A new study for global asymptotic stability of a fractional-order hepatitis B epidemic model

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Abstract

The purpose of this work is to provide a rigorous mathematical study for global asymptotic stability (GAS) of a recognized fractional-order hepatitis B epidemic model, which was proposed in a recent work. We use a simple approach to establish the GAS of the fractional-order hepatitis B model. This approach is based on extensions of the Lyapunov stability theory and the fractional Barbalat's lemma in combination with some nonstandard techniques for fractional dynamical systems. As an important consequence, the GAS of disease free and disease endemic equilibrium points is determined fully. The obtained results not only improve but also generalize some existing works. In addition, a set of numerical experiments is performed to support and illustrate the constructed theoretical results.

Keywords: HBV, Caputo fractional derivative, Fractional differential equations, Stability analysis, Global asymptotic stability, Lyapunov functions

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1. Introduction

Hepatitis B is an infectious disease caused by the hepatitis B virus (HVB), which can attack the liver and can cause both acute and chronic diseases. Nowadays, hepatitis B has become a major global health problem. This leads to urgent requests for strategies and measures to prevent and control the HBV. For this purpose, many mathematicians and epidemiologists have proposed a large number of mathematical models, which are based on epidemiological principles, to discover characteristics and transmission mechanisms of the HBV (see, for instance, [3, 10, 11, 12, 15, 16, 17, 20, 22, 27, 28, 29, 30, