Finite-Time Stabilization by a continuous Feedback of a Class of Nonlinear Systems

Abdessalem Bouzidi Thouraya Kharrat

University of Sfax
Faculty of Sciences of Sfax-Tunisia
Soukra Road, km 3.5, 3000 Sfax - Tunisia

Email adresses: abdessalembouzidi1@gmail.com, thouraya.kharrat@fss.rnu.tn

Abstract: This paper investigates the problem of finite-time stabilization of a class of nonlinear systems with dilations. The key of this work is the development of a recursive design procedure to design a state feedback which stabilizes the nonlinear system in finite time. The proof is based on the finite-time Lyapunov stability theorem.

AMS Classification: 93D15, 93D40.

Keywords: Finite time stability, nonlinear systems, stabilization, feedback.

1 Introduction

The problem of finite-time stability of nonlinear systems has been studied by many researchers, see for instance [4], [13], [17]. The key of the previous works is the finite-time Lyapunov stability theorem. In [7], the authors have constructed a finite-time stabilizing state feedback controller for a class of nonlinear systems. They developed systematic algorithms to design continuous state feedback and a control Lyapunov function, to achieve global finite-time stabilization for the considered systems. In [17], the authors consider the problem of finite time stabilization of a class of nonlinear systems in the p-normal form.

9 Nov 2022 23:16:35 PST
220703-Kharrat Version 3 - Submitted to Rocky Mountain J. Math.

In this paper, we consider a family of nonlinear systems described by:

$$\begin{cases} \dot{x}_1 = x_2^{r_2} + f_1(x,u,t) \\ \dot{x}_2 = x_3^{r_3} + f_2(x,u,t) \\ \vdots \\ \dot{x}_{n-1} = x_n^{r_n} + f_{n-1}(x,u,t) \\ \dot{x}_n = u + f_n(x,u,t) \end{cases} \tag{1}$$

where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and the control input respectively, $f_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuously differentiable function, for $i = 1, 2, ..., n$ with $f_i(0,0,t) = 0$, $\forall t \geq 0$, and $\{r_i,\ 1 \leq i \leq n\}$ is a family of positive reals satisfying, for any $1 \leq i \leq n$, $0 < r_{i+1} < r_i < 1$ and $r_i = \frac{p_i}{m_i}$, where p_i and m_i are odd integers. In ([1],[3]), the authors consider a family of nonlinear systems in the form (1) and study the problem of asymptotic stabilization in the case where $r_1 = r_2 = ... = r_n = p > 1$ with p is an odd integer. In [10], the authors study the case where $(r_1 = r_2 = ... = r_n = r)$ with $0 < r < 1$ and $r = \frac{p}{q}$, $(p < q)$ with p, q are odd integers. The novelty in this work is to consider a family of positive reals r_i that are not necessary equal.
The main contribution of this paper is the development of a recursive design method that yields a feedback controller stabilizing in finite-time the nonlinear system (1). The state feedback stabilizer is constructed using a recursive design method, inspired from the adding of a power integrator technique [10], [11].
The paper is organized as follows. After recalling the notion of finite-time stability and stabilization, we give sufficient conditions for stabilization of nonlinear systems (1) in finite time. The idea of the proof is to use a Lyapunov function and a recursive design method that yields a state feedback stabilizer. Then, a numerical example is given to illustrate the main result.

2 Preliminary results

We consider the non-autonomous system:

$$\dot{x}(t) = f(t,x),\ t \geq 0,\ x \in \mathbb{R}^n, \tag{2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We denote by $x(t, t_0, x_0)$ the solution of the system (2) starting from $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 2.1 *[13](finite time stability)*
The origin of the system (2) is finite-time stable if:
a) it is stable;
b) for all $t_0 \geq 0$, there exists $\eta(t_0) > 0$, such that if $\|x_0\| < \eta(t_0)$, then

i) $x(t, t_0, x_0)$ is defined for $t \geq t_0$,

ii) there exists $0 \leq T(t_0, x_0) < +\infty$ such that $x(t, t_0, x_0) = 0$ for all $t \geq t_0 + T(t_0, x_0)$.

Denote

$$T_0(x_0) = \inf\{T(t_0, x_0) \geq 0 : \ x(t, t_0, x_0) = 0, \ \forall t \geq t_0 + T(t_0, x_0)\},$$

$T_0(x_0)$ is called the settling time of the solution $x(t, t_0, x_0)$.

Definition 2.2 *[12](Stabilization in finite time)*
The system (2) is finite-time stabilizable if there exists a continuous feedback function u such that:

a) $u(0) = 0$,

b) the origin of the closed loop system $\dot{x} = f(t, x, u(x))$ is finite-time stable.

