ON NEW REFINEMENTS AND GENERALIZATIONS OF q-HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. The aim of this work is to prove generalized estimations for the q-Hermite-Hadamard inequality for convex functions using a parameter. By special choice of this parameter, we show that our results reduce the earlier obtained q-Hermite-Hadamard inequalities. We also present some more new refinements of these q-Hermite-Hadamard inequalities. Furthermore, we establish a new lemma and, by using this, we obtain some new quantum inequalities that generalize quantum midpoint and quantum trapezoid inequalities for convex functions.

1. Introduction

Recently, many studies have been conducted in quantum analysis. In 2018, one of these is the following q-Hermite-Hadamard inequality proved by Alp et al. [4]:

Theorem 1. *Let $\phi : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then, we have*

$$\phi\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a}\int_a^b \phi(x) \ {}_a d_q x \ \leq \frac{q\phi(a)+\phi(b)}{[2]_q}, \tag{1.1}$$

where $[2]_q = 1+q$.

Bermudo et al. [6] proved the following new version of q-Hermite-Hadamard inequality by using q^b-integral.

Theorem 2. *If $\phi : [a,b] \to \mathbb{R}$ is a convex differentiable function on $[a,b]$. Then, we have the following q-Hermite-Hadamard inequalities*

$$\phi\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a}\int_a^b \phi(x) \ {}^b d_q x \ \leq \frac{\phi(a)+q\phi(b)}{[2]_q}. \tag{1.2}$$

Remark 1. *It is very easy to observe that by adding (1.1) and (1.2), we have the following q-Hermite-Hadamard inequality (see, [6]):*

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_a^b \phi(x) \ {}_a d_q x \ + \int_a^b \phi(x) \ {}^b d_q x \ \right] \leq \frac{\phi(a)+\phi(b)}{2}. \tag{1.3}$$

Recently, Ali et al. [2] and Sitthiwirattham et al. [16] used new techniques to prove the following two different and new versions of Hermite-Hadamard type inequalities:

Theorem 3. [2, 16] *For a convex mapping $\phi : [a,b] \to \mathbb{R}$, the following inequalities hold:*

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\left[\int_a^{\frac{a+b}{2}} \phi(x) \ {}^{\frac{a+b}{2}} d_q x \ + \int_{\frac{a+b}{2}}^b \phi(x) \ {}_{\frac{a+b}{2}} d_q x \ \right] \leq \frac{\phi(a)+\phi(b)}{2}$$

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and

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\left[\int_a^{\frac{a+b}{2}} \phi(x) \ {}_a d_q x + \int_{\frac{a+b}{2}}^{b} \phi(x) \ {}^b d_q x\right] \leq \frac{\phi(a)+\phi(b)}{2}. \tag{1.4}$$

On the other side, many studies have recently been carried out in the field of q-analysis, starting with Euler due to high demand for mathematics that models quantum computing q-calculus appeared as a connection between mathematics and physics. It has several applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other sciences, quantum theory, mechanics, and the theory of relativity [9, 10, 11, 12]. Apparently, Euler was the founder of this branch of mathematics, by using the parameter q in Newton's work on infinite series. Later, Jackson was the first to develop q-calculus that knows without limit calculus in a systematic way [9]. In 1908-1909, Jackson defined the general q-integral and q-difference operator [11]. In 1969, Agarwal described the q-fractional derivative for the first time [1]. In 1966-1967 Al-Salam introduced a q-analogue of the Riemann-Liouville fractional integral operator and q-fractional integral operator [3]. In 2004, Rajkovic [15] gave a definition of the Riemann-type q-integral which generalized the Jackson q-integral. In 2013, Tariboon introduced ${}_aD_q$-difference operator [4]. In [13, 14], Noor et al. presented some trapezoid type inequalities for quantum integrals. Recently, in 2020, Bermudo et al. introduced the notion of ${}^bD_q$ derivative and integral [6].

Recently, Alp et al. [4, 5] obtained Wirtinger type q-integral inequalities and quantum estimations for q-midpoint type inequalities. For more recent results on q-integral inequalities, one can read [7, 8].

Inspired by the quantum analysis we mentioned above, in this paper we prove generalized and improved q-Hermite-Hadamard inequalities for convex functions using the quantum integrals. Also, we achieve combined q-midpoint and q-trapezoidal inequalities into a single inequality. Special cases of our results obtained yield all previous works.

2. Preliminaries of q-Calculus

In this section, we present some required definitions about q-calculus. Throughout the paper, we consider that $0<q<1$ and use the following notation (see, [12]):

$$[n]_q = \frac{q^n-1}{q-1}.$$

Definition 1. [17] *The left or q_a-derivative of $\phi:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:*

$$ {}_aD_q\phi(x) = \frac{\phi(x)-\phi(qx+(1-q)a)}{(1-q)(x-a)}, \ x\neq a. \tag{2.1}$$

Definition 2. [15] *The left or q_a-integral of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:*

$$\int_a^x \phi(t) \ {}_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n \phi(q^n x+(1-q^n)a) = (x-a)\int_0^1 \phi((1-t)a+tx) \ d_q t \ . \tag{2.2}$$

Definition 3. [6] *The right or q^b-derivative of $\phi:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:*

$$ {}^bD_q\phi(x) = \frac{\phi(qx+(1-q)b)-\phi(x)}{(1-q)(b-x)}, \ x\neq b.$$

Definition 4. [6] *The right or q^b-integral of $\phi:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:*

$$\int_x^b \phi(t) \ {}^b d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n \phi(q^n x+(1-q^n)b) = (b-x)\int_0^1 \phi(tb+(1-t)x) \ {}^1 d_q t \ .$$

Lemma 1. *We have the equality for q_a-integrals*

$$\int_a^b (x-a)^{\alpha} \ {}_a d_q x = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q}$$

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for $\alpha \in \mathbb{R}\backslash\{-1\}$.

