb{R}} \hat{K}_0(\omega)^m\,d\omega.$$

This implies

$$\mathrm{tr}[P_{[-n,n]} K^m_0] = 2n(2\pi)^{\frac{m-2}{2}} \int_{\mathbb{R}} \left[\sqrt{\frac{\pi}{2}} \frac{1}{\cosh(\frac{\pi\omega}{2})}\right]^m d\omega = 2n\pi^{m-2} \int_{\mathbb{R}} \frac{1}{[\cosh(\omega)]^m}\,d\omega$$

which proves the lemma. □

We give now a new proof of Proposition 5.7. We formulate only the relevant part.

Proposition 6.5. *The special Hilbert matrix $H_{N,1}$, cf.* (11)*, satisfies*

$$-\liminf_{N\to\infty}\frac{2}{\ln(N)}\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,1})\big)\leq\sum_{m=0}^{\infty}\gamma_{2^m}.$$

Proof. We start from (58) but use now Lemmas 6.2 through 6.4. These imply (we only need even exponents)

$$\frac{1}{\pi^{2k}}\operatorname{tr}[H_{N,1}^{2k}]\leq\frac{2n_\delta(N)}{\pi^2}2^{2k\delta}\int_{\mathbb{R}}\frac{1}{[\cosh(\omega)]^{2k}}\,d\omega,\ 0<\delta\leq\frac{1}{3},\ n_\delta(N)=\frac{1}{4}\ln\frac{N+\delta}{\delta},$$

which can be further estimated with the aid of (62)

$$\frac{1}{\pi^{2k}}\operatorname{tr}[H_{N,1}^{2k}]\leq\frac{2n_\delta(N)}{\pi^2}2^{2k\delta}\frac{2}{\sqrt{k-1}},\ k\geq 2.$$

In order to compensate the exponentially growing prefactor we choose $\delta=\frac{1}{k}$,

$$\frac{1}{\pi^{2k}}\operatorname{tr}[H_{N,1}^{2k}]\leq\frac{16}{\pi^2}\frac{n_{\frac{1}{k}}(N)}{\sqrt{k-1}},\ n_{\frac{1}{k}}(N)=\frac{1}{4}\ln[(N+\frac{1}{k})k].$$

Now we can estimate the infinite sum in (58)

$$\begin{aligned}\sum_{m=M+1}^{\infty}\frac{1}{\pi^{2^m}}\operatorname{tr}[H_{N,1}^{2^m}]&\leq\frac{16}{\pi^2}\sum_{m=M+1}^{\infty}\frac{1}{\sqrt{2^{m-1}-1}}\frac{1}{4}\ln\big((N+\frac{1}{2^{m-1}})2^{m-1}\big)\\&\leq\frac{4}{\pi^2}\sum_{m=M}^{\infty}\frac{1}{\sqrt{2^m-1}}\big\{m\ln(2)+\ln(N+\frac{1}{2^m})\big\}\\&\leq C_1\sum_{m=M}^{\infty}\frac{m}{\sqrt{2^m-1}}+C_2\ln(N+1)\sum_{m=M}^{\infty}\frac{1}{\sqrt{2^m-1}}.\end{aligned}$$

This yields the analogue of (60)

$$-\liminf_{N\to\infty}\frac{2}{\ln(N)}\ln\big(\det(\mathbb{1}-\frac{1}{\pi}H_{N,1})\big)\leq\sum_{m=0}^{M}\gamma_{2^m}+C_3\sum_{m=M}^{\infty}\frac{1}{\sqrt{2^m-1}}.$$

Letting $M\to\infty$ we obtain the statement. □

Appendix A. Integrals

Lemma A.1. *Let $m\in\mathbb{N}$. Then,*

$$I_{2m}:=\int_{\mathbb{R}}\frac{1}{\cosh(x)^{2m}}\,dx=2\prod_{k=1}^{m-1}\frac{2k}{2k+1}=2\frac{4^{m-1}[(m-1)!]^2}{(2m-1)!},\tag{61}$$

which can be estimated

$$I_{2m+2}\leq\frac{2}{\sqrt{m}},\ m\in\mathbb{N}.\tag{62}$$

Proof. We note $\frac{d}{dx}\tanh(x) = 1/\cosh(x)^2$ and integrate by parts

$$\begin{aligned}
I_{2m+2} &= \int_{\mathbb{R}} \frac{1}{[\cosh(x)]^{2m}} \frac{1}{[\cosh(x)]^2}\,dx \\
&= \left[\frac{1}{[\cosh(x)]^{2m}} \frac{\sinh(x)}{\cosh(x)}\right]_{-\infty}^{\infty} + 2m\int_{\mathbb{R}} \frac{\sinh(x)}{[\cosh(x)]^{2m+1}} \frac{\sinh(x)}{\cosh(x)}\,dx \\
&= 2m\int_{\mathbb{R}} \frac{[\cosh(x)]^2}{[\cosh(x)]^{2m+2}}\,dx - 2m\int_{\mathbb{R}} \frac{1}{[\cosh(x)]^{2m+2}}\,dx \\
&= 2mI_{2m} - 2mI_{2m+2}.
\end{aligned}$$