Definition 2.3 *[12](uniform finite time stability)*
The origin of the system(2) is uniformly finite-time stable if :
a) it is uniformly asymptotically stable,
b) it is finite-time stable,
c) there exists a positive definite continuous function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the settling time of the system (2): $T_0(t, x) \leq \alpha(\| x \|), \ \forall x \in \mathbb{R}^n$.

Theorem 2.4 *[13]**Lyapunov Theory for Finite-Time Stability***
Consider the system

$$\dot{x} = f(x) \tag{3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous non-Lipschitz and $f(0) = 0$.
Suppose that there exists a continuously differentiable function $V(x)$ defined on a neighborhood $\overline{U} \subset \mathbb{R}^n$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$ satisfying

1) *$V(x)$ is positive definite on $\overline{U}$;*

2) *$\dot{V}(x) + cV^{\alpha}(x) \leq 0$, $\forall x \in \overline{U}$;*

then the origin of the system (3) is locally finite-time stable.
The settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T_x(x_0) \leq \frac{V^{1-\alpha}(x_0)}{c(1-\alpha)}$$

for all x_0 in some open neighborhood of the origin.
If $\overline{U} = \mathbb{R}^n$ and $V(x)$ is also radially unbounded, then the origin is globally finite-time stable.

Notation: For $r > 0$, denote $\mathbf{B}_F(0, r) = \{x \in \mathbb{R}^n, ||x|| \leq r\}$.

3 Finite-time stabilization of nonlinear systems

The goal of this paper is the design of a stabilizing feedback of the system (1). For this, we introduce the following assumption.

Assumption 3.1
For $i = 1, 2, ..., n$, there exists bounded continuously differentiable function $\gamma_i : \mathbb{R}^i \to \mathbb{R}^+$ for all $(x_1, x_2, ..., x_i) \in \mathbb{R}^i$ and $\forall (x, u, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$ one has

$$| f_i(x, u, t) | \leq (| x_1 |^{r_1} + | x_2 |^{r_2} + ... + | x_i |^{r_i}) \gamma_i(x_1, x_2, ..., x_i). \tag{4}$$

Remark
We assumed that $\gamma_i(.)$ is a positive bounded function, then $\gamma_i(x_1, ..., x_i) \leq \bar{a}$, for all $(x_1, x_2, ..., x_i) \in \mathbb{R}^i$, $i \in \{1, ..., n\}$, where $\bar{a}$ is a positive real number.

Theorem 3.2
If the assumption (3.1) is satisfied, then the nonlinear system (1) is locally finite-time stabilizable by a continuous feedback.

For the proof of the main result, we need the following steps.

First step:
In the first step, we prove the following result:

Proposition 3.3 .
Let $V_1(x) = \frac{1}{2}x_1^2$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and the virtual control ${x^*}_2^{r_2} = -(2\bar{a})x_1^{r_1}$. Then V_1 is non negative and the derivative of V_1 along the trajectories of the system (1) satisfies :

$$\dot{V}_1(x) \leq -\bar{a} | \xi_1 |^{1+r_1} + \xi_1(x_2^{r_2} - {x^*}_2^{r_2}) \tag{5}$$

where $\xi_1 = x_1 - x_1^*$ and $x_1^* = 0$.

Proof.
Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, it is clear that $V_1(x) = \dfrac{1}{2}x_1^2 > 0$ for $x_1 \neq 0$.
The derivative of V_1 along the trajectories of the system (1) is

$$\begin{array}{rcl} \dot{V}_1(x) & = & x_1 \dot{x}_1 \\ & = & x_1 {x_2}^{r_2} + x_1 f_1(u, x, t) \\ & \leq & x_1 x_2^{r_2} + \bar{a} \mid x_1 \mid^{1+r_1} \\ & \leq & x_1(x_2^{r_2} - {x^*}_2^{r_2}) + x_1 x_2^{*r_2} + \bar{a} \mid x_1 \mid^{1+r_1} \end{array}$$

If we introduce the virtual control as :

$$x_2^{*r_2} = -(2\bar{a})x_1^{r_1},$$

By the hypothesis that p_1 and m_1 are odd integers, we get:

$$x_1 x_2^{*r_2} = -(2\bar{a})x_1^{1+r_1} = -(2\bar{a})x_1^{\overline{\frac{m_1+p_1}{m_1}}} = -(2\bar{a})|x_1|^{\overline{\frac{m_1+p_1}{m_1}}}$$

Therefore,

$$\dot{V}_1(x) \leq -\bar{a} \mid \xi_1 \mid^{1+r_1} + \xi_1(x_2^{r_2} - x_2^{*r_2})$$

where $\xi_1 = x_1 - x_1^*$, $x_1^* = 0$, and $V_1(x) = \frac{1}{2}x_1^2 \leq x_1^2 \leq \xi_1^2$.
□