3. Generalizations of q-Hermite-Hadamard inequalities

In this section, we prove the following generalized versions of q-Hermite-Hermite inequalities.

Theorem 4. *Let $\phi : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities hold:*

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \phi(t)\ {}_a d_q t + \int_{\lambda a+(1-\lambda)b}^b \phi(t)\ {}^b d_q t\right) \leq \frac{\phi(a)+\phi(b)}{2}. \tag{3.1}$$

Proof. Since ϕ is a convex function, we have

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x)+\phi(y)}{2}.$$

Let's consider $x = (1-\lambda t)a + \lambda tb$ and $y = t\lambda a + (1-\lambda t)b$ for $t \in [0,1]$ and $\lambda \in (0,1]$ Then, we obtain

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{\phi((1-\lambda t)a+\lambda tb)+\phi(t\lambda a+(1-\lambda t)b)}{2}. \tag{3.2}$$

By q-integrating the inequality (3.2) with respect to t over $[0,1]$, we obtain

$$\phi\left(\frac{a+b}{2}\right) \leq \int_0^1 \frac{\phi((1-\lambda t)a+\lambda tb)+\phi(t\lambda a+(1-\lambda t)b)}{2} d_q t. \tag{3.3}$$

By Definition 2 and Definition 4, we have the following equalities

$$\begin{aligned}
&\int_0^1 \frac{\phi((1-\lambda t)a+\lambda tb)+\phi(t\lambda a+(1-\lambda t)b)}{2} d_q t \\
= &\ (1-q)\sum_{n=0}^{\infty} q^n \frac{\phi((1-\lambda q^n)a+\lambda q^n b)+\phi(q^n\lambda a+(1-\lambda q^n)b)}{2} \\
= &\ \frac{(1-q)}{2}\sum_{n=0}^{\infty} q^n \phi([\lambda b+(1-\lambda)a]q^n+(1-q^n)a) \\
&+\frac{(1-q)}{2}\sum_{n=0}^{\infty} q^n \phi([\lambda a+(1-\lambda)b]q^n+(1-q^n)b) \\
= &\ \frac{1}{2\lambda(b-a)}\lambda(b-a)(1-q)\sum_{n=0}^{\infty} q^n \phi([\lambda b+(1-\lambda)a]q^n+(1-q^n)a) \\
&+\frac{1}{2\lambda(b-a)}\lambda(b-a)(1-q)\sum_{n=0}^{\infty} q^n \phi([\lambda a+(1-\lambda)b]q^n+(1-q^n)b) \\
= &\ \frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \phi(t)\ {}_a d_q t + \int_{\lambda a+(1-\lambda)b}^b \phi(t)\ {}^b d_q t\right).
\end{aligned} \tag{3.4}$$

This gives the first inequality of (3.1). On the other hand, by convexity of ϕ, we can write

$$\begin{aligned}
&\frac{\phi(\lambda ta+(1-\lambda t)b)+\phi((1-\lambda t)a+\lambda tb)}{2} \\
\leq &\ \frac{\lambda t\phi(a)+(1-\lambda t)\phi(b)+(1-\lambda t)\phi(a)+\lambda t\phi((b)}{2} \\
= &\ \frac{\phi(a)+\phi(b)}{2}.
\end{aligned} \tag{3.5}$$

By q-integrating (3.5) with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \frac{\phi((1-\lambda t)a+\lambda tb)+\phi(\lambda ta+(1-\lambda t)b)}{2} d_q t \\
\leq &\ \frac{\phi(a)+\phi(b)}{2}\int_0^1 d_q t = \frac{\phi(a)+\phi(b)}{2}.
\end{aligned} \tag{3.6}$$

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From (3.4) and (3.6), we obtain the second inequality of (3.1). This completes the proof. □

Remark 2. *If we choose $\lambda = \frac{1}{2}$ in Theorem 4, then the inequalities (3.1) reduce to the inequalities (1.4).*

Corollary 1. *If we take $\lambda = 1$ in Theorem 4, then the inequalities (3.1) reduce to the inequalities (1.3).*

Corollary 2. *If we assign $\lambda = \frac{1}{[2]_q}$ in Theorem 4, then we have the following new quantum Hermite-Hadamard inequalities*

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{[2]_q}{2(b-a)}\left(\int_a^{\frac{qa+b}{[2]_q}} \phi(t) \ {}_a d_q t \ + \int_{\frac{a+qb}{[2]_q}}^b \phi(t) \ {}^b d_q t\right) \leq \frac{\phi(a)+\phi(b)}{2}.$$