We solve for I_{2m+2} to obtain the recursion formula

$$I_{2(m+1)} = \frac{2m}{2m+1} I_{2m}$$

which immediately yields

$$I_{2(m+1)} = 2\prod_{k=1}^{m} \frac{2k}{2k+1} = 2\prod_{k=1}^{m} \frac{k}{k+\frac{1}{2}}$$

since $I_2 = 2$. This implies (61). In order to derive the bound we use the inequality between the geometric and arithmetic mean

$$I_{2(m+1)} = 2\frac{\sqrt{m}}{m+\frac{1}{2}} \frac{\sqrt{m}\sqrt{m-1}}{m-\frac{1}{2}} \frac{\sqrt{m-1}\sqrt{m-2}}{m-\frac{3}{2}} \cdots \frac{\sqrt{2}\sqrt{1}}{1+\frac{1}{2}}\sqrt{1} \leq 2\frac{\sqrt{m}}{m+\frac{1}{2}} \leq \frac{2}{\sqrt{m}}.$$

This proves (62). □

The following integral is a special case of an integral that appeared in the study of the ground state energy of the free Fermi gas [11]. We evaluate it here for the sake of completeness.

Lemma A.2. *Let $\beta \in \mathbb{C} \setminus [1,\infty[$. Then,*

$$I(\beta) := \int_0^\infty \ln\big(1 - \frac{\beta}{\cosh(x)}\big)\,dx = \frac{1}{2}[\operatorname{arcosh}(-\beta)]^2 + \frac{\pi^2}{8}. \tag{63}$$

Here, arcosh *is the principal branch on the cut plane $\mathbb{C} \setminus]-\infty,-1[$.*

Proof. First of all, we transform the integral into a form that can be treated by standard methods. To this end, we write $f(x) = \cosh(x) - 1$ for short. Note that $f(0) = 0$, $f(\infty) = \infty$, and $f'(x) > 0$ for $x > 0$. Therefore,

$$x = f^{-1}(y),\ dx = \frac{d}{dy}(f^{-1}(y))\,dy,$$

is a well-defined substitution. Hence,

$$I(\beta) = \int_0^\infty \ln(1 - \frac{\beta}{f(x)+1})\,dx = \int_0^\infty \ln(1 - \frac{\beta}{y+1})\frac{d}{dy}(f^{-1}(y))\,dy.$$

An integration by parts yields

$$I(\beta) = -\beta\int_0^\infty \frac{1}{y+1-\beta}\frac{1}{y+1} f^{-1}(y)\,dy = \int_0^\infty \big[\frac{1}{y+1} - \frac{1}{y+1-\beta}\big] f^{-1}(y)\,dy.$$

The integral is of the type

$$I(\beta) = \int_0^\infty r(y)g(y)\,dy,\ g(y) := f^{-1}(y)$$

where the rational function r does not have poles in $[0,\infty[$. Such integrals can be evaluated by standard methods if one finds a function h with a certain jump at $[0,\infty[$. In our case

$$h(z) := -\frac{1}{4\pi i}[\operatorname{arcosh}(-z-1)]^2.$$

Then, via the residue theorem

$$\begin{aligned}
I(\beta) &= 2\pi i \sum_{z\in\mathbb{C}\setminus[0,\infty[} \operatorname{res}(r(z)h(z)) \\
&= \frac{1}{2}\sum_{z\in\mathbb{C}\setminus[0,\infty[} \operatorname{res}\left[\frac{1}{z+1-\beta}[\operatorname{arcosh}(-z-1)]^2\right] - \frac{1}{2}\sum_{z\in\mathbb{C}\setminus[0,\infty[} \operatorname{res}\left[\frac{1}{z+1}[\operatorname{arcosh}(-z-1)]^2\right] \\
&= \frac{1}{2}[\operatorname{arcosh}(-\beta)]^2 - \frac{1}{2}[\operatorname{arcosh}(0)]^2
\end{aligned}$$

which yields (63). □

The method used to prove the preceding lemma does not work in the case $\beta = 1$. One could use a continuity argument to cover this case as well. Instead, we transform the integral into a well-known integral.

Lemma A.3. *Let $\beta = 1$ in Lemma A.2. Then,*

$$I(1) = \int_0^\infty \ln\Big(1 - \frac{1}{\cosh(x)}\Big)\,dx = -\frac{3\pi^2}{8}. \tag{64}$$

Proof. Despite the singularity at $x = 0$ the integral is well-defined since the logarithm $x \mapsto \ln(x)$ is integrable a $x = 0$. We integrate by parts and use some standard formulae for the hyperbolic functions

$$\begin{aligned}
I(1) &= -\int_0^\infty \frac{x}{\cosh(x)-1}\frac{\sinh(x)}{\cosh(x)}\,dx \\
&= -\int_0^\infty \frac{x}{[\cosh(x)]^2-1}\frac{(\cosh(x)+1)\sinh(x)}{\cosh(x)}\,dx \\
&= -\int_0^\infty \frac{x}{\sinh(x)}\,dx - \int_0^\infty \frac{x}{\sinh(x)\cosh(x)}\,dx \\
&= -\frac{3}{2}\int_0^\infty \frac{x}{\sinh(x)}\,dx.
\end{aligned}$$

The latter integral is well-known and has the value $\frac{\pi^2}{4}$. It can be evaluated via Cauchy's integral theorem and an appropriate integration contour. A possible choice is the rectangle with vertices $\pm R$ and $\pm R + i\pi$ with a small half circle at $i\pi$ cut out. □

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