Inductive step:
Let $1 \leq k \leq n-1$, we assume at step $k-1$,
i) there exist a set of parameters $q_i \in]0, 1[$ and continuous virtual controls $x_1^*, ..., x_k^*$, defined by:

$$\left\{ \begin{array}{l} x_1^* = 0 \\ x_2^* = -q_2 \xi_1^{\overline{\frac{r_1}{r_2}}} \\ \vdots \\ x_k^* = -q_k \xi_{k-1}^{\overline{\frac{r_{k-1}}{r_k}}} \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{l} \xi_1 = x_1 - x_1^* \\ \\ \vdots \\ \\ \xi_k = x_k - x_k^* = x_k + q_k \xi_{k-1}^{\overline{\frac{r_{k-1}}{r_k}}} \end{array} \right.$$

ii) there exists a positive and proper continuously differentiable function $V_{k-1}(x)$, which satisfies:

$$\begin{array}{l} V_{k-1}(x) = \alpha_{k-1}(x_1, ..., x_{k-1}) \\ V_{k-1}(x) > 0, \ \text{for}(x_1, ..., x_{k-1}) \neq 0 \\ V_{k-1}(x) \leq (\xi_1^2 + ... + \xi_{k-1}^2) \end{array} \tag{6}$$

and

$$\dot{V}_{k-1}(x) \leq -(\bar{a} + n - k) \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \xi_{k-1}(x_k^{r_k} - x_k^{*r_k}) \tag{7}$$

We have the following result.

Proposition 3.4
We consider

$$V_k(x) = \alpha_k(x_1, ..., x_k) = V_{k-1}(x) + W_k(x_1, ..., x_k) \tag{8}$$

where $W_k(x_1, ..., x_k) = \displaystyle\int_{x_k^*}^{x_k} (s - x_k^*)ds$. *Then the derivative of* V_k *along the trajectories of the system (1) satisfies:*

$$\dot{V}_k(x) \leq -(2\bar{a} + n - k) \sum_{i=1}^{k} \mid \xi_i \mid^{1+r_i} + \xi_k (x_{k+1}^{r_{k+1}} - {x_{k+1}^*}^{r_{k+1}}) \tag{9}$$

Before the proof of the previous result, we need the following lemmas:

Lemma 3.5 *([7],[11])*
For $x, y \in \mathbb{R}$, *and* $0 < b \leq 1$, *we have the inequality*

$$(\mid x \mid + \mid y \mid)^b \leq \mid x \mid^b + \mid y \mid^b \tag{10}$$

Therefore, for all reals x_i, $i = 1, 2, ..., n$, *one has*

$$(\mid x_1 \mid + \mid x_2 \mid + ... + \mid x_n \mid)^b \leq \mid x_1 \mid^b + \mid x_2 \mid^b + ... + \mid x_n \mid^b \tag{11}$$

where $b = \frac{p}{q} \leq 1$, *with* $p, q > 0$ *are odd integers. Moreover, we have:*

$$\mid x^b + y^b \mid \leq 2^{1-b} \mid x + y \mid^b \tag{12}$$

Lemma 3.6 *([7],[11])*
Let s, t *be positive real numbers and* a, b, d *continuous and positive real functions, then for* $c > 0$ *we have :*

$$\mid a \mid^s \mid b \mid^t d \leq c. \mid a \mid^{s+t} + \frac{t}{s+t} \left[\frac{s}{c(s+t)} \right]^{\frac{s}{t}} \mid b \mid^{s+t} d^{\frac{s+t}{t}} \tag{13}$$

Lemma 3.7 *([19])*
For $a, b, c \in \mathbb{R}$, *such that* $0 < a \leq b \leq c$, *then we have the inequality:*

$$\mid x \mid^b \leq \mid x \mid^a + \mid x \mid^c, \ \forall x \in \mathbb{R}$$

Lemma 3.8
For $k = 2, ..., n$, we have

$$\mid x_k \mid^{r_k} \leq \mid \xi_k \mid^{r_k} + \mid \xi_{k-1} \mid^{r_{k-1}} \tag{14}$$

Proof.
We have $\xi_k = x_k - x_k^*$, then

$$\begin{aligned} |x_k|^{r_k} &= |\xi_k + x_k^*|^{r_k} \\ &= |\xi_k - q_k \xi_{k-1}^{\frac{r_{k-1}}{r_k}}|^{r_k} \\ &\leq |\xi_k|^{r_k} + |\xi_{k-1}|^{r_{k-1}} \end{aligned}$$

