ON THE GENERAL TRIPLE CORRELATION SUMS FOR $GL_2 \times GL_2 \times GL_2$

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ABSTRACT. Fix $X \geq 2$. Let f be a Hecke newform of prime level p. In this paper, we investigate the general triple correlation sum

$$\sum_{h\geq 1}\sum_{l\geq 1}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)\lambda_f(n+l)\,U\Big(\frac{n}{X}\Big)V\Big(\frac{h}{H}\Big)R\Big(\frac{l}{L}\Big)$$

for $H, L \geq 1$ in the level aspect. As a result, we prove a non-trivial bound for any H, L satisfying that $L > X^{1/4}$ and $\max\{L^3X^{-2}, \sqrt{L}, X^{1/4}\} < H < \min\{X^{2/3}L^{1/3}, L^2\}$. It can be shown that there exist certain newforms such the non-trivial bound for the triple sum can be achieved, so long as $\max\{H, L\} \geq X^{1/4+\varepsilon}$. Particularly, whenever $L = H$, we present a non-trivial estimate for any p such that $H^2/X \leq p < \min\{H^2X^{-1/2}, H\}$, and further obtain the more significant cancellations for these sums in the different segments of H.

1. Introduction

In number theory, a basic question is to explore the nature of the associated Fourier coefficients of cusp forms, a challenging topic of which being the shifted correlation sums problem. This, however, plays a tremendously important rôle in many other related topics, such as the moments of L-functions (or zeta-functions), subconvexity, the Gauss circle problem and the Quantum Unique Ergodicity (QUE) conjecture, etc (see, for instance, [21, 10, 2, 5, 9, 7, 8, 16, 13, 12] and the references therein).

While a lot of of attention was being paid to the bounds for the double correlation sums, yet much less is known for the triple sums problem in the literature, on account of the extra complexity of its own. In the classic case of all the arithmetic functions being the divisor functions, in 2011, Browning [4] showed that, if $H \geq X^{3/4+\varepsilon}$,

$$\sum_{1\leq h\leq H}\sum_{1\leq n\leq X} d(n)d(n+h)d(n+2h) = \frac{11}{8}\phi(h)\prod_p\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)HX\log^3 X + o\big(HX\log^3 X\big)$$

up to an explicit multiplicative function $\phi(h)$. After that, Blomer [3] used the spectral decomposition for partially smoothed triple correlation sums to establish an asymptotic formula that

$$\sum_{h\geq 1}\sum_{1\leq n\leq X} W\left(\frac{h}{H}\right)d(n)d_l(n+h)d(n+2h) = XH\widetilde{W}(1)P_{l+1}(\log X) + O\left(X^{\varepsilon}\left(H^2 + HX^{1-\frac{1}{l+2}} + X\sqrt{H} + \frac{X^{\frac{3}{2}}}{\sqrt{H}}\right)\right),$$

for any $l \in \mathbb{N}$, where W is a smooth function supported on $[1/2, 5/2]$, $\widetilde{W}$ denotes its Mellin transform of W, d_l is the l-th fold divisor function and P_l is a polynomial of degree l. Notice that, here, Blomer

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improved the range of H substantially to $H \geq X^{1/3+\varepsilon}$, and produced a power saving error term. In addition, in [3], he was able to attain a more general version that, for any complex sequence $\mathbf{a} = \{a(n)\}$,

$$\sum_{h\geq 1}\sum_{1\leq n\leq X} W\left(\frac{h}{H}\right)d(n)a(n+h)d(n+2h) = H\widetilde{W}(1)\sum_{n\leq X} a(n)\sum_{d\geq 1}\frac{S(2n,0;d)}{d^2}$$
$$(\log n + 2\gamma - 2\log d)^2 + O\left(X^{\varepsilon}\left(\frac{H^2}{\sqrt{X}} + HX^{\frac{1}{4}} + \sqrt{XH} + \frac{X}{\sqrt{H}}\right)\|\mathbf{a}\|_2\right),$$

where $\|\mathbf{a}\|_2$ is the ℓ^2-norm. Let $k, k' \geq 2$ be any even integers. Let $f_1 \in \mathcal{B}_k^*(1)$ and $f_2 \in \mathcal{B}_{k'}^*(1)$ be two Hecke newforms on GL_2 with $\lambda_{f_1}(n)$ and $\lambda_{f_2}(n)$ being their n-th Hecke eigenvalues, respectively (see §2 for definitions). Subsequently, Lin [22] proved that

$$\sum_{h\geq 1}\sum_{1\leq n\leq X} W\left(\frac{h}{H}\right)\lambda_{f_1}(n)a(n+h)\lambda_{f_2}(n+2h) \ll \frac{X^{1+\varepsilon}}{H}\left(\sqrt{XH} + \frac{X}{\sqrt{H}}\right)\|\mathbf{a}\|_2,$$

which, however, beats the "trivial" bound barrier $O(X^{\varepsilon}H\sqrt{X}\|\mathbf{a}\|_2)$, provided that $H \geq X^{2/3+\varepsilon}$. Here and thereafter, the trivial bound means to take absolute value for each summand followed by using Deligne's bound. As an immediate consequence, one has seen that

$$\sum_{h\geq 1}\sum_{1\leq n\leq X} W\left(\frac{h}{H}\right)\lambda_{f_1}(n)\lambda_{f_2}(n+h)\lambda_{f_3}(n+2h) \ll X^{\varepsilon}\left(XH, \frac{X^2}{\sqrt{H}}\right)$$

for any $f_3 \in \mathcal{B}_{k''}^*(1)$ with $k'' \in 2\mathbb{N}$. In contrast to Lin' s work, recently, Singh [30] was able to attain

$$\sum_{h\geq 1}\sum_{n\geq 1} W_1\left(\frac{h}{H}\right)W_2\left(\frac{n}{X}\right)\lambda_{f_1}(n)\lambda_{f_2}(n+h)\lambda_{f_3}(n+2h) \ll X^{\varepsilon}\left(\sqrt{X}H + X^{\frac{3}{2}}\right),$$

extending the range of H to $H \geq X^{1/2+\varepsilon}$, where W_1, W_2 are two smooth bump functions supported on the interval $[1/2, 5/2]$. Until now, the best result is due to Lü-Xi [23, 24] who achieved that

$$\sum_{1\leq n\leq X} W\left(\frac{h}{H}\right)a(n)b(n+h)\lambda_{f_1}(n+2h) \ll X^{\varepsilon}\Delta_1(X,H)\,\|\mathbf{a}\|_2\,\|\mathbf{b}\|_2$$

for any complex sequence $\mathbf{b} = \{b(n)\}$, which allows one to take $H \geq X^{2/5+\varepsilon}$; the definition of $\Delta_1(X,H)$, however, can be referred to [24, Theorem 3.1]. More recently, Hulse et al. [14] successfully attained

$$\sum_{h\geq 1}\sum_{n\geq 1}\lambda_{g_1}(n)\lambda_{g_2}(h)\lambda_{g_3}(2n-h)\exp\left(-\frac{h}{H} - \frac{n}{X}\right) \ll X^{\kappa-1+\vartheta+\frac{1}{2}+\varepsilon}H^{\frac{\kappa-1}{2}-\vartheta+\frac{1}{2}+\varepsilon},$$

where $\vartheta < 7/64$ denotes the currently best approximation towards the Generalized Ramanujan Conjecture. Here, $\lambda_{g_1}(n)$, $\lambda_{g_2}(n)$ and $\lambda_{g_3}(n)$ denote the n-th non-normalized coefficients of holomorphic cusp forms g_1, g_2 and g_3, each of weight $\kappa \geq 2$, level $M \geq 2$ and trivial nebentypus. It is noticeable that, just lately, Munshi [29] considered the more involved problem of pursuing the most intrinsic cancellations of the correlation sums with the levels of the associated forms being allowed to vary. As a result, he achieved that, for any newform $f \in \mathcal{B}_k^*(p)$ of weight k and level p, whenever $X^{1/3+\varepsilon} \leq p \leq X$,

$$\sum_{1\leq n\leq X}\lambda_f(n)\lambda_f(n+ph) \ll p^{\frac{1}{4}}X^{\frac{3}{4}+\varepsilon}$$

for any fixed integer h such that $|h| \leq X/p$. It is reasonable to expect that there exist certain families of forms which reveal strong cancellations, and produce fairly wider ranges for H securing the non-trivial estimates for the triple sums. This, on the other hand, is the motivation of the paper.