Theorem 5. *Let $\phi : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then we have the following inequalities:*

$$\begin{aligned} (3.7) \qquad & \phi\left(\frac{a+b}{2}\right) \\ \leq & \ \frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right) + \phi\left(\frac{qa+b}{[2]_q}\right)\right] \\ \leq & \ \frac{1+q^2}{2q\lambda[2]_q(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \phi(t) \ {}_a d_q t \ + \int_{\lambda a+(1-\lambda)b}^b \phi(t) \ {}^b d_q t\right) \\ & -\frac{(1-q)}{2q[2]_q}\left[\phi(\lambda a+(1-\lambda)b) + \phi((1-\lambda)a+\lambda b)\right] \\ \leq & \ \frac{\phi(a)+\phi(b)}{2}. \end{aligned}$$

Proof. By convexity of ϕ, we have

$$\begin{aligned} \phi\left(\frac{a+qb}{[2]_q}\right) &= \phi\left(\frac{((1-q\lambda t)a+q\lambda tb) + q(t\lambda a+(1-\lambda t)b)}{[2]_q}\right) \\ &\leq \frac{\phi((1-q\lambda t)a+q\lambda tb) + q\phi(t\lambda a+(1-\lambda t)b)}{[2]_q} \\ &\leq \frac{\phi(a)+q\phi(b)}{[2]_q}, \end{aligned}$$

which implies

$$\begin{aligned} (3.8) \qquad & \phi\left(\frac{a+qb}{[2]_q}\right) \\ \leq & \ \frac{\phi((1-q\lambda t)a+q\lambda tb) + q\phi(t\lambda a+(1-\lambda t)b)}{[2]_q} \\ \leq & \ \frac{\phi(a)+q\phi(b)}{[2]_q}. \end{aligned}$$

Then q-integrating (3.8) with respect to t over $[0,1]$, we obtain

$$(3.9) \qquad \phi\left(\frac{a+qb}{[2]_q}\right) \leq \int_0^1 \frac{\phi((1-q\lambda t)a+q\lambda tb) + q\phi(t\lambda a+(1-\lambda t)b)}{[2]_q} d_q t \leq \frac{\phi(a)+q\phi(b)}{[2]_q}.$$

The mid term of (3.9) can be calculated as follows:

$$(3.10) \qquad \int_0^1 \frac{\phi((1-q\lambda t)a+q\lambda tb) + q\phi(t\lambda a+(1-\lambda t)b)}{[2]_q} d_q t$$

$$\begin{aligned}
&= (1-q)\sum_{n=0}^{\infty} q^n \frac{\phi\left((1-\lambda q^{n+1})a+\lambda q^{n+1}b\right)+q\phi\left(q^n\lambda a+(1-\lambda q^n)b\right)}{[2]_q} \\
&= \frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty} q^n\phi\left((1-\lambda q^{n+1})a+\lambda q^{n+1}b\right)+\frac{q(1-q)}{[2]_q}\sum_{n=0}^{\infty} q^n\phi\left(q^n\lambda a+(1-\lambda q^n)b\right) \\
&= \frac{(1-q)}{q[2]_q}\sum_{n=1}^{\infty} q^n\phi\left((1-\lambda q^{n})a+\lambda q^{n}b\right)+\frac{q(1-q)}{[2]_q}\sum_{n=0}^{\infty} q^n\phi\left(q^n\lambda a+(1-\lambda q^n)b\right) \\
&= \frac{(1-q)}{q[2]_q}\sum_{n=0}^{\infty} q^n\phi\left((1-\lambda q^{n})a+\lambda q^{n}b\right)+\frac{q(1-q)}{[2]_q}\sum_{n=0}^{\infty} q^n\phi\left(q^n\lambda a+(1-\lambda q^n)b\right) \\
&\quad -\frac{(1-q)}{q[2]_q}\phi\left((1-\lambda)a+\lambda b\right) \\
&= \frac{1}{\lambda q[2]_q(b-a)}\lambda(b-a)(1-q)\sum_{n=0}^{\infty} q^n\phi\left([\lambda b+(1-\lambda)a]q^n+(1-q^n)a\right) \\
&\quad +\frac{q}{\lambda[2]_q(b-a)}\lambda(b-a)(1-q)\sum_{n=0}^{\infty} q^n\phi\left([\lambda a+(1-\lambda)b]q^n+(1-q^n)b\right) \\
&\quad -\frac{(1-q)}{q[2]_q}\phi\left((1-\lambda)a+\lambda b\right) \\
&= \frac{1}{\lambda[2]_q(b-a)}\left(\frac{1}{q}\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +q\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^bd_qt\right) \\
&\quad -\frac{(1-q)}{q[2]_q}\phi\left((1-\lambda)a+\lambda b\right).
\end{aligned}$$

So we can write

$$\begin{aligned}
&\phi\left(\frac{a+qb}{[2]_q}\right) \\
&\leq \frac{1}{\lambda[2]_q(b-a)}\left(\frac{1}{q}\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +q\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^bd_qt\right) \\
&\quad -\frac{(1-q)}{q[2]_q}\phi\left((1-\lambda)a+\lambda b\right) \\
&\leq \frac{\phi(a)+q\phi(b)}{[2]_q}.
\end{aligned} \tag{3.11}$$

On the other hand, by using similar method, we can write

$$\begin{aligned}
&\phi\left(\frac{qa+b}{[2]_q}\right) \\
&\leq \frac{q\phi\left((1-\lambda t)a+\lambda tb\right)+\phi\left(qt\lambda a+(1-q\lambda t)b\right)}{[2]_q} \\
&\leq \frac{q\phi(a)+\phi(b)}{[2]_q}.
\end{aligned} \tag{3.12}$$