□

Proof. [**Proposition**]
Let $x \in \mathbb{R}^n$, we have

$$V_k(x) = V_{k-1}(x) + W_k(x_1, ..., x_k) \tag{15}$$

and

$$\frac{\partial W_k}{\partial x_k}(x) = (x_k - x_k^*) = \xi_k$$

It is clear that $W_k(x_1, ..., x_k) > 0$, for $(x_1, ..., x_k) \neq 0$. Consequently $V_k(x) > 0$ for $(x_1, ..., x_k) \neq 0$.
The derivative of V_k along the trajectories of the system (1) gives

$$\dot{V}_k(x) = \dot{V}_{k-1}(x) + \sum_{l=1}^{k} \frac{\partial W_k}{\partial x_l} \dot{x}_l$$

We get

$$\dot{V}_k(x) \leq -(\bar{a} + n - k) \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \xi_{k-1}(x_k^{r_k} - {x^*}_k^{r_k}) + \frac{\partial W_k}{\partial x_k} \dot{x}_k + \sum_{l=1}^{k-1} \frac{\partial W_k}{x_l} \dot{x}_l,$$

then

$$\dot{V}_k(x) \leq -(\bar{a}+n-k) \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \xi_{k-1}(x_k^{r_k} - {x^*}_k^{r_k}) + \xi_k(x_{k+1}^{r_{k+1}} + f_k(x,u,t)) + \sum_{l=1}^{k-1} \frac{\partial W_k}{x_l} \dot{x}_l,$$

so

$$\begin{aligned} \dot{V}_k(x) \quad \leq \quad & -(\bar{a} + n - k) \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \underbrace{\xi_{k-1}(x_k^{r_k} - {x^*}_k^{r_k})}_{\textbf{(a)}} \\ & + \xi_k(x_{k+1}^{r_{k+1}} - {x^*}_{k+1}^{r_{k+1}}) + \xi_k {x^*}_{k+1}^{r_{k+1}} + \underbrace{\xi_k f_k(x,u,t)}_{\textbf{(b)}} + \underbrace{\sum_{l=1}^{k-1} \frac{\partial W_k}{x_l} \dot{x}_l}_{\textbf{(c)}} \end{aligned}$$

Now, according to lemmas (3.5)and (3.6),we estimate the expression(**a**):
We have

$$\begin{array}{rcl} \xi_{k-1}(x_k^{r_k} - {x^*}_k^{r_k}) & \leq & \mid \xi_{k-1} \mid 2^{1-r_k} \mid x_k - x_k^* \mid^{r_k} \\ & \leq & \mid \xi_{k-1} \mid 2^{1-r_k} \mid \xi_k \mid^{r_k} \\ & \leq & 2\xi_{k-1}^{1+r_{k-1}} + \rho_1 \xi_k^{1+r_k} \end{array}$$

where $\rho_1 = \dfrac{1}{1+r_k}\left(\frac{r_k}{2(1+r_k)}\right)^{r_k}(2^{1-r_k})^{1+r_k}$

Then, according to lemmas (3.5)-(3.8), we estimate the expression (**b**):

$$\begin{array}{rcl} \xi_k f_k(x,u,t) & \leq & \mid \xi_k \mid\mid f_k(x,u,t) \mid \\ & \leq & \mid \xi_k \mid \bar{a}(\mid x_1 \mid^{r_1} + ... + \mid x_k \mid^{r_k}) \\ & \leq & \mid \xi_k \mid 2\bar{a}(\mid \xi_1 \mid^{r_1} + ... + \mid \xi_k \mid^{r_k}) \\ & \leq & \dfrac{\bar{a}}{3}\displaystyle\sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \rho_2 \sum_{i=1}^{k} \xi_k^{1+r_i} \end{array}$$

where $\rho_2 = \max(c_1, c_2, ..., c_{k-1}, 2\bar{a})$ and $c_i = \frac{1}{1+r_i}\left(\frac{r_i}{1+r_i}\right)^{r_i}(2\bar{a})^{1+r_i}$, for $i = 1, ..., k-1$.

In the next, according to lemmas (3.5)-(3.8), we estimate the expression (**c**):

$$\mid \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l}\dot{x}_l \mid \leq \mid x_k - x_k^* \mid \sum_{l=1}^{k-1} \mid \frac{\partial x_k^*}{\partial x_l}\dot{x}_l \mid \tag{16}$$

$$\leq \mid \xi_k \mid \sum_{l=1}^{k-1} \mid \frac{\partial x_k^*}{\partial x_l}\dot{x}_l \mid \tag{17}$$