In the present paper, we shall go further to explore the more general types of the triple correlation sums. The main result is the following:

Theorem 1.1. *Fix $X \geq 2$. Let $H, L \geq 2$ and $p \geq 2$ a prime satisfying that $\max\{H, L\} \leq \sqrt{Xp}$ and $p \leq X$. Let U, V, R be three smooth bump functions supported $[1/2, 5/2]$. Then, for any newform $f \in \mathcal{B}_k^*(p)$, we have*

$$\sum_{h\geq 1}\sum_{l\geq 1}\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+h)\lambda_f(n+l)\, U\Big(\frac{n}{X}\Big)V\Big(\frac{h}{H}\Big)R\Big(\frac{l}{L}\Big) \ll X^{\varepsilon}\Big[X^{\frac{3}{2}}p + \frac{XHL}{\sqrt{p}} + Xp\sqrt{HL} + X^{\frac{5}{4}}p^{\frac{1}{4}}(H+L)\Big] \tag{1.1}$$

as $k \to \infty$, where the implied constant depends merely on the weight k and ε.

Observing that the triple sum is trivially $O(X^{1+\varepsilon}HL)$, the upper-bound in (1.1) is seen to be non-trivial, so long as

$$\max\left\{\frac{H^2}{X}, \frac{L^2}{X}\right\} \leq p < \min\left\{\frac{HL}{\sqrt{X}}, \sqrt{HL}, \frac{H^4}{X}, \frac{L^4}{X}\right\}, \tag{1.2}$$

with $L > X^{1/4}$ and $\max\{L^3X^{-2}, \sqrt{L}, X^{1/4}\} < H < \min\{X^{2/3}L^{1/3}, L^2\}$. Meanwhile, it can be seen that the there exist certain newforms $f \in \mathcal{B}_k^*(p)$ such that the bound above is non-trivial for any $H, L \geq 1$ satisfying that $\min\{H, L\} \geq X^{1/4+\varepsilon}$. In the special case where $L = H$, one sees that the estimate in (1.1) is non-trivial for $H^2/X \leq p < \min\{H^2X^{-1/2}, H\}$ with $H > X^{1/4}$. Particularly, as a direct application of Theorem 1.1, we obtain:

Corollary 1.2. *For $X^{1/4+\varepsilon} \leq H \leq \sqrt{X}$ and $X^{3/4} \leq H < X$, there exists a family of newforms $f \in \mathcal{B}_k^*(p)$ with $p \asymp H^{4/3+\varepsilon}X^{-1/3}$, such that*

$$\sum_{h\geq 1}\sum_{l\geq 1}\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+h)\lambda_f(n+l)\, U\Big(\frac{n}{X}\Big)V\Big(\frac{h}{H}\Big)R\Big(\frac{l}{H}\Big) \ll_{k,\varepsilon} \max\left\{X^{\frac{7}{6}+\varepsilon}H^{\frac{4}{3}}, X^{\frac{2}{3}+\varepsilon}H^{\frac{7}{3}}\right\}$$

as $k \to \infty$; while, on the other hand, for $\sqrt{X} < H < X^{3/4}$, there, however, exists a family of newforms $f \in \mathcal{B}_k^*(p)$ with $p \asymp H^{2/3+\varepsilon}$, such that

$$\sum_{h\geq 1}\sum_{l\geq 1}\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+h)\lambda_f(n+l)\, U\Big(\frac{n}{X}\Big)V\Big(\frac{h}{H}\Big)R\Big(\frac{l}{H}\Big) \ll_{k,\varepsilon} X^{1+\varepsilon}H^{\frac{5}{3}}$$

as $k \to \infty$.

Notations. *Throughout the paper, ε always denotes an arbitrarily small positive constant which might not be the same at each occurrence. $n \sim X$ means that $X/2 < n \leq X$ for any positive integer $n \geq 1$; μ is the Möbius function and $d(n)$ is the divisor function of n. We introduce the characteristic function $\mathbf{1}_{\mathcal{S}}$ which equals one, if the assertion $\mathcal{S}$ holds true, and zero otherwise. The symbol $\mathbb{N}$ denotes the ring of positive integers. As usual, we denote by $S(m, n; c)$ the Kloosterman sum which is given in the following way $S(m, n; c) = \sum^*_{x \bmod c} e((m\overline{x} + nx)/c)$ for any positive integers m, n and c, where $*$ indicates that the summation is restricted to $(x, c) = 1$, and $\overline{x}$ is the inverse of x modulo c.*

2. Preliminaries

2.1. **Modular forms.** We will first give a recap of the theory of modular forms for $SL_2(\mathbb{Z})$. Let $k \geq 2$ be an even integer, and $N > 0$ an integer. Let χ be a primitive character to modulus q such that $N|q$, satisfying $\chi(-1) = (-1)^k$. We denote by $\mathcal{S}_k(N, \chi)$ the vector space of holomorphic cusp forms on $\Gamma_0(N)$ with nebentypus χ and weight k. For any $f \in \mathcal{S}_k(N, \chi)$, one has

$$f(z) = \sum_{n\geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$$

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for $z \in \mathfrak{h}$. Here, $e(z)$ means $e^{2\pi i z}$ for any $z \in \mathbb{C}$, and $\mathfrak{h}$ is the upper half-plane. Observe that $\mathcal{S}_k(N,\chi)$ is a finite dimensional Hilbert spaces which can be equipped with the Petersson inner products

$$\langle f_1, f_2\rangle = \int_{\Gamma_0(N)\backslash\mathfrak{h}} f_1(z)\overline{f_2(z)}y^{k-2}\mathrm{d}x\mathrm{d}y.$$

Let us recall the Hecke operators $\{T_n\}$ with $(n,N)=1$, which satisfy the multiplicativity relation

$$T_nT_m = \sum_{d|(n,m)} \chi(d)T_{\frac{nm}{d^2}}. \tag{2.3}$$

It thus follows that, for any $f_1, f_2 \in \mathcal{S}_k(N,\chi)$, one has $\langle T_nf_1, f_2\rangle = \chi(n)\langle f_1, T_nf_2\rangle$ for all $(n,N)=1$. One can also find an orthogonal basis $\mathcal{B}_k(N,\chi)$ of $\mathcal{S}_k(N,\chi)$ consisting of common eigenfunctions of all the Hecke operators T_n with $(n,N)=1$. For each $f \in \mathcal{B}_k(N,\chi)$, denote by $\lambda_f(n)$ the n-th Hecke eigenvalue, which satisfies the relation $T_nf(z) = \lambda_f(n)f(z)$ for all $(n,N)=1$. It thus follows from (2.3) that

$$\psi_f(m)\lambda_f(n) = \sum_{d|(n,m)} \chi(d)\psi_f\left(\frac{mn}{d^2}\right)$$

for any $m,n \geq 1$ with $(n,N)=1$. In particular, $\psi_f(1)\lambda_f(n) = \psi(n)$, if $(n,N)=1$. It is therefore can be enunciated that

$$\overline{\lambda_f(n)} = \overline{\chi(n)}\lambda_f(n), \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \tag{2.4}$$

whenever $(mn,N)=1$.

The Hecke eigenbasis $\mathcal{B}_k(N,\chi)$ also contains a subset of newforms $\mathcal{B}_k^*(N,\chi)$, those forms which are simultaneous eigenfunctions of all the Hecke operators T_n for any $n \geq 1$, and normalized to have first Fourier coefficient $\psi_f(1)=1$. The elements of $\mathcal{B}_k^*(N,\chi)$ are usually called primitive forms (the symbol is simply abbreviated to $\mathcal{B}_k^*(N)$, if χ is trivial). In particular, for any primitive form $f \in \mathcal{B}_k^*(N,\chi)$, the relations in (2.4) holds for any $m,n \geq 1$, from which one may have the exact factorization that $\lambda_f(dm) = \lambda_f(d)\lambda_f(m)$ for $d|N$. It is, on the other hand, worth to record that, for general $n \geq 1$, Deligne's bound asserts that $|\lambda_f(n)| \leq d(n)$; while, the Rankin-Selberg theory implies

$$\sum_{1\leq n\leq X} |\lambda_f(n)|^2 \ll_k (XN)^{\varepsilon}X \tag{2.5}$$

uniformly in any $X \geq 2$.

2.2. GL_2 **Voronoĭ formula.** We will have a need of the following Voronoĭ-type summation formula; see, for instance, [20, Theorem A.4].

Lemma 2.1. *Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let $f \in \mathcal{B}_k^*(N)$ be a newform. For $(a,q)=1$ set $N_2 := N/(N,q)$. If $h \in \mathbb{C}^\infty(\mathbb{R}^{\times,+})$ is a Schwartz function vanishing in a neighborhood of zero, then there exists a complex number $\mathfrak{l}$ of modulus one, which depends on a, q and f, and a newform $f^\star \in \mathcal{B}_k^*(N)$ such that*

$$\sum_{n\geq 1} \lambda_f(n)e\left(\frac{an}{q}\right)h\left(\frac{n}{X}\right) = \frac{2\pi\mathfrak{l}}{q\sqrt{N_2}}\sum_{n\geq 1} \lambda_{f^\star}(n)e\left(-\frac{\overline{aN_2}n}{q}\right)\mathfrak{J}\left(\frac{nX}{q^2N_2};h\right), \tag{2.6}$$

where

$$\mathfrak{J}(x;h) = \int_{\mathbb{R}^+} h(\xi)J_{k-1}(4\pi\sqrt{x\xi})\mathrm{d}\xi.$$

For any $x > 0$, one may write $J_{k-1}(x) = x^{-1/2}(H_k^+(x)e(x) + H_k^-(x)e(-x))$ for some smooth functions $H^\pm$ satisfying that $x^j H_k^{\pm(j)}(x) \ll_k x/(1+x)^{3/2}$ for any $j \geq 0$; the existence is guaranteed, for instance, by [31, Section 6.5], if $x < 1$ and [31, Section 3.4], if $x \geq 1$.