By q-integrating (3.12) and calculating the mid term similar to (3.10), we obtain

$$\begin{aligned}
&\phi\left(\frac{qa+b}{[2]_q}\right) \\
&\leq \frac{1}{\lambda[2]_q(b-a)}\left(q\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +\frac{1}{q}\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^bd_qt\right)
\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
& -\frac{(1-q)}{q\,[2]_q}\phi\left(\lambda a+(1-\lambda)b\right) \\
\leq\ & \frac{q\phi(a)+\phi(b)}{[2]_q}.
\end{aligned}
$$

By adding (3.11) and (3.13), we obtain

$$
\begin{aligned}
& \phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right) \\
\leq\ & \frac{1+q^2}{q\lambda\,[2]_q\,(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ \ {}_ad_qt\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ \ {}^bd_qt\right) \\
& -\frac{(1-q)}{q\,[2]_q}\left[\phi\left(\lambda a+(1-\lambda)b\right)+\phi\left((1-\lambda)a+\lambda b\right)\right] \\
\leq\ & \phi(a)+\phi(b).
\end{aligned}
$$

which gives the second and third inequalities of (3.7). By convexity of ϕ, we can write

$$
\phi\left(\frac{a+b}{2}\right)=\phi\left(\frac{a+qb}{2\,[2]_q}+\frac{qa+b}{2\,[2]_q}\right)\leq\frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right].
$$

This proves the first inequality of (3.7). Thus the proof is completed. □

The following Corollary gives the refinements of the inequality (1.4).

Corollary 3. *If we choose $\lambda=\frac{1}{2}$ in Theorem 5, then we have the following refinement of Hermite-Hadamard inequality*

$$
\begin{aligned}
& \phi\left(\frac{a+b}{2}\right) \\
\leq\ & \Psi_1(q) \\
\leq\ & \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}}\phi(t)\ \ {}_ad_qt\ +\int_{\frac{a+b}{2}}^{b}\phi(t)\ \ {}^bd_qt\right) \\
\leq\ & \Psi_2(q) \\
\leq\ & \frac{\phi(a)+\phi(b)}{2},
\end{aligned}
\tag{3.14}
$$

where

$$
\Psi_1(q)=\frac{q\,[2]_q}{2\left(1+q^2\right)}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right]+\frac{(1-q)}{1+q^2}\phi\left(\frac{a+b}{2}\right)
$$

and

$$
\Psi_2(q)=\frac{q\,[2]_q}{1+q^2}\frac{\phi(a)+\phi(b)}{2}+\frac{(1-q)}{1+q^2}\phi\left(\frac{a+b}{2}\right).
$$

Proof. By $\lambda=\frac{1}{2}$ in (3.7), we have

$$
\begin{aligned}
& \phi\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right] \\
\leq\ & \frac{1+q^2}{q\,[2]_q\,(b-a)}\left(\int_a^{\frac{a+b}{2}}\phi(t)\ \ {}_ad_qt\ +\int_{\frac{a+b}{2}}^{b}\phi(t)\ \ {}^bd_qt\right) \\
& -\frac{(1-q)}{q\,[2]_q}\phi\left(\frac{a+b}{2}\right)
\end{aligned}
$$

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$$\leq \frac{\phi(a)+\phi(b)}{2}$$

which gives

$$\begin{aligned} &\frac{1+q^2}{q[2]_q}\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right]+\frac{(1-q)}{q[2]_q}\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1+q^2}{q[2]_q(b-a)}\left(\int_a^{\frac{a+b}{2}}\phi(t)\ {}_ad_qt+\int_{\frac{a+b}{2}}^b\phi(t)\ {}^bd_qt\right) \\ \leq\ & \frac{\phi(a)+\phi(b)}{2}+\frac{(1-q)}{q[2]_q}\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1+q^2}{q[2]_q}\frac{\phi(a)+\phi(b)}{2}. \end{aligned} \tag{3.15}$$

If we multiply the inequality (3.7) by $\frac{q[2]_q}{1+q^2}$, then we obtain the desired result. □

The following Corollary gives the refinements of the inequality (1.3).

Corollary 4. *If we assign $\lambda=1$ in Theorem 5, then we have the following refinement of Hermite-Hadamard inequality*

$$\begin{aligned} &\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \Psi_3(q) \\ \leq\ & \frac{1}{2(b-a)}\left(\int_a^b\phi(t)\ {}_ad_qt+\int_a^b\phi(t)\ {}^bd_qt\right) \\ \leq\ & \frac{\phi(a)+\phi(b)}{2}, \end{aligned}$$

where

$$\Psi_3(q)=\frac{q[2]_q}{2(1+q^2)}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right]+\frac{(1-q)}{1+q^2}[\phi(a)+\phi(b)].$$

Proof. If we take $\lambda=1$ in Theorem 5, then we have

$$\begin{aligned} &\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right] \\ \leq\ & \frac{1+q^2}{2q[2]_q(b-a)}\left(\int_a^b\phi(t)\ {}_ad_qt+\int_a^b\phi(t)\ {}^bd_qt\right) \\ & -\frac{(1-q)}{2q[2]_q}[\phi(a)+\phi(b)] \\ \leq\ & \frac{\phi(a)+\phi(b)}{2}. \end{aligned}$$