But

$$\begin{array}{rcl} \displaystyle\sum_{l=1}^{k-1} \mid \frac{\partial x_k^*}{\partial x_l}\dot{x}_l \mid & = & \mid \dfrac{\partial x_k^*}{\partial x_1}\dot{x}_1 \mid + ... + \mid \dfrac{\partial x_k^*}{\partial x_{k-1}}\dot{x}_{k-1} \mid \\ & = & q_k \mid \dfrac{\partial \xi_{k-1}^{\frac{r_{k-1}}{r_k}}}{\partial x_1}\dot{x}_1 \mid + ... + q_k \mid \dfrac{\partial \xi_{k-1}^{\frac{r_{k-1}}{r_k}}}{\partial x_{k-1}}\dot{x}_{k-1} \mid \\ & = & q_k \dfrac{r_{k-1}}{r_k} \mid \xi_{k-1} \mid^{\frac{r_{k-1}}{r_k}-1} \mid \dfrac{\partial \xi_{k-1}}{\partial x_1}\dot{x}_1 \mid + ... + q_k \dfrac{r_{k-1}}{r_k} \mid \xi_{k-1} \mid^{\frac{r_{k-1}}{r_k}-1} \mid \dfrac{\partial \xi_{k-1}}{\partial x_{k-1}}\dot{x}_{k-1} \mid \\ & \leq & G_{k-1}(x_1, ..., x_{k-1}) \mid x_2^{r_2} + f_1(x,u,t) \mid + ... + G_{k-1}(x_1, ..., x_{k-1}) \mid x_k^{r_k} + f_{k-1}(x,u,t) \mid \\ & \leq & 2\bar{a}G_{k-1}(x_1, ..., x_{k-1})(\mid \xi_1 \mid^{r_1} + \mid \xi_k \mid^{r_k}) \end{array}$$

where G_{k-1} is a positive continuous function.

Then

$$\begin{aligned} \mid \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l} \dot{x}_l \mid &\leq \mid \xi_k \mid \sum_{l=1}^{k-1} \mid \frac{\partial x_k^*}{\partial x_l} \dot{x}_l \mid \\ &\leq \mid \xi_k \mid 2\bar{a} G_{k-1}(x_1, ..., x_{k-1})(\mid \xi_1 \mid^{r_1} + ... + \mid \xi_k \mid^{r_k}) \\ &\leq \frac{\bar{a}}{3} \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \widehat{G}_{k-1}(x_1, ..., x_{k-1}) \sum_{i=1}^{k} \xi_k^{1+r_i} \end{aligned}$$

where $\widehat{G}_{k-1} = \max(m_1, ..., m_{k-1}, (2\bar{a}G_{k-1}))$ and $m_i = \frac{1}{1+r_i}\left(\frac{r_i}{\bar{a}/3(1+r_i)}\right)^{r_i}(2\bar{a}G_{k-1})^{1+r_i}$, $i = 1, ..., k-1$.
So, we get

$$\begin{aligned} \dot{V}_k(x) &\leq -(2\bar{a} + n - k) \sum_{i=1}^{k-1} \mid \xi_i \mid^{1+r_i} + \xi_k {x^*}_{k+1}^{r_k+1} + \xi_k (x_{k+1}^{r_{k+1}} - {x^*}_{k+1}^{r_{k+1}}) \\ &+ (\rho_1 + \rho_2 \sum_{i=1}^{k-1} \mid \xi_k \mid^{r_i - r_k} + \widehat{G}_{k-1}(x_1, ..., x_{k-1}) \sum_{i=1}^{k-1} \mid \xi_k \mid^{r_i - r_k}) \mid \xi_k \mid^{1+r_k} \end{aligned}$$

We choose

$${x^*}_{k+1}^{r_{k+1}} = -\left(2\bar{a} + n - k + \rho_1 + \rho_2 \sum_{i=1}^{k-1} \mid \xi_k \mid^{r_i - r_k} + \widehat{G}_{k-1}(x_1, ..., x_{k-1}) \sum_{i=1}^{k-1} \mid \xi_k \mid^{r_i - r_k}\right) \xi_k^{r_k}$$

We obtain

$$\dot{V}_k(x) \leq -(2\bar{a} + n - k) \sum_{i=1}^{k} \mid \xi_i \mid^{1+r_i} + \xi_k (x_{k+1}^{r_{k+1}} - {x^*}_{k+1}^{r_{k+1}}) \tag{18}$$

□

Using the inductive argument above, at the n-th step, we obtain the following result.