2.3. **The Wilton-type bounds.** Let $X \geq 2$. Suppose that the function $w(y)$ satisfies that

$$\begin{cases} w(y) \text{ is smooth with support in the dyadic interval } [X, 2X], \\ y^j |w^{(j)}(y)| \leq c_j \end{cases}$$

for all $j \geq 0$ and some positive real numbers c_j. We call $w(y)$ an X-*dyadic weight function*. We now have the following Wilton-type bound involving the cusp forms on GL_2, which we shall use after a while, and from which the final bound in our main theorem would follow; see [15, Corollary 1.8].

Lemma 2.2. *Let $X \geq 2$ and $w(y)$ be an X-dyadic weight function. Then, for any $\alpha \in \mathbb{R}$ and newform $f \in \mathcal{B}_k^*(N)$ with square-free level N, we have*

$$\sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} e(n\alpha) w(n) \ll_{k,\varepsilon,c_j} X^\varepsilon N^{\frac{1}{4}+\varepsilon}. \tag{2.7}$$

2.4. **The delta method.** The δ-symbol method was developed in [5, 6] as variant of the circle method. Further development and applications can be found in Jutila [18, 19], Heath-Brown [11], Munshi [27], and more recently [1] to name a few. The main purpose is to express $\delta(n, 0)$ the Dirac symbol at 0 (restricted to the integers n in some given range: $|n| \leq X$), in terms of 'harmonics' $e(an/q)$ for some integers a, q satisfying $(a, q) = 1$ and $q \leq Q$, with Q being any fixed positive real number. In order to be of practical use, one expects the δ-symbol method should be capable of providing an expression for $\delta(n, 0)$ in terms of harmonics of a small moduli. Nevertheless, the modulus in the circle method cannot be less than $\sqrt{X}$, which corresponds to using Dirichlet's approximation theorem to produce values $q \leq Q$ (see [11]).

Instead of directly appealing to the version due to Duke, Friedlander and Iwaniec (see, for instance, [17, Chapter 20]), in this paper, we shall exploit an important new input - the 'conductor lowering mechanism' due to Munshi; see [25], [26] or the survey [28].

Lemma 2.3. *Let $Q \geq 1$. Then, for any n up to X and $\mathfrak{K}|n$, one might thus detect the symbol $\delta(n, 0)$ in the following manner*

$$\delta(n, 0) = \frac{1}{\mathfrak{K}Q} \sum_{q \leq Q} \frac{1}{q} \sum_{\substack{a \bmod q\mathfrak{K} \\ (a,q)=1}} e\left(\frac{an}{q\mathfrak{K}}\right) \int_{\mathbb{R}} g(q, \tau) e\left(\frac{n\tau}{qQ\mathfrak{K}}\right) \mathrm{d}\tau,$$

where

$$g(q, \tau) = 1 + h(q, \tau) \quad \textit{with} \quad h(q, \tau) = O\left(\frac{1}{qQ}\left(|\tau| + \frac{q}{Q}\right)\right)^A,$$

$$\tau^j \frac{\partial^j}{\partial \tau^j} g(q, \tau) \ll \log Q \min\left(\frac{Q}{q}\frac{1}{|\tau|}\right) \textit{ for any integer } j \geq 0,$$

and $g(q, \tau) \ll |\tau|^{-A}$ for any sufficiently large A. In particular, the effective range of the τ-integral is $[-X^\varepsilon, X^\varepsilon]$.

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3. Proof of theorem 1.1

3.1. **Initial configuration.** In this section, we are dedicated to the proof of Theorem 1.1. Recall that we shall be concerned about the triple sum

$$\sum_{h\geq 1}\sum_{l\geq 1}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)\lambda_f(n+l)\,W\Big(\frac{n}{X}\Big)V^\flat\Big(\frac{h}{H}\Big)V^\natural\Big(\frac{l}{L}\Big) \tag{3.8}$$

for $H, L \leq X$. Appealing to $\delta(n,0)$, the Dirac symbol at 0, one may re-write the above as

$$\sum_{m\geq 1}\sum_{n\geq 1}\sum_{t\geq 1}\sum_{h\geq 1}\sum_{l\geq 1}\lambda_f(n)\lambda_f(m)\lambda_f(t)\delta(m-n-h,0)\,\delta(t-n-l,0)\,U^\flat\Big(\frac{m}{X}\Big)$$
$$W^\flat\Big(\frac{n}{X}\Big)U^\natural\Big(\frac{t}{X}\Big)V^\flat\Big(\frac{h}{H}\Big)V^\natural\Big(\frac{l}{L}\Big),$$

where $U^\flat, U^\natural, V^\flat, V^\natural, W$ are five smooth functions supported $[1/2, 5/2]$ with bounded derivatives, respectively. We manage to detect the shifts $m = n+h$ and $t = n+l$ by invoking Lemma 2.3 with $\mathfrak{K} = p$. We are thus led to an alternative form for the sum in (3.8) as follows:

$$\Xi(p,H,L,X) = \sum_{1\leq q_1,q_2\leq \mathcal{Q}}\;\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2p\\ (\alpha,q_1)=1,(\beta,q_2)=1}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2p), \tag{3.9}$$

where the multiple sum $\mathcal{K}$ is defined as

$$\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,\ell_1,\ell_2) = \frac{1}{(p\mathcal{Q})^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2}\sum_{m\geq 1}\lambda_f(m)e\Big(\frac{\alpha m}{\ell_1}\Big)$$
$$U^\flat_{\tau_1}\Big(\frac{m}{X}\Big)\sum_{t\geq 1}\lambda_f(t)e\Big(\frac{\beta t}{\ell_2}\Big)U^\natural_{\tau_2}\Big(\frac{t}{X}\Big)\sum_{n\geq 1}\lambda_f(n)e\Big(-\frac{(\alpha\ell_2+\beta\ell_1)n}{\ell_1\ell_2}\Big)$$
$$W_{\tau_1,\tau_2}\Big(\frac{n}{X}\Big)\sum_{h\geq 1}\sum_{l\geq 1}e\Big(-\frac{\alpha h}{\ell_1}\Big)V^\flat_{\tau_1}\Big(\frac{h}{H}\Big)e\Big(-\frac{\beta l}{\ell_2}\Big)V^\natural_{\tau_2}\Big(\frac{l}{L}\Big)\,\mathrm{d}\tau_1\mathrm{d}\tau_2$$

for any $\ell_1, \ell_2 \in \mathbb{N}$, with the functions $U^\flat, U^\natural, V^\flat, V^\natural, W$ being

$$U^\flat_{\tau_1}(m) = U^\flat(m)e\Big(\frac{mX\tau_1}{pq\mathcal{Q}}\Big),\quad U^\natural_{\tau_2}(t) = U^\natural(t)e\Big(\frac{tX\tau_2}{pq\mathcal{Q}}\Big),\quad W_{\tau_1,\tau_2}(n) = W(n)e\Big(-\frac{nX(\tau_1+\tau_2)}{pq\mathcal{Q}}\Big)$$
$$V^\flat_{\tau_1}(h) = V^\flat(h)e\Big(-\frac{hH\tau_1}{pq\mathcal{Q}}\Big),\quad V^\natural_{\tau_2}(l) = V^\natural(l)e\Big(-\frac{lL\tau_2}{pq\mathcal{Q}}\Big).$$

Here, the parameter $\mathcal{Q}$ will be taken as

$$\mathcal{Q} = \sqrt{\frac{X}{p}} \tag{3.10}$$

which is below the square-root of the length of the summation X over n in (3.8); this, however, is what the philosophy of the 'conductor lowering mechanism' embodies.