This gives, by $\phi\left(\frac{a+b}{2}\right)\leq\frac{\phi(a)+\phi(b)}{2}$,

$$\begin{aligned} &\frac{1+q^2}{q[2]_q}\phi\left(\frac{a+b}{2}\right) \\ \leq\ & \phi\left(\frac{a+b}{2}\right)+\frac{(1-q)}{2q[2]_q}[\phi(a)+\phi(b)] \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left[\phi\left(\frac{a+qb}{[2]_q}\right)+\phi\left(\frac{qa+b}{[2]_q}\right)\right]+\frac{(1-q)}{2q\,[2]_q}\left[\phi(a)+\phi(b)\right]\\
&\leq \frac{1+q^2}{2q\,[2]_q\,(b-a)}\left(\int_a^b \phi(t)\ {}_a d_q t\ +\int_a^b \phi(t)\ {}^b d_q t\right)\\
&\leq \frac{1+q^2}{2q\,[2]_q}\left[\phi(a)+\phi(b)\right].
\end{aligned}
$$

If we multiply the inequality (3.16) by $\frac{q[2]_q}{1+q^2}$, then we obtain the required result. □

4. Parameterized Quantum integral inequalities

Lemma 2. *Let $\phi:[a,b]\to\mathbb{R}$ be a q-differentiable function. If ${}_aD_q\phi$ and ${}^bD_q\phi$ are continuous and q-integrable over $[a,b]$, then the following new equality holds:*

$$
\begin{aligned}
&\frac{\lambda(b-a)}{2}\int_0^1 qt\left[\ {}^bD_q\phi\left((1-\lambda t)b+\lambda ta\right)\ -\ {}_aD_q\phi\left((1-\lambda t)a+\lambda tb\right)\ \right]d_qt\\
=\ &\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\right)\\
&-\frac{\phi\left(\lambda b+(1-\lambda)a\right)+\phi\left(\lambda a+(1-\lambda)b\right)}{2}.
\end{aligned}
$$

Proof. Assume that

$$
\begin{aligned}
&\frac{\lambda(b-a)}{2}\left(\int_0^1 qt\ {}^bD_q\phi\left((1-\lambda t)b+\lambda ta\right)\ d_qt-\int_0^1 qt\ {}_aD_q\phi\left((1-\lambda t)a+\lambda tb\right)\ d_qt\right) \qquad (4.1)\\
=\ &\frac{\lambda(b-a)}{2}\left(I_1-I_2\right).
\end{aligned}
$$

By Definition 3, we have

$$
\begin{aligned}
I_1 &= \int_0^1 qt\ {}^bD_q\phi\left((1-\lambda t)b+\lambda ta\right)\ d_qt \qquad (4.2)\\
&= \int_0^1 qt\frac{\phi\left((1-\lambda t)b+\lambda ta\right)-\phi\left(q\lambda ta+(1-q\lambda t)b\right)}{(1-q)(a-b)\lambda t}d_qt\\
&= \frac{q}{(a-b)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^n a+(1-\lambda q^n)b\right)-\frac{q}{(a-b)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^{n+1}a+(1-\lambda q^{n+1})b\right)\\
&= \frac{q}{(a-b)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^n a+(1-\lambda q^n)b\right)-\frac{1}{(a-b)\lambda}\sum_{n=1}^{\infty}q^n\phi\left(\lambda q^n a+(1-\lambda q^n)b\right)\\
&= \frac{1-q}{(b-a)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^n a+(1-\lambda q^n)b\right)-\frac{1}{(b-a)\lambda}\phi\left(\lambda a+(1-\lambda)b\right)\\
&= \frac{1}{(b-a)^2\lambda^2}\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ -\frac{1}{(b-a)\lambda}\phi\left(\lambda a+(1-\lambda)b\right).
\end{aligned}
$$

Similarly by Definition 1, we get

$$
\begin{aligned}
I_2 &= \int_0^1 qt\ {}_aD_q\phi\left((1-\lambda t)a+\lambda tb\right)\ d_qt\\
&= \int_0^1 qt\frac{\phi\left((1-\lambda t)a+\lambda tb\right)-\phi\left(q\lambda tb+(1-q\lambda t)a\right)}{(1-q)(b-a)\lambda t}d_qt\\
&= \frac{q}{(b-a)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^n b+(1-\lambda q^n)a\right)-\frac{q}{(b-a)\lambda}\sum_{n=0}^{\infty}q^n\phi\left(\lambda q^{n+1}b+(1-\lambda q^{n+1})a\right)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{q}{(b-a)\lambda}\sum_{n=0}^{\infty} q^n\phi\left(\lambda q^n b+(1-\lambda q^n)a\right) \\
&\quad -\frac{1}{(b-a)\lambda}\sum_{n=0}^{\infty} q^n\phi\left(\lambda q^n b+(1-\lambda q^n)a\right)+\frac{1}{(b-a)\lambda}\phi\left(\lambda b+(1-\lambda)a\right) \\
&= \frac{1}{(b-a)\lambda}\phi\left(\lambda b+(1-\lambda)a\right)-\frac{1-q}{(b-a)\lambda}\sum_{n=0}^{\infty} q^n\phi\left(\lambda q^n b+(1-\lambda q^n)a\right) \\
&= \frac{1}{(b-a)\lambda}\phi\left(\lambda b+(1-\lambda)a\right)-\frac{1}{(b-a)^2\lambda^2}\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ .
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\frac{\lambda(b-a)}{2}\left(I_1-I_2\right) \\
&= \frac{1}{2(b-a)\lambda}\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ -\frac{1}{2}\phi\left(\lambda a+(1-\lambda)b\right) \\
&\quad -\frac{1}{2}\phi\left(\lambda b+(1-\lambda)a\right)+\frac{1}{2(b-a)\lambda}\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t \\
&= \frac{1}{2\lambda(b-a)}\left(\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ +\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ \right) \\
&\quad -\frac{\phi\left(\lambda b+(1-\lambda)a\right)+\phi\left(\lambda a+(1-\lambda)b\right)}{2}.
\end{aligned}$$