Proposition 3.9
Let

$$V_n(x_1, ..., x_n) = V_{n-1}(x_1, ..., x_{n-1}) + W_n(x_1, ..., x_n),$$

where $W_n(x) = \int_{x_n^*}^{x_n} (s - x_n^*) ds$, *and consider a non-Lipschitz continuous state feedback control law of the form:*

$$u(x) = -\beta_n(x) \xi_n^{r_n}$$

with $\beta_n(x) = 2\bar{a} + \rho_1 + \rho_2 \sum_{i=1}^{n-1} \mid \xi_n \mid^{r_i - r_n} + \widehat{G}_{n-1}(x) \sum_{i=1}^{n-1} \mid \xi_n \mid^{r_i - r_n}$.
Then V_n *is a finite-time Lyapunov function for the closed loop system (1) by the feedback* u *and the closed- loop system (1) is locally finite-time stable.*

Proof. We have:

$$V_n(x_1,...,x_n) = V_{n-1}(x_1,...,x_{n-1}) + W_n(x_1,...,x_n)$$

where $W_n(x) = \int_{x_n^*}^{x_n}(s-x_n^*)ds$, then the derivative of V_n along the trajectories of the system (1) gives

$$\dot{V}_n(x) = \dot{V}_{n-1}(x) + \sum_{l=1}^{n}\frac{\partial W_n}{\partial x_l}\dot{x}_l$$

Therefore

$$\dot{V}_n(x_1,...,x_n) \leq -2\bar{a}\sum_{i=1}^{n-1}|\xi|_i^{1+r_i} + \xi_{n-1}(x_n^{r_n} - x^{*r_n}_n) + u(x)\xi_n + \xi_n f_n(x,u,t) + \sum_{l=1}^{n-1}\frac{\partial W_n}{\partial x_l}\dot{x}_l$$

So,

$$\begin{aligned}\dot{V}_n(x_1,...,x_n) &\leq -2\bar{a}\sum_{i=1}^{n-1}\xi_i^{1+r_i} + u(x)\xi_n \\ &+ (\rho_1 + \rho_2\sum_{i=1}^{n-1}\mid\xi_n\mid^{r_i-r_n} + \widehat{G}_{n-1}(x)\sum_{i=1}^{n-1}\mid\xi_n\mid^{r_i-r_n})\mid\xi_n\mid^{1+r_n}\end{aligned}$$

We conclude that there exists a non-Lipschitz continuous state feedback control law of the form:

$$u(x_1,...,x_n) = -(2\bar{a} + \rho_1 + \rho_2\sum_{i=1}^{n-1}\mid\xi_n\mid^{r_i-r_n} + \widehat{G}_{n-1}(x)\sum_{i=1}^{n-1}\mid\xi_n\mid^{r_i-r_n})\xi_n^{r_n}$$

that satisfies

$$\dot{V}_n(x_1,...,x_n) \leq -(\mid\xi_1\mid^{1+r_1} + ... + \mid\xi_n\mid^{1+r_n})$$

and

$$V_n(x_1,...,x_n) \leq (\xi_1^2 + ... + \xi_n^2)$$

We choose $c = \frac{1+r_1}{2} \in]0,1[$, we obtain

$$V^c(x_1,...,x_n) \leq (\mid\xi_1\mid^{1+r_1} + \mid\xi_2\mid^{1+r_1+r_2-r_2} + ... + \mid\xi_n\mid^{1+r_1-r_n+r_n})$$

So

$$\dot{V}_n(x_1,...,x_n) + V_n^c(x_1,...,x_n) \leq \Big[|\xi_2|^{1+r_2}(|\xi_2|^{r_1-r_2}-1) + ... + |\xi_n|^{1+r_n}(|\xi_n|^{r_1-r_n}-1)\Big]$$

Then

$$\dot{V}_n(x_1,...,x_n) + V_n^c(x_1,...,x_n) \leq 0, \forall x = (x_1,...,x_n) \in \mathbf{B}_F(0,r)$$

with $0 < r < 1$.

Then the closed-loop system (1)-(19) is locally finite-time stable.

□

4 Example

Consider the following nonlinear system of the form

$$\begin{cases} \dot{x_1} = x_2^{\frac{7}{13}} \\ \dot{x_2} = u + f_2(x_1, x_2) \end{cases} \tag{19}$$

where f_2 is a continuously differentiable function defined by:

$$f_2(x_1, x_2) = \frac{x_2^2}{3 + 3x_2^2}$$

It is easy to verify that

$$\mid f_2(x_1, x_2) \mid \leq \frac{1}{3}(\mid x_1 \mid^{\frac{11}{13}} + \mid x_2 \mid^{\frac{7}{13}})$$

First, choose $V_1(x) = \frac{1}{2}x_1^2$, whose time derivative satisfies

$$\dot{V}_1(x) \leq x_1(x_2^{\frac{7}{13}} - {x^*}_2^{\frac{7}{13}}) + x_1 {x^*}_2^{\frac{7}{13}}$$