We shall now proceed to distinguish whether $(\alpha,p) = 1$ (resp. $(\beta,p) = 1$) or not in the following analysis. We are thus led to nine parts for Ξ, i.e., the 'degenerate term' $\Xi^{\text{Deg.}}$, the 'non-degenerate' term $\Xi^{\text{Non-de.}}$, the two 'cross terms' $\Xi^{\text{Cros1.}}, \Xi^{\text{Cros2.}}$ and the error term $\Xi^{\text{Err.}}$, with $\Xi = \Xi^{\text{Deg.}} + \Xi^{\text{Non-de.}} + \Xi^{\text{Cros1.}} + \Xi^{\text{Cros2.}} + \Xi^{\text{Err.}}$. Here, the degenerate and non-degenerate terms are respectively defined as

$$\Xi^{\text{Deg.}}(p,H,L,X) = \sum_{\substack{1\leq q_1,q_2\leq\mathcal{Q}\\ (p,q_1q_2)=1}}\;\sideset{}{^*}\sum_{\substack{\alpha \bmod q_1\\ \beta \bmod q_2}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1,q_2), \tag{3.11}$$

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$$\Xi^{\text{Non-de.}}(p,H,L,X)=\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_1q_2)=1}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2p}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2p), \tag{3.12}$$

and the two cross terms are defined as

$$\Xi^{\text{Cros1.}}(p,H,L,X)=\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_1q_2)=1}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2), \tag{3.13}$$

$$\Xi^{\text{Cros2.}}(p,H,L,X)=\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_1q_2)=1}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1\\ \beta \bmod q_2p}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1,q_2p); \tag{3.14}$$

while, the remaining error term $\Xi^{\text{Err.}}(p,H,L,X)$ is given by the following

$$\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_1)=1\\ p|q_2}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2p}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2p)+\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_1)=1\\ p|q_2}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1\\ \beta \bmod q_2p}}$$
$$\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1,q_2p)+\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_2)=1\\ p|q_1}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2p}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2p)+\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(p,q_2)=1\\ p|q_1}}$$
$$\sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2)+\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\ p|q_1,p|q_2}}\ \sideset{}{^*}\sum_{\substack{\alpha \bmod q_1p\\ \beta \bmod q_2p}}\mathcal{K}_{q_1,q_2}(H,L,X;\alpha,\beta,q_1p,q_2p).$$

To take care of the subsums above now will be the objectives of the remaining parts of this paper. We shall begin with $\Xi^{\text{deg.}}$; the analysis of the term $\Xi^{\text{Non-de.}}$ and then the two cross terms $\Xi^{\text{Cros1.}}$, $\Xi^{\text{Cros2.}}$ will be postponed to the end of this paper. While, truly $\Xi^{\text{Err.}}$ serves as a noisy error term, which provides a relatively small magnitude to Ξ by an entirely analogous argument as that for the major term $\Xi^{\text{Non-de.}}$ in §3.3.

3.2. **Treatment of $\Xi^{\text{Deg.}}$.** In this part, we deal with the multiple sum $\Xi^{\text{Deg.}}$, as shown in (3.11). For any $\iota,\nu,\upsilon,\rho,\varsigma\in\mathbb{R}$, write

$$\mathcal{W}_{\tau_1,\tau_2}(\iota,\nu,\upsilon,\rho,\varsigma)=U^{\flat}_{\tau_1}(\iota)\,U^{\natural}_{\tau_2}(\nu)\,W_{\tau_1,\tau_2}(\upsilon)\,V^{\flat}_{\tau_1}(\rho)V^{\natural}_{\tau_2}(\varsigma). \tag{3.15}$$

One finds $\Xi^{\text{Deg.}}$ is dominated by

$$\frac{X}{(p\mathcal{Q})^2}\sup_{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon}\Bigg|\sum_{\substack{1\le q_1,q_2\le\mathcal{Q}\\(q_1q_2,p)=1}}\frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2}\ \sideset{}{^*}\sum_{\alpha \bmod q_1}\ \sideset{}{^*}\sum_{\beta \bmod q_2}\sum_{m\ge1}\frac{\lambda_f(m)}{\sqrt{m}}$$
$$e\left(\frac{\alpha m}{q_1}\right)\sum_{t\ge1}\frac{\lambda_f(t)}{\sqrt{t}}e\left(\frac{\beta t}{q_2}\right)\sum_{n\ge1}\lambda_f(n)e\left(-\frac{(\alpha q_2+\beta q_1)n}{q_1q_2}\right) \tag{3.16}$$
$$\sum_{h\ge1}\sum_{l\ge1}e\left(-\frac{\alpha h}{q_1}-\frac{\beta l}{q_2}\right)\mathcal{W}_{\tau_1,\tau_2}\left(\frac{m}{X},\frac{t}{X},\frac{n}{X},\frac{h}{H},\frac{l}{L}\right)\Bigg|.$$

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We will now apply the Voronoĭ formula, Lemma 2.1, to transform the sums over m, t into their dualized forms, which reveals that the expression in (3.16) would be no more than

$$\frac{1}{p}\sup_{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon}\left|\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(q_1q_2,p)=1}}\frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2}\sum_{m,t\ll_\varepsilon X^\varepsilon}\frac{\lambda_f(m)\lambda_f(t)}{\sqrt{mt}}\sum_{n\ge1}\lambda_f(n)\right.$$
$$\left.\sum_{h\ge1}\sum_{l\ge1}S(-\overline{p}m,-(n+h);q_1)S(-\overline{p}t,-(n+l);q_2)\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{mX}{q_1^2p},\frac{n}{X},\frac{h}{H}\right)\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{tX}{q_2^2p},\frac{n}{X},\frac{l}{L}\right)\right|. \tag{3.17}$$

Here, for any $\rho,\upsilon,\nu\in\mathbb{R}^+$, $\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}$ is explicitly given by

$$\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}(\rho,\upsilon,\nu)=W_{\tau_1,\tau_2}(\upsilon)\,V^\flat_{\tau_1}(\nu)\int_{\mathbb{R}^+}U^\flat_{\tau_1}(\xi)\left(4\pi\sqrt{\rho\xi}\right)^{-\frac{1}{2}}H^{\pm}_k\left(4\pi\sqrt{\rho\xi}\right)e\left(\pm4\pi\sqrt{\rho\xi}\right)\mathrm{d}\xi,$$

upon combining with the basic approximations of J-Bessel functions in §2.2; while, $\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}$ corresponds to the exact form of $\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}$, with $V^\flat_{\tau_1}$ (resp. $U^\flat_{\tau_1}$) replaced by $V^\natural_{\tau_2}$ (resp. $U^\natural_{\tau_2}$). By repeated integration by parts for enough times, one quickly sees that essentially $\rho\ll_\varepsilon X^\varepsilon\cdot(1+X\tau_1/(q_1p\mathcal{Q}))^2$ and

$$\rho^j\frac{\partial^j}{\rho^j}U^{\flat\ (j)}_{\tau_1}\left(\frac{\xi}{\rho}\right)\ll_j\left(1+\frac{X\tau_1}{q_1p\mathcal{Q}}\right)^j$$

for any $j\ge0$. It thus follows that

$$\left|\rho^j\frac{\partial^j}{\rho^j}\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}(\rho,\upsilon,\nu)\right|+\left|\rho^j\frac{\partial^j}{\rho^j}\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}(\rho,\upsilon,\nu)\right|\ll_{k,j}\frac{\rho^{\frac{1}{4}}}{(1+\sqrt{\rho})^{\frac{3}{2}}}\left(1+\frac{X\tau_1}{q_1p\mathcal{Q}}\right)^j, \tag{3.18}$$

as $k\to\infty$, upon changing the variable in the integral above. Moreover, it can be seen that essentially $m\ll_\varepsilon q_1^2p/X^{1-\varepsilon}\cdot(1+X\tau_1/(q_1p\mathcal{Q}))^2$ and $t\ll_\varepsilon q_2^2p/X^{1-\varepsilon}\cdot(1+X\tau_2/(q_2p\mathcal{Q}))^2$. It is also remarkable that, here and in the sequel, one might identify $\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}}$ and $\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}$ as two Schwarz functions with rapid decay, respectively.

To proceed further, let us pay attention to the case where $q_1=q_2$ in (3.17). It is verifiable that an argument which has the flavors of that for the dominated case of $q_1\ne q_2$, however, indicates the much less importance of this scenario (as far as the contribution is concerned). Indeed, if $q_1=q_2=q$, say, the secondary application of the Cauchy-Schwarz inequality shows an contribution by an amount

$$\frac{1}{p}\sum_{1\le q\le\mathcal{Q}}\frac{\mathcal{A}_1^{\frac{1}{2}}(q^2p/X,H;q)\mathcal{A}_2^{\frac{1}{2}}(q^2p/X,L;q)}{q^2} \tag{3.19}$$

to $\Xi^{\mathrm{Deg.}}$, where, for any $V\ge2$,

$$\mathcal{A}_1(V,H;q)=\sum_{n\ge1}\sum_{m\ge1}\left|\sum_{h\ge1}S(-\overline{p}m,-(n+h);q)\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{m}{V},\frac{n}{X},\frac{h}{H}\right)\right|^2, \tag{3.20}$$

and $\mathcal{A}_2$ indicates the same expression with $\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}$ replaced by $\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}$. Possion shows that the $\mathcal{A}_1$-sum (resp. the $\mathcal{A}_2$-sum) is bounded by $\ll XVHq$. This implies that (3.19) is no more than $O(X\sqrt{HL}/p)$ which is well controlled by the estimate (3.26) below.