Which completes the proof. □

Theorem 6. *We assume that the conditions of Lemma 2 hold. If the* $\left|\ {}_aD_q\phi\ \right|$ *and* $\left|\ {}^bD_q\phi\ \right|$ *are convex* $[a,b]$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ \right)\right. \\
&\quad \left.-\frac{\phi\left(\lambda b+(1-\lambda)a\right)+\phi\left(\lambda a+(1-\lambda)b\right)}{2}\right| \\
&\leq \frac{\lambda q(b-a)}{2[2]_q[3]_q}\left[\left([3]_q-\lambda[2]_q\right)\left[\left|\ {}^bD_q\phi(b)\ \right|+\left|\ {}_aD_q\phi(a)\ \right|\right]\right. \\
&\quad \left.+\lambda[2]_q\left[\left|\ {}^bD_q\phi(a)\ \right|+\left|\ {}_aD_q\phi(b)\ \right|\right]\right].
\end{aligned}$$

Proof. With the help of Lemma 2, we get

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ \right)\right. \\
&\quad \left.-\frac{\phi\left(\lambda b+(1-\lambda)a\right)+\phi\left(\lambda a+(1-\lambda)b\right)}{2}\right| \\
&\leq \frac{\lambda(b-a)}{2}\int_0^1 qt\left|\ {}^bD_q\phi\left((1-\lambda t)b+\lambda ta\right)\ \right|d_qt+\frac{\lambda(b-a)}{2}\int_0^1 qt\left|\ {}_aD_q\phi\left((1-\lambda t)a+\lambda tb\right)\ \right|d_qt.
\end{aligned}\tag{4.3}$$

By using the convexity of $\left|\ {}_aD_q\phi\ \right|$ and $\left|\ {}^bD_q\phi\ \right|$, we have

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_a d_q t\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ {}^b d_q t\ \right)\right. \\
&\quad \left.-\frac{\phi\left(\lambda b+(1-\lambda)a\right)+\phi\left(\lambda a+(1-\lambda)b\right)}{2}\right|
\end{aligned}$$

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$$\begin{aligned}
&\leq \frac{\lambda(b-a)}{2}\int_0^1 qt\left((1-\lambda t)\left|\ {}^{b}D_q\phi(b)\ \right|+\lambda t\left|\ {}^{b}D_q\phi(a)\ \right|\right)d_qt \\
&+\frac{\lambda(b-a)}{2}\int_0^1 qt\left((1-\lambda t)\left|\ {}_{a}D_q\phi(a)\ \right|+\lambda t\left|\ {}_{a}D_q\phi(b)\ \right|\right)d_qt \\
&= \frac{\lambda(b-a)}{2}\left[\left[\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right]\int_0^1 (qt-q\lambda t^2)d_qt\right. \\
&\left.+\left[\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right]\int_0^1 q\lambda t^2 d_qt\right] \\
&= \frac{\lambda q(b-a)}{2}\left[\left(\frac{1}{[2]_q}-\frac{\lambda}{[3]_q}\right)\left[\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right]\right. \\
&\left.+\frac{\lambda}{[3]_q}\left[\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right]\right] \\
&= \frac{\lambda q(b-a)}{2[2]_q[3]_q}\left[\left([3]_q-\lambda[2]_q\right)\left[\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right]\right. \\
&\left.+\lambda[2]_q\left[\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right]\right].
\end{aligned}$$

Thus, the proof is completed. □

Remark 3. *If we assign $\lambda=1$ in Theorem 6, then we have the following trapezoid type inequality*

$$\begin{aligned}
&\left|\frac{1}{2(b-a)}\left(\int_a^b \phi(t)\ {}_{a}d_qt+\int_a^b \phi(t)\ {}^{b}d_qt\right)-\frac{\phi(a)+\phi(b)}{2}\right| \\
&\leq \frac{q(b-a)}{2[2]_q[3]_q}\left[q^2\left[\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right]\right. \\
&\left.+[2]_q\left[\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right]\right].
\end{aligned}$$

Remark 4. *If we assign $\lambda=\frac{1}{2}$ in Theorem 6, then we have the following Midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} \phi(t)\ {}_{a}d_qt+\int_{\frac{a+b}{2}}^b \phi(t)\ {}^{b}d_qt\right)-\phi\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{q(b-a)}{8[2]_q[3]_q}\left[\left([3]_q+q^2\right)\left[\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right]\right. \\
&\left.+[2]_q\left[\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right]\right]
\end{aligned}$$

which is proved by Ali et al. in [2].