The virtual controller $x_2^* = -q_2 x_1^{11/7}$, with $q_2 = 0.5 \in]0, 1[$, gives

$$\dot{V}_1(x_1) \leq -0.5\xi_1^{\frac{24}{13}} + \xi_1(x_2^{\frac{7}{13}} - {x^*}_2^{\frac{7}{13}})$$

with $\xi_1 = x_1$.
Next, let

$$\xi_2 = x_2 - x_2^* \quad \text{and} \quad V_2(x_1, x_2) = V_1(x_1) + \int_{x_2^*}^{x_2} (s - x_2^*) ds$$

where

$$W_2(x_1, x_2) = \int_{x_2^*}^{x_2} (s - x_2^*) ds$$

is a positive continuously differentiable function .
Then

$$\begin{aligned} \dot{V}_2(x_1, x_2) &= \dot{V}_1(x_1) + \xi_2 u + \xi_2 f(x_1, x_2) + \xi_2 \frac{\partial x_2^*}{\partial x_1} \dot{x}_1 \\ &\leq -0.5\xi_1^{\frac{24}{13}} + \underbrace{\xi_1(x_2^{\frac{7}{13}} - {x^*}_2^{\frac{7}{13}})}_{e_1} + \xi_2 u + \underbrace{\xi_2 f(x_1, x_2)}_{e_2} + \underbrace{\xi_2 \frac{\partial x_2^*}{\partial x_1} \dot{x}_1}_{e_3} \end{aligned}$$

According to expressions$(a),(b)$ and (c), we estimate $(e_1),(e_2)$ and (e_3).
We obtain:

$$\dot{V}_2(x_1,x_2) \leq -0.50\xi_1^{\frac{24}{13}} + u\xi_2 + 2,81(\xi_2^{\frac{4}{13}}+1)\xi_2^{\frac{7}{13}+1} + \left(0,75(0.73x_1^{\frac{3}{7}})^{\frac{24}{13}}\xi_2^{\frac{4}{13}} + 0.73x_1^{\frac{3}{7}}\right)\xi_2^{\frac{7}{13}+1}$$

By choosing

$$u(x_1,x_2) = -\left(-0.50 + 2,81(\xi_2^{\frac{4}{13}}+1) + (0,75(0.73x_1^{\frac{3}{7}})^{\frac{24}{13}}\xi_2^{\frac{4}{13}} + 0.73x_1^{\frac{3}{7}})\right)\xi_2^{\frac{7}{13}},$$

for $x \in B_F(0,r)$ with $0<r<1$, the same steps of the proof of proposition 3.9 give

$$\dot{V}_2(x_1,x_2) \leq -(\mid \xi_1 \mid^{1+\frac{11}{13}} + \mid \xi_2 \mid^{1+\frac{7}{13}})$$

and

$$V_2(x_1,x_2) \leq (\xi_1^2+\xi_2^2) \tag{20}$$

If we choose $\alpha = \dfrac{1+\frac{11}{13}}{2} = \frac{12}{13} \in]0,1[$, we obtain

$$\dot{V}_2(x_1,x_2) + V_2^{\alpha}(x_1,x_2) \leq 0 \tag{21}$$

which means that the system (19) is locally finite-time stabilizable. The neighborhood around the origin where the system converges in finite time is $B_F(0,r)$, $0<r<1$.
The settling time depending on the initial condition $(-0.3,0.5)$ is:

$$\begin{aligned} T_x(-0.3,0.5) &\leq \frac{V_2^{1/13}(-0.3,0.5)}{1/13} \\ &\leq 26(\xi_1^2+\xi_2^2)^{1/13} \\ &\leq 26((-0.3)^2+(0.5+0.50(-0.3)^{22/7})^{1/13} \\ &\leq 23.80 \end{aligned}$$

The following figure illustrates the example for initial condition $(-0.3,0.5)$.