Next, we shall be devoted to the analysis of the typical scenario where q_1 differs always from q_2 in (3.17). One applies the Cauchy-Schwarz inequality, which produces (essentially)

$$\Xi^{\text{Deg.}}(p,H,L,X) \ll \frac{1}{p} \sup_{\substack{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon \\ m,t\ll_\varepsilon X^\varepsilon}} \Omega_1^{\frac{1}{2}}(p,m,H,X)\,\Omega_2^{\frac{1}{2}}(p,t,L,X) \tag{3.21}$$

with

$$\begin{aligned}
\Omega_1(p,m,H,X) &= \sum_{n\geq 1}\left|\sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{g(\tau,q)}{q}\sum_{h\geq 1} S(-\overline{p}m,-(n+h);q)\,\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q^2p},\frac{n}{X},\frac{h}{H}\right)\right|^2,\\
\Omega_2(p,t,L,X) &= \sum_{n\geq 1}\left|\sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{g(\tau,q)}{q}\sum_{h\geq 1} S(-\overline{p}t,-(n+l);q)\,\widetilde{\mathcal{W}_{2,\tau_1,\tau_2}^{\pm}}\left(\frac{tX}{q^2p},\frac{n}{X},\frac{l}{L}\right)\right|^2.
\end{aligned} \tag{3.22}$$

In the following analysis, it suffices to evaluate Ω_1; the same argument works for Ω_2, observing that they bear a striking resemblance to each other. Here, more precisely, one has

$$\begin{aligned}
\Omega_1(p,m,H,X) = \sum_{1\leq q_1,q_2\leq \mathcal{Q}} \frac{g(\tau,q_1)\overline{g(\tau,q_2)}}{q_1q_2} \sum_{h_1,h_2\geq 1}\sum_{n\geq 1} S(-\overline{p}m,-(n+h_1);q_1)\\
\overline{S(-\overline{p}m,-(n+h_2);q_2)}\,\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q_1^2p},\frac{n}{X},\frac{h_1}{H}\right)\overline{\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q_2^2p},\frac{n}{X},\frac{h_2}{H}\right)},
\end{aligned} \tag{3.23}$$

As is customary in studying the multiple sum Ω_1, we shall identify it depending on whether q_1 and q_2 are equal or not. The contributions from both cases to Ω_1 are denoted by Ω_1^0 and $\Omega_1^{\neq}$, respectively. We shall now begin with Ω_0. In the case of $q_1=q_2=q$, say, after an application of the Poisson to the n-sum (with the modulus q), it is presented in the following form

$$\begin{aligned}
X\sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{|g(\tau,q)|^2}{q^3}\sum_{h_1,h_2\geq 1}\sum_{\gamma \bmod q} S(-\overline{p}m,-(\gamma+h_1);q)\overline{S(-\overline{p}m,-(\gamma+h_2);q)}\\
\int_{\mathbb{R}^+}\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q^2p},\xi,\frac{h_1}{H}\right)\overline{\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q^2p},\xi,\frac{h_2}{H}\right)}\mathrm{d}\xi.
\end{aligned}$$

It is remarkable that, here, the non-zero frequencies do not exist in practice, in view of that $q<X^{1+\varepsilon}$. Opening the Kloosterman sums, and executing the γ-sum shows that the inner-sum on the first line is roughly $q^2\mathbf{1}_{h_1\equiv h_2 \bmod q}$, upon employing the relation involving Ramanujan sum that $S(n,0;q)=\sum_{ab=q}\mu(a)\sum_{\beta \bmod q} e(\beta n/b)$. We thus find

$$\Omega_1^0(p,m,H,X) \ll X^{1+\varepsilon}\sum_{1\leq q\leq \mathcal{Q}}\frac{H}{q}\left(1+\frac{H}{q}\right) \ll X^{1+\varepsilon}H^2. \tag{3.24}$$

Now, let us move on to the investigation of $\Omega_1^{\neq}$. The initial procedure is to invoke the Poisson (with the modulus q_1q_2), which transforms the multiple sum $\Omega_1^{\neq}$ into

$$\begin{aligned}
X\sum_{\substack{1\leq q_1,q_2\leq \mathcal{Q}\\(q_1q_2,p)=1}} \frac{g(\tau,q_1)\overline{g(\tau,q_2)}}{(q_1q_2)^2}\sum_{h_1,h_2\geq 1}\sum_{\delta \bmod q_1q_2} S(-\overline{p}m,-(\delta+h_1);q_1)\\
\overline{S(-\overline{p}m,-(\delta+h_2);q_2)}\int_{\mathbb{R}^+}\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q_1^2p},\xi,\frac{h_1}{H}\right)\overline{\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}\left(\frac{mX}{q_2^2p},\xi,\frac{h_2}{H}\right)}\mathrm{d}\xi.
\end{aligned}$$

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The non-zero frequencies do not exist as well, since of $q_1q_2 < X$. We claim actually the display above vanishes. To show this, one proceeds by writing $q_1 = q_1'\hbar$, $q_2 = q_2'\hbar$, with $(q_1, q_2) = \hbar$ and $(q_1', q_2') = 1$. Notice that $\hbar$ is co-prime with one of factors q_1', q_2'; without loss of generality, we assume that $(\hbar, q_1') = 1$. We find the δ-sum thus can be expressed as

$$\sum_{\alpha \bmod q_1'} \sideset{}{^*}\sum_{x \bmod q_1'\hbar} \sideset{}{^*}\sum_{y \bmod q_2'\hbar} e\left(-\frac{\overline{px}m + (\alpha + h_1)x}{q_1'\hbar} + \frac{\overline{py}m + (\alpha + h_2)y}{q_2'\hbar}\right)$$

$$= \sideset{}{^*}\sum_{x_1 \bmod q_1'} \sideset{}{^*}\sum_{x_2 \bmod q_2'\hbar} \sideset{}{^*}\sum_{y_1 \bmod \hbar} \sum_{\alpha \bmod q_1'} e\left(-\frac{\overline{p\hbar x_1} + m(\alpha + h_1)\overline{\hbar}x_1}{q_1'}\right)$$

$$\sum_{\beta \bmod q_2'\hbar} e\left(-\frac{\overline{pq_1'y_1}m + (\beta + h_1)\overline{q_1'}y_1}{\hbar} + \frac{\overline{px_2}m + (\beta + h_2)x_2}{q_2'\hbar}\right) = 0.$$

This, however, confirms the prior assertion immediately. Altogether, one arrives at

$$\Omega_1(p, m, H, X) \ll X^{1+\varepsilon}H^2, \quad \Omega_2(p, t, L, X) \ll X^{1+\varepsilon}L^2 \tag{3.25}$$

for any $\varepsilon > 0$, from which, it thus can be inferable that

$$\Xi^{\text{Deg.}}(p, H, L, X) \ll \frac{X^{1+\varepsilon}HL}{p}, \tag{3.26}$$

upon recalling (3.21).

3.3. **Treatment of $\Xi^{\text{Non-de.}}$.** Now, let us concentrate on the analysis of $\Xi^{\text{Non-de.}}$. Recall (3.12). One finds that the quantity we are faced with is the following

$$\begin{aligned}\frac{X}{(p\mathcal{Q})^2} \sup_{\tau_1,\tau_2 \ll_\varepsilon X^\varepsilon} \Bigg| \sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(q_1q_2,p)=1}} \frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2} \sideset{}{^*}\sum_{\alpha \bmod pq_1} \sideset{}{^*}\sum_{\beta \bmod pq_2} \sum_{m\ge1} \frac{\lambda_f(m)}{\sqrt{m}} \\ e\left(\frac{\alpha m}{q_1p}\right) \sum_{t\ge1} \frac{\lambda_f(t)}{\sqrt{t}} e\left(\frac{\beta t}{q_2p}\right) \sum_{n\ge1} \lambda_f(n) e\left(-\frac{(\alpha q_2 + \beta q_1)n}{q_1q_2p}\right) \\ \sum_{h\ge1}\sum_{l\ge1} e\left(-\frac{\alpha h}{q_1p} - \frac{\beta l}{q_2p}\right) \mathcal{W}_{\tau_1,\tau_2}\left(\frac{m}{X}, \frac{t}{X}, \frac{n}{X}, \frac{h}{H}, \frac{l}{L}\right)\Bigg|,\end{aligned} \tag{3.27}$$

where $\mathcal{W}_{\tau_1,\tau_2}$ is defined as in (3.15). We intend to apply the Voronoĭ formula, Lemma 2.1, again, the multiple sum in the absolute value thus being recast as