Corollary 5. *In Theorem 6, assume $\lambda=\frac{1}{[2]_q}$, then, we reach as follows*

$$\begin{aligned}
&\left|\frac{[2]_q}{(b-a)}\left(\int_a^{\frac{qa+b}{[2]_q}} \phi(t)\ {}_{a}d_qt+\int_{\frac{a+qb}{[2]_q}}^b \phi(t)\ {}^{b}d_qt\right)-\left(\phi\left(\frac{qa+b}{[2]_q}\right)+\phi\left(\frac{a+qb}{[2]_q}\right)\right)\right| \\
&\leq \frac{q(b-a)}{[2]_q^2[3]_q}\left\{q[2]_q\left(\left|\ {}^{b}D_q\phi(b)\ \right|+\left|\ {}_{a}D_q\phi(a)\ \right|\right)+\left(\left|\ {}^{b}D_q\phi(a)\ \right|+\left|\ {}_{a}D_q\phi(b)\ \right|\right)\right\}.
\end{aligned}$$

Theorem 7. *We assume that the conditions Lemma 2 hold. If the functions $\left|{}_{a}D_q\phi\right|^s$ and $\left|{}^{b}D_q\phi\right|^s$, $s>1$ are convex, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \phi(t)\ {}_{a}d_qt+\int_{\lambda a+(1-\lambda)b}^b \phi(t)\ {}^{b}d_qt\right)\right. \\
&\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|
\end{aligned}$$

$$\leq \frac{\lambda q(b-a)}{2}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{[2]_q-\lambda}{[2]_q}\left|\ {}^{b}D_q\phi(b)\ \right|^s+\frac{\lambda}{[2]_q}\left|\ {}^{b}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{[2]_q-\lambda}{[2]_q}\left|\ {}_{a}D_q\phi(a)\ \right|^s+\frac{\lambda}{[2]_q}\left|\ {}_{a}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right],$$

where $s^{-1}+r^{-1}=1$.

Proof. By applying Hölder's inequality in (4.3), we have

$$\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ \ {}_{a}d_qt\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ \ {}^{b}d_qt\ \right)\right.$$
$$\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|$$
$$\leq \frac{\lambda(b-a)}{2}\left[\left(\int_0^1(qt)^r\,d_qt\right)^{\frac{1}{r}}\left(\int_0^1\left|\ {}^{b}D_q\phi((1-\lambda t)b+\lambda ta)\ \right|^s d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1(qt)^r\,d_qt\right)^{\frac{1}{r}}\left(\int_0^1\left|\ {}_{a}D_q\phi((1-\lambda t)a+\lambda tb)\ \right|^s d_qt\right)^{\frac{1}{s}}\right].$$

Since the functions $\left|{}_{a}D_q\phi\right|^s$ and $\left|{}^{b}D_q\phi\right|^s$ are convex, we have

$$\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ \ {}_{a}d_qt\ +\int_{\lambda a+(1-\lambda)b}^{b}\phi(t)\ \ {}^{b}d_qt\ \right)\right.$$
$$\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|$$
$$\leq \frac{\lambda(b-a)}{2}\left(\int_0^1(qt)^r\,d_qt\right)^{\frac{1}{r}}$$
$$\times\left[\left(\int_0^1\left[(1-\lambda t)\left|\ {}^{b}D_q\phi(b)\ \right|^s+\lambda t\left|\ {}^{b}D_q\phi(a)\ \right|^s\right]d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left[(1-\lambda t)\left|\ {}_{a}D_q\phi(a)\ \right|^s+\lambda t\left|\ {}_{a}D_q\phi(a)\ \right|^s\right]d_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{\lambda q(b-a)}{2}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{[2]_q-\lambda}{[2]_q}\left|\ {}^{b}D_q\phi(b)\ \right|^s+\frac{\lambda}{[2]_q}\left|\ {}^{b}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{[2]_q-\lambda}{[2]_q}\left|\ {}_{a}D_q\phi(a)\ \right|^s+\frac{\lambda}{[2]_q}\left|\ {}_{a}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

Remark 5. *If we assign* $\lambda=1$ *in Theorem 7, then we have the following trapezoid type inequality*

$$\left|\frac{1}{2(b-a)}\left(\int_a^b\phi(t)\ \ {}_{a}d_qt\ +\int_a^b\phi(t)\ \ {}^{b}d_qt\ \right)-\frac{\phi(a)+\phi(b)}{2}\right|$$
$$\leq \frac{q(b-a)}{2}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q\left|\ {}^{b}D_q\phi(b)\ \right|^s+\left|\ {}^{b}D_q\phi(a)\ \right|^s}{[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{q\left|\ {}_{a}D_q\phi(a)\ \right|^s+\left|\ {}_{a}D_q\phi(a)\ \right|^s}{[2]_q}\right)^{\frac{1}{s}}\right].$$

Remark 6. *If we assign $\lambda=\frac{1}{2}$ in Theorem 7, then we have the following Midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}}\phi(t)\ {}_ad_qt\ +\int_{\frac{a+b}{2}}^b\phi(t)\ {}^bd_qt\ \right)-\phi\left(\frac{a+b}{2}\right)\right|\\
\le\ & \frac{q(b-a)}{4}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{\left([2]_q+q\right)\left|\ {}^bD_q\phi(b)\ \right|^s+\left|\ {}^bD_q\phi(a)\ \right|^s}{[2]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left([2]_q+q\right)\left|\ {}_aD_q\phi(a)\ \right|^s+\left|\ {}_aD_q\phi(a)\ \right|^s}{[2]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}$$

which is proved by Ali et al. in [2].