9 Nov 2022 23:16:35 PST
220703-Kharrat Version 3 - Submitted to Rocky Mountain J. Math.

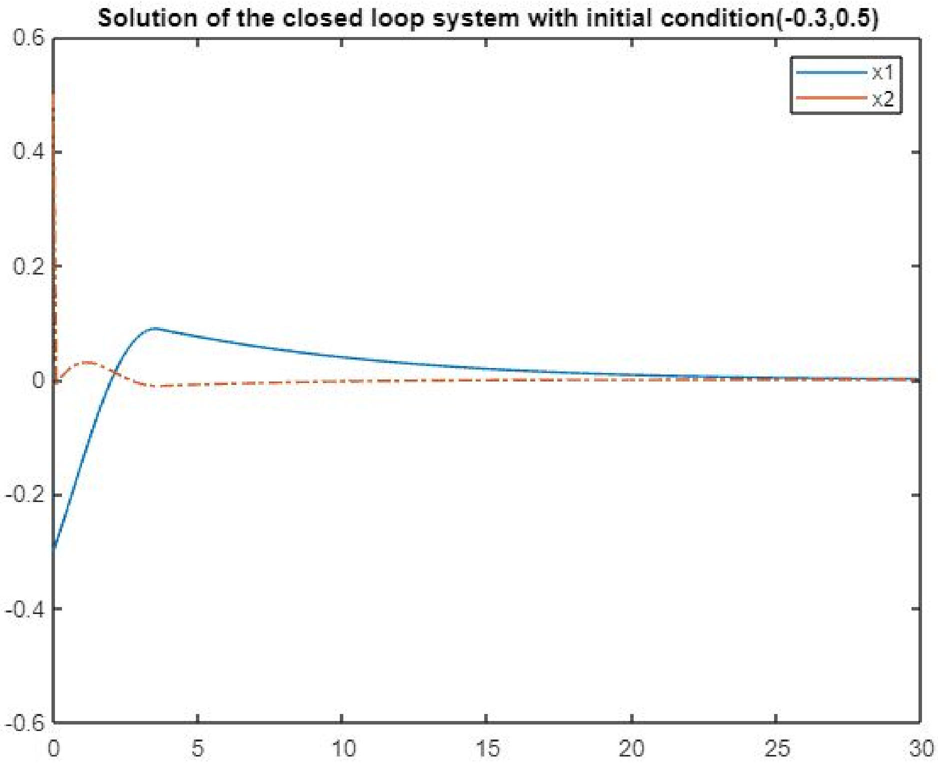

Figure 1: Trajectories of the closed-loop system(19) with $(x_1(0), x_2(0)) = (-0.3, 0.5)$

5 Conclusion

In this paper, we gave a simpler design method for achieving locally finite-time stabilization of a family of nonlinear systems (1), under the assumption (3.1). Finally, a numerical example has also been given to prove the use of our main result.

References

[1] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Analysis: Theory, Methods & Applications, vol. 7, p. 1163-1173, (1983).

[2] M. Athans and P.L. Falb, *Optimal Control: An Introduction to the theory and its Applications*, (2007).

[3] A. Bacciotti, *Local stabilizability on nonlinear control systems*, Singapore: World Scientific, (1991).

[4] S. Bhat and D. Bernstein, *Finite-time stability of homogeneous systems*, Proceedings of the 1997 American Control Conference, vol.4, p. 2513-2514 , (1997).

[5] R. A. Freeman, P. Kokotović, *Robust Nonlinear Control Design State-Space and Lyapunov Techniques*, (1996).

[6] W. Hahn, *Stability of Motion*, New York: springer-Verlag, (1967).

[7] X. Huang, W. Lin and B. Yang *Finite-time Stabilization in the large for Uncertain Nonlinear Systems*, Proceedings of the 2004 American Control Conference, vol.2, p. 1073-1078, (2004).

[8] M. Kawski, *Homogeneous stabilizing feedback laws*, Control-Theory and Advanced Technology, (1990).

[9] H. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, (2002).

[10] J. Li and C. Qian, *Global Finite-Time Stabilization by Dynamic Output Feedback for a Class of Continuous Nonlinear Systems*, IEEE Transactions on Automatic Control, vol. 51, p. 879-884, (2006).

[11] J. Li and C. Qian, *Global Finite-Time Stabilization of a class of uncertain Nonlinear systems using output feedback*, Proceedings of the 44th IEEE Conference on Decision and Control, p. 2652-2657, (2005).

[12] E. Moulay, *Une contribution à l'étude de la stabilité en temps fini et de la stabilisation*, thesis, (2005).

[13] E. Moulay and W. Perruquetti, *Finite-time stability of nonlinear systems*, 42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475), vol.4, p. 3641-3646, (2003).

[14] C. Qian and W. Lin, *A continuous feedback approach to global strong stabilization of nonlinear systems*, IEEE trans. Automat. Contr., vol. 46, p. 1061-1079, (2001).

[15] L. Rosier, *Homogeneous Lyapunov function for Homogeneous continuous vector field*, Systems & Control Letters, vol. 19, p. 467-473, (1992).

[16] E. D. Sontag, *A 'universal' construction of Artstein's theorem on nonlinear stabilization*, Systems & Control Letters, vol. 13, p. 117-123, (1989).

[17] X. Wang, S. Huang, Z. Xiang, *Output feedback finite-time stabilization of a class of nonlinear time-delay systems in the p-normal form*, Int J Robust Nonlinear Control, vol. 30, p. 4418–4432, (2020).

[18] X. Wang and Z. Xiang, *Global finite-time stabilisation of high-order nonlinear systems: a dynamic gain-based approach*, International Journal of Systems Science, vol. 50, p. 1677-1687, (2019).

[19] B. Yang, *Output Feedback Control for nonlinear Systems with unstabilizable and undetectable linearization*, Case Western Reserve University, (2006).