$$\begin{aligned}\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(q_1q_2,p)=1}} \frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2} \sum_{m,t\ll_\varepsilon p^{1+\varepsilon}} \frac{\lambda_f(m)\lambda_f(t)}{\sqrt{mt}} \sum_{n\ge1} \lambda_f(n) \sum_{h\ge1}\sum_{l\ge1} S(-m, -(n+h); q_1p) \\ S(-t, -(n+l); q_2p)\, \widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{mX}{(q_1p)^2}, \frac{n}{X}, \frac{h}{H}\right) \widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{tX}{(q_2p)^2}, \frac{n}{X}, \frac{l}{L}\right),\end{aligned} \tag{3.28}$$

up to a multiplier factor of modulus $O(X^\varepsilon)$ at most. We first come to extracting the contribution from the case where $q_1 = q_2 = q$, say, in (3.28). In that case, (3.28) reads

$$\sum_{\substack{1\le q\le \mathcal{Q}\\(q,p)=1}} \frac{g(q,\tau_1)g(q,\tau_2)}{q^2} \sum_{m,t\ll_\varepsilon p^{1+\varepsilon}} \frac{\lambda_f(m)\lambda_f(t)}{\sqrt{mt}} \sum_{n\ge1} \lambda_f(n) \sum_{h\ge1}\sum_{l\ge1} S(-m, -(n+h); qp)$$

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$$S(-t,-(n+l);qp)\,\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{mX}{(qp)^2},\frac{n}{X},\frac{h}{H}\right)\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{tX}{(qp)^2},\frac{n}{X},\frac{l}{L}\right).$$

To see quickly what will be the shape of the transformed expression, one may apply the Cauchy-Schwarz inequality twice bounding the display above by

$$\sum_{1\leq q\leq \mathcal{Q}}\frac{\mathcal{B}_1^{\frac{1}{2}}((qp)^2/X,H;qp)\mathcal{B}_2^{\frac{1}{2}}((qp)^2/X,L;qp)}{q^2}, \tag{3.29}$$

where, as in (3.20), for any $T\geq 2$, $\mathcal{B}_1$ is given by

$$\mathcal{B}_1(T,H;q)=\sum_{n\geq 1}\sum_{m\geq 1}\left|\sum_{h\geq 1}S(-m,-(n+h);q)\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m}{T},\frac{n}{X},\frac{h}{H}\right)\right|^2, \tag{3.30}$$

and $\mathcal{B}_2$ means the same expression just with $\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}$ replaced by $\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}$. Notice that there holds the estimate that $\mathcal{B}_1(T,H;q)\ll XTHq$ which follows by an application of the Possion, whence the sum in (3.29) is $O(Xp^2\sqrt{HL})$. The contribution from the case of $q_1=q_2$ to $\Xi^{\text{Non-de.}}$ is majorized by $O(pX\sqrt{HL})$, upon recalling (3.27).

As presented before, here, the salient point is to analyze the scenario where $q_1\neq q_2$ in (3.28). Akin to (3.21), the Cauchy-Schwarz inequality thus implies

$$\Xi^{\text{Non-de.}}(p,H,L,X)\ll\frac{1}{p}\sup_{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon}\Psi_1^{\frac{1}{2}}(p,H,X)\,\Psi_2^{\frac{1}{2}}(p,L,X)+pX\sqrt{HL} \tag{3.31}$$

with Ψ_1,Ψ_2 being taking the following forms

$$\Psi_1(p,H,X)=\sum_{n\geq 1}\left|\sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}}\frac{g(\tau,q)}{q}\sum_{h\geq 1}\sum_{m\ll_\varepsilon p^{1+\varepsilon}}\frac{\lambda_f(m)}{\sqrt{m}}S(-m,-(n+h);qp)\,\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{mX}{(qp)^2},\frac{n}{X},\frac{h}{H}\right)\right|^2,$$

and

$$\Psi_2(p,L,X)=\sum_{n\geq 1}\left|\sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}}\frac{g(\tau,q)}{q}\sum_{t\ll_\varepsilon p^{1+\varepsilon}}\frac{\lambda_f(t)}{\sqrt{t}}\sum_{l\geq 1}S(-t,-(n+l);qp)\,\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{tX}{(qp)^2},\frac{n}{X},\frac{l}{L}\right)\right|^2,$$

respectively. Here, the term $pX\sqrt{HL}$ on the right-hand side of (3.31) means the contribution from the case of $q_1=q_2$ in (3.27). We shall now merely consider Ψ_1 in what follows; the argument for Ψ_2 follows similarly. Upon expanding the square, one sees, more explicitly,

$$\Psi_1(p,H,X)=\sum_{n\geq 1}\sum_{\substack{1\leq q_1,q_2\leq \mathcal{Q}\\(q_1q_2,p)=1}}\frac{g(\tau,q_1)\overline{g(\tau,q_2)}}{q_1q_2}\sum_{h_1,h_2\geq 1}\sum_{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}$$
$$S(-m_1,-(n+h_1);q_1p)\overline{S(-m_2,-(n+h_2);q_2p)}$$
$$\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_1X}{(q_1p)^2},\frac{n}{X},\frac{h_1}{H}\right)\overline{\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_2X}{(q_2p)^2},\frac{n}{X},\frac{h_2}{H}\right)}.$$

In analogy to Ω_1, we now proceed from the non-generic terms $q_1=q_2$ and the generic terms $q_1\neq q_2$, the contributions to Ψ_1 from the both being denoted by Ψ_1^0 and $\Psi_1^{\neq}$, respectively (in other words, one has the decomposition $\Psi_1 = \Psi_1^0 + \Psi_1^{\neq}$). We first treat Ψ_1^0. Assume that $q_1 = q_2 = q$. One has already seen that

$$\Psi_1^0(p,H,X) = \sum_{n\geq 1} \sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{|g(\tau,q)|^2}{q^2} \sum_{h_1,h_2\geq 1} \sum_{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}} S(-m_1,-(n+h_1);qp)\,\overline{S(-m_2,-(n+h_2);qp)}\, \widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_1X}{(qp)^2},\frac{n}{X},\frac{h_1}{H}\right)\overline{\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_2X}{(qp)^2},\frac{n}{X},\frac{h_2}{H}\right)}.$$

–*Contribution from* $h_1 = h_2$. Let us have a look at the non-generic situation where $h_1 = h_2 = h$, say. An application of the Poisson to the n-sum reduces the right-hand side to

$$Xp \sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{|g(\tau,q)|^2}{q} \sum_{h\geq 1} \sum_{\substack{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}\\ m_1\equiv m_2 \bmod qp}} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}} \int_{\mathbb{R}^+} \widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_1X}{(qp)^2},\xi,\frac{h}{H}\right)\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{m_2X}{(qp)^2},\xi,\frac{h}{H}\right)\mathrm{d}\xi \ll HX^{1+\varepsilon}p. \tag{3.32}$$

–*Contribution from* $h_1 \neq h_2$. Next, we turn to the generic situation where $h_1 \neq h_2$. Assembling with the Poisson (to the h_1, h_2-sums with the modulus qp this time), we are thus, however, led to

$$H^2 \sum_{n\geq 1} \sum_{\substack{1\leq q\leq \mathcal{Q}\\(q,p)=1}} \frac{|g(\tau,q)|^2}{q^2} \sum_{|l_1|\ll_\varepsilon qp/H^{1-\varepsilon}} e\left(\frac{-nl_1}{qp}\right) \sum_{|l_2|\ll_\varepsilon qp/H^{1-\varepsilon}} e\left(\frac{-nl_2}{qp}\right) \sum_{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}} e\left(\frac{-m_1\overline{l_1}-m_2\overline{l_2}}{qp}\right)\mathcal{Y}\left(\frac{m_1X}{(qp)^2},\frac{n}{X},\frac{Hl_1}{qp}\right)\overline{\mathcal{Y}\left(\frac{m_2X}{(qp)^2},\frac{n}{X},-\frac{Hl_2}{qp}\right)}, \tag{3.33}$$

where the resulting integral $\mathcal{Y}$ is defined as

$$\mathcal{Y}(x,y,l) = \int_{\mathbb{R}^+} \widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}(x,y,\xi)\,e(-\xi l)\,\mathrm{d}\xi$$

for any $x, y \in \mathbb{R}^+$ and $l \in \mathbb{Z}$ with $l \neq 0$. We now wish to apply the Wilton-type bound in Lemma 2.2 to the inner sums over m_1, m_2 in (3.33). To this end, we denote by $\Upsilon(m_1,m_2;n,l_1,l_2)$ this double sum, and decompose dyadically it in the m_1, m_2-variables such that

$$\Upsilon(m_1,m_2;n,l_1,l_2) = \sum_{Z_1\geq 1}\sum_{Z_2\geq 1} \Upsilon_{Z_1,Z_2}(m_1,m_2;n,l_1,l_2)) \tag{3.34}$$

with Υ_{Z_1,Z_2} being a smooth function of m_1, m_2 supported on $m_1 \sim Z_1$ and $m_2 \sim Z_2$, where Z_1 (resp. Z_2) runs through the powers of 2 independently and satisfies that $Z_1 \ll p^{1+\varepsilon}$ (resp. $Z_2 \ll p^{1+\varepsilon}$). We thus infer that the expression in (3.33) is

$$\ll (XH)^\varepsilon X\mathcal{Q}p^2 \sup_{\substack{Z_1\ll p^{1+\varepsilon}\\ Z_2\ll p^{1+\varepsilon}}} \left|\sum_{Z_1\geq 1}\sum_{Z_2\geq 1} \Upsilon_{Z_1,Z_2}(m_1,m_2;n,l_1,l_2)\right|. \tag{3.35}$$