Theorem 8. *We assume that the conditions Lemma 2 hold. If the functions $|{}_aD_q\phi|^s$ and $\left|{}^bD_q\phi\right|^s$, $s\ge1$ are convex, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +\int_{\lambda a+(1-\lambda)b}^b\phi(t)\ {}^bd_qt\ \right)\right.\\
&\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|\\
\le\ & \frac{\lambda q(b-a)}{2[2]_q}\left[\left(\frac{\left([3]_q-\lambda[2]_q\right)\left|\ {}^bD_q\phi(b)\ \right|^s+\lambda[2]_q\left|\ {}^bD_q\phi(a)\ \right|^s}{[3]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left([3]_q-\lambda[2]_q\right)\left|\ {}_aD_q\phi(a)\ \right|^s+\lambda[2]_q\left|\ {}_aD_q\phi(a)\ \right|^s}{[3]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Proof. By using the power mean inequality in (4.3), we have

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +\int_{\lambda a+(1-\lambda)b}^b\phi(t)\ {}^bd_qt\ \right)\right.\\
&\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|\\
\le\ & \frac{\lambda(b-a)}{2}\left[\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{s}}\left(\int_0^1 qt\left|\ {}^bD_q\phi((1-\lambda t)b+\lambda ta)\ \right|^sd_qt\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{s}}\left(\int_0^1 qt\left|\ {}_aD_q\phi((1-\lambda t)a+\lambda tb)\ \right|^sd_qt\right)^{\frac{1}{s}}\right].
\end{aligned}$$

By using the convexity of the functions $|{}_aD_q\phi|^s$ and $\left|{}^bD_q\phi\right|^s$, we have

$$\begin{aligned}
&\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\phi(t)\ {}_ad_qt\ +\int_{\lambda a+(1-\lambda)b}^b\phi(t)\ {}^bd_qt\ \right)\right.\\
&\left.-\frac{\phi(\lambda b+(1-\lambda)a)+\phi(\lambda a+(1-\lambda)b)}{2}\right|\\
\le\ & \frac{\lambda(b-a)}{2}\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{s}}\left[\left(\int_0^1 qt\left[(1-\lambda t)\left|\ {}^bD_q\phi(b)\ \right|^s+\lambda t\left|\ {}^bD_q\phi(a)\ \right|^s\right]d_qt\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\int_0^1 qt\left[(1-\lambda t)\left|\ {}_aD_q\phi(a)\ \right|^s+\lambda t\left|\ {}_aD_q\phi(a)\ \right|^s\right]d_qt\right)^{\frac{1}{s}}\right]
\end{aligned}$$

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$$= \frac{\lambda(b-a)}{2}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\left(\frac{q}{[2]_q}-\frac{q\lambda}{[3]_q}\right)\left|\ {}^{b}D_q\phi(b)\ \right|^s+\frac{q\lambda}{[3]_q}\left|\ {}^{b}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\left(\frac{q}{[2]_q}-\frac{q\lambda}{[3]_q}\right)\left|\ {}_{a}D_q\phi(a)\ \right|^s+\frac{q\lambda}{[3]_q}\left|\ {}_{a}D_q\phi(a)\ \right|^s\right)^{\frac{1}{s}}\right].$$

Thus, the proof is completed. □

Remark 7. *If we assign $\lambda=1$ in Theorem 8, then we have the following trapezoid type inequality*

$$\left|\frac{1}{2(b-a)}\left(\int_a^b\phi(t)\ \ {}_{a}d_qt\ +\int_a^b\phi(t)\ \ {}^{b}d_qt\ \right)-\frac{\phi(a)+\phi(b)}{2}\right|$$

$$\leq \frac{q(b-a)}{2[2]_q}\left[\left(\frac{q^2\left|\ {}^{b}D_q\phi(b)\ \right|^s+[2]_q\left|\ {}^{b}D_q\phi(a)\ \right|^s}{[3]_q}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\frac{q^2\left|\ {}_{a}D_q\phi(a)\ \right|^s+[2]_q\left|\ {}_{a}D_q\phi(a)\ \right|^s}{[3]_q}\right)^{\frac{1}{s}}\right].$$

Remark 8. *If we assign $\lambda=\frac{1}{2}$ in Theorem 8, then we have the following Midpoint type inequality*

$$\left|\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}}\phi(t)\ \ {}_{a}d_qt\ +\int_{\frac{a+b}{2}}^b\phi(t)\ \ {}^{b}d_qt\ \right)-\phi\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{q(b-a)}{4[2]_q}\left[\left(\frac{\left([3]_q+q^2\right)\left|\ {}^{b}D_q\phi(b)\ \right|^s+[2]_q\left|\ {}^{b}D_q\phi(a)\ \right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\frac{\left([3]_q+q^2\right)\left|\ {}_{a}D_q\phi(a)\ \right|^s+[2]_q\left|\ {}_{a}D_q\phi(a)\ \right|^s}{[3]_q}\right)^{\frac{1}{s}}\right],$$

which is proved by Ali et al. in [2].

5. Conclusion

In this research, we prove generalized quantum estimations for q-Hermite-Hadamard inequality for convex functions involving two kinds of quantum integrals. Special cases of our results obtained yield all previous works. It is an interesting and new problem that the upcoming researchers may use the techniques of this research and prove q-fractional inequalities and similar inequalities or similar our results can be obtained for different kinds of convexities.

Availability of data and material

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Competing Interests

The authors declare that they have no competing interests.

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Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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