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One finds, from (3.18), that $\Upsilon_{Z_1,Z_2}(m_1,m_2;n,l_1,l_2)$ is a Z_1 (resp. Z_2)-dyadic weight function with respect to the variable m_1 (resp. m_2), with

$$Z_1^i \frac{\partial^i}{m_1^i}\mathcal{Y}\left(\frac{m_1X}{(qp)^2},\frac{n}{X},\frac{Hl}{qp}\right) \ll_{k,i} \frac{qp(Z_1X)^{\frac14}}{\left(qp+\sqrt{Z_1X}\right)^{\frac32}}\left(1+\frac{X\tau_1}{qp\mathfrak{Q}}\right)^i$$

and

$$Z_2^j \frac{\partial^j}{m_2^j}\mathcal{Y}\left(\frac{m_2X}{(qp)^2},\frac{n}{X},\frac{Hl}{qp}\right) \ll_{k,j} \frac{qp(Z_2X)^{\frac14}}{\left(qp+\sqrt{Z_2X}\right)^{\frac32}}\left(1+\frac{X\tau_1}{qp\mathfrak{Q}}\right)^j$$

for any $i,j\geq 0$. An application of Lemma 2.2 finally shows that the contribution from $h_1\neq h_2$ in Ψ_1^0 is bounded by

$$\ll (XH)^\varepsilon X\mathfrak{Q}p^2\cdot(Xp)^\varepsilon p^{\frac12+\varepsilon} \ll (pH)^\varepsilon X^{\frac32+\varepsilon}p^2. \tag{3.36}$$

Having established the estimates for Ψ_1^0, we are left with $\Psi_1^{\neq}$. To evaluate this term, one invokes the Poisson (with the modulus pq_1q_2), so that it can be verifiable that actually there holds the following

$$\begin{aligned}\Psi_1^{\neq}(p,H,X) = X\sum_{\jmath\in\mathbb{Z}}\sum_{\substack{1\leq q_1,q_2\leq\mathfrak{Q}\\(q_1q_2,p)=1}}\frac{g(\tau,q_1)\overline{g(\tau,q_2)}}{(q_1q_2)^2p}\sum_{h_1,h_2\geq1}\sum_{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}\\ \mathfrak{F}(h_1,h_2,m_1,m_2,\jmath;p,q_1,q_2)\,\mathcal{Y}^\dagger\left(\frac{h_1}{H},\frac{h_2}{H},\frac{m_1X}{(q_1p)^2},\frac{m_2X}{(q_2p)^2},\frac{\jmath X}{pq_1q_2}\right)+O(X^{-A})\end{aligned} \tag{3.37}$$

for any sufficiently large A, where the exponential sum $\mathfrak{F}$ is given by

$$\mathfrak{F}(h_1,h_2m_1,m_2,\jmath;p,q_1,q_2) = \sum_{\gamma \bmod pq_1q_2} S(-m_1,-(\gamma+h_1);q_1p)\,\overline{S(-m_2,-(\gamma+h_2);q_2p)}\,e\left(\frac{\gamma\jmath}{pq_1q_2}\right),$$

and the resulting integral $\mathcal{Y}^\dagger$ is of the form

$$\mathcal{Y}^\dagger(h_1,h_2,m_1,m_2,\jmath) = \int_{\mathbb{R}^+}\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}(m_1,\xi,h_1)\,\overline{\widetilde{\mathcal{W}_{1,\tau_1,\tau_2}^{\pm}}(m_2,\xi,h_2)}\,e(-\xi\jmath)\,\mathrm{d}\xi.$$

It can be enunciated that, by repeated integration by parts for many times, that (essentially) $\jmath$ is truncated at $|\jmath|\ll_\varepsilon X^\varepsilon\mathfrak{Q}^2p/X^{1-\varepsilon}\ll X^\varepsilon$. Here, of course, $\mathcal{Y}^\dagger$ enjoys the analogous properties with that for a Schwarz function, which is controlled by $O_\varepsilon(X^\varepsilon)$ for any $\varepsilon>0$. As already hinted §3.2 in handling $\Omega_1^{\neq}$, it suffices to investigate the focal case where $(q_1,q_2)=1$ (from which the dominated contribution thus can be captured). In this sense, from now on, we shall carry out the discussions under the assumption of the coprimality between q_1 and q_2. Now, if one writes $\gamma = q_1q_2\overline{q_1q_2}x+q_1p\overline{q_1p}y+q_2p\overline{q_2p}z$, with $x \bmod p$, $y \bmod q_2$ and $z \bmod q_1$, such that $(x,p)=1$, $(y,q_2)=1$ and $(z,q_1)=1$, the sum $\mathfrak{F}$ thus reads

$$\begin{aligned}\sum_{x\bmod p}S(-m_1\overline{q_1},-(x+h_1)\overline{q_1};p)\,\overline{S(-m_2\overline{q_2},-(x+h_2)\overline{q_2};p)}\,e\left(\frac{x\overline{q_1q_2}\jmath}{p}\right)\sum_{y\bmod q_2}e\left(\frac{y\overline{pq_1}\jmath}{q_2}\right)\\ \overline{S(-m_2\overline{p},-(y+h_2)\overline{p};q_2)}\sum_{z\bmod q_1}S(-m_1\overline{p},-(z+h_1)\overline{p};q_1)\,e\left(\frac{z\overline{pq_2}\jmath}{q_1}\right)\end{aligned}$$

which is equal to

$$\begin{aligned}q_1q_2p\,e\left(-\frac{m_2q_1\cdot\overline{p\jmath}+h_2\jmath\cdot\overline{pq_1}}{q_2}-\frac{m_1q_2\cdot\overline{p\jmath}+h_1\jmath\cdot\overline{pq_2}}{q_1}\right)\\ \sum_{\varpi\bmod p}e\left(\frac{m_2\cdot\overline{q_2^2\varpi}-m_1\cdot\overline{q_1^2(\varpi+\overline{q_1q_2}\jmath)}-h_1(\varpi+\overline{q_1q_2}\jmath)+h_2\varpi}{p}\right).\end{aligned}$$

We proceed by applying the Poisson (to the h_1, h_2-sums) again. It turns out that the right-hand side of (3.37) thus can be dominated by

$$XH^2 \sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(q_1q_2,p)=1}} \frac{g(\tau,q_1)\overline{g(\tau,q_2)}}{q_1q_2} \sum_{|\jmath|\ll_\varepsilon X^\varepsilon} \sum_{\substack{|s_1|\ll_\varepsilon pq_1/H^{1-\varepsilon}\\(s_1,p)=1\\ s_1\equiv \overline{q_2}\jmath \bmod q_1}} \sum_{\substack{|s_2|\ll_\varepsilon pq_2/H^{1-\varepsilon}\\(s_2,p)=1\\ s_2\equiv \overline{q_1}\jmath \bmod q_2\\ q_2s_1+q_1s_2-\jmath\equiv 0 \bmod p}} \sum_{m_1,m_2\ll_\varepsilon p^{1+\varepsilon}} \frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}$$
$$e\left(-\frac{m_2q_1\cdot\overline{p\jmath}}{q_2}-\frac{m_1q_2\cdot\overline{p\jmath}}{q_1}-\frac{m_1\cdot\overline{q_1s_1}+m_2\cdot\overline{q_2s_2}}{p}\right)\mathcal{Y}^\ddagger\left(\frac{m_1X}{(q_1p)^2},\frac{m_2X}{(q_2p)^2},\frac{X\jmath}{pq_1q_2},\frac{Hs_1}{q_1p},\frac{Hs_2}{q_2p}\right) \tag{3.38}$$

with the weight function $\mathcal{Y}^\ddagger$ being given by

$$\mathcal{Y}^\ddagger(\rho_1,\rho_2,\jmath,s_1,s_2)=\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}(\rho_1,\xi_1,\xi_2)\overline{\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}(\rho_2,\xi_1,\xi_3)}\,e\left(-\xi_1\jmath\right.$$
$$\left.-\xi_2s_1-\xi_3s_2\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

for any $\rho_1,\rho_2\in\mathbb{R}^+$ and $s_1,s_2,\jmath\in\mathbb{Z}$. As done in estimating the multiple sum in (3.33), we shall employ Lemma 2.2 again. In a similar vein, one might proceed to denote by $\Theta(m_1,m_2;s_1,s_2,\jmath,q_1,q_2)$ the sums over m_1,m_2 in (3.38), and decompose dyadically this double sum such that

$$\Theta(m_1,m_2;s_1,s_2,\jmath,q_1,q_2)=\sum_{R_1\ge1}\sum_{R_2\ge1}\Theta_{R_1,R_2}(m_1,m_2;s_1,s_2,\jmath,q_1,q_2).$$

Here, Θ_{R_1,R_2} is a smooth function of m_1,m_2 supported on $m_1\sim R_1$ and $m_2\sim R_2$, with R_1 (resp. R_2) running through the powers of 2 independently, and satisfying that $R_1\ll p^{1+\varepsilon}$ (resp. $R_2\ll p^{1+\varepsilon}$). It thus follows that

$$\Psi_1^{\neq}(p,H,X)\ll XH^2\sum_{\substack{1\le q_1,q_2\le \mathcal{Q}\\(q_1q_2,p)=1}}\frac{|g(\tau,q_1)\overline{g(\tau,q_2)}|}{q_1q_2}\sum_{|\jmath|\ll_\varepsilon X^\varepsilon}\sum_{\substack{|s_1|\ll_\varepsilon pq_1/H^{1-\varepsilon}\\(s_1,p)=1\\ s_1\equiv\overline{q_2}\jmath \bmod q_1}}$$
$$\sum_{\substack{|s_2|\ll_\varepsilon pq_2/H^{1-\varepsilon}\\(s_2,p)=1\\ s_2\equiv\overline{q_1}\jmath \bmod q_2\\ q_2s_1+q_1s_2-\jmath\equiv0 \bmod p}}\sup_{\substack{R_1\ll p^{1+\varepsilon}\\R_2\ll p^{1+\varepsilon}}}\left|\sum_{R_1\ge1}\sum_{R_2\ge1}\Theta_{R_1,R_2}(m_1,m_2;s_1,s_2,\jmath,q_1,q_2)\right|.$$

Moreover, one might verify that Θ_{R_1,R_2} is a R_1 (resp. R_2)-dyadic weight function in the variable m_1 (resp. m_2), which enjoys the entirely analogous features as that for Υ_{Z_1,Z_2} in (3.34). Now, an application of Lemma 2.2 gives

$$\Psi_1^{\neq}(p,H,X)\ll X^{1+\varepsilon}H^{2+\varepsilon}p^{\frac12+\varepsilon}\sum_{\substack{1\le q_1,q_2\le\mathcal{Q}\\(q_1q_2,p)=1}}\frac{|g(\tau,q_1)\overline{g(\tau,q_2)}|}{q_1q_2}\left(1+\frac{p}{H^{1-\varepsilon}}\right)$$
$$\ll X^{1+\varepsilon}H^{2+\varepsilon}p^{\frac12+\varepsilon}\left(1+\frac{p}{H^{1-\varepsilon}}\right).$$

From this and (3.32) together with (3.36), it would be concluded that

$$\Psi_1(p,H,X)\ll(XpH)^\varepsilon\left(X^{\frac32}p^2+XH^2\sqrt{p}\right),\quad \Psi_2(p,L,X)\ll(XpL)^\varepsilon\left(X^{\frac32}p^2+XL^2\sqrt{p}\right),$$

upon recalling that $\max\{H,L\}\le\sqrt{Xp}$. Thus, we are allowed eventually to deduce

$$\Xi^{\text{Non-de.}}(p,H,L,X)\ll X^\varepsilon\left[X^{\frac32}p+\frac{XHL}{\sqrt{p}}+Xp\sqrt{HL}+X^{\frac54}p^{\frac14}(H+L)\right], \tag{3.39}$$

upon combining with (3.31).

3.4. **Treatments of $\Xi^{\mathbf{Cros1.}},\Xi^{\mathbf{Cros2.}}$.** At the end of the paper, let us devote ourselves to exploring the two cross terms $\Xi^{\mathrm{Cros1.}}$ and $\Xi^{\mathrm{Cros2.}}$, whereby to complete the proof of Theorem 1.1. It is remarkable that these two terms are not indispensable to contribute fairly large magnitudes to Ξ in (3.9). To illustrate this, upon recalling (3.13) and (3.14), one sees that actually $\Xi^{\mathrm{Cros1.}}$ is boiled down to evaluating

$$\frac{X}{(p\mathcal{Q})^2}\sup_{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon}\Bigg|\sum_{\substack{1\leq q_1,q_2\leq \mathcal{Q}\\(q_1q_2,p)=1}}\frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2}\sideset{}{^*}\sum_{\alpha \bmod pq_1}\sideset{}{^*}\sum_{\beta \bmod q_2}\sum_{m\geq1}\frac{\lambda_f(m)}{\sqrt{m}}$$
$$e\left(\frac{\alpha m}{q_1p}\right)\sum_{t\geq1}\frac{\lambda_f(t)}{\sqrt{t}}e\left(\frac{\beta t}{q_2}\right)\sum_{n\geq1}\lambda_f(n)e\left(-\frac{(\alpha q_2+\beta q_1 p)n}{q_1q_2p}\right)$$
$$\sum_{h\geq1}\sum_{l\geq1}e\left(-\frac{\alpha h}{q_1p}-\frac{\beta l}{q_2}\right)\mathcal{W}_{\tau_1,\tau_2}\left(\frac{m}{X},\frac{t}{X},\frac{n}{X},\frac{h}{H},\frac{l}{L}\right)\Bigg|.$$

After an application of the Voronoĭ formula, the quantity in the absolute value is converted into (essentially)

$$\sum_{\substack{1\leq q_1,q_2\leq \mathcal{Q}\\(q_1q_2,p)=1}}\frac{g(q_1,\tau_1)g(q_2,\tau_2)}{q_1q_2}\sum_{\substack{t\ll_\varepsilon X^\varepsilon\\ m\ll_\varepsilon p^{1+\varepsilon}}}\frac{\lambda_f(m)\lambda_f(t)}{\sqrt{mt}}\sum_{n\geq1}\lambda_f(n)\sum_{h\geq1}\sum_{l\geq1}S(-m,-(n+h);q_1p)$$
$$S(-\overline{p}t,-(n+l);q_2p)\,\widetilde{\mathcal{W}^{\pm}_{1,\tau_1,\tau_2}}\left(\frac{mX}{(q_1p)^2},\frac{n}{X},\frac{h}{H}\right)\widetilde{\mathcal{W}^{\pm}_{2,\tau_1,\tau_2}}\left(\frac{tX}{q^2p},\frac{n}{X},\frac{l}{L}\right). \tag{3.40}$$

At the moment, the preceding discussions in §3.2 can be adapted to show that

$$\Xi^{\mathrm{Cros1.}}(p,H,L,X)\ll\frac{1}{p}\sup_{\substack{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon\\ t\ll_\varepsilon X^\varepsilon}}\Psi_1^{\frac12}(p,H,X)\,\Omega_2^{\frac12}(p,t,L,X)$$
$$+\frac{1}{p}\sum_{1\leq q\leq\mathcal{Q}}\frac{\mathcal{B}_1^{\frac12}((qp)^2/X,H;qp)\mathcal{A}_2^{\frac12}(q^2p/X,L;q)}{q^2}.$$

Here, the first one on the right-hand side stems from the contribution of the generic terms $q_1\neq q_2$ in (3.40); while, the second one is related to the non-generic terms $q_1=q_2$. Analogously, one might find

$$\Xi^{\mathrm{Cros2.}}(p,H,L,X)\ll\frac{1}{p}\sup_{\substack{\tau_1,\tau_2\ll_\varepsilon X^\varepsilon\\ m\ll_\varepsilon X^\varepsilon}}\Omega_1^{\frac12}(p,m,H,X)\,\Psi_2^{\frac12}(p,L,X)$$
$$+\frac{1}{p}\sum_{1\leq q\leq\mathcal{Q}}\frac{\mathcal{A}_1^{\frac12}(q^2p/X,H;q)\mathcal{B}_2^{\frac12}((qp)^2/X,L;qp)}{q^2}.$$

Notice that Ω_1 (resp. Ω_2) is dominated by Ψ_1 (resp. Ψ_2). These two upper-bounds above are thus well controlled by the estimate in (3.39).

Now, upon recalling the decomposition at the beginning of this section, one collects the bounds (3.26) and (3.39), from which the desired estimate in (1.1) follows immediately, and hence Theorem 1.1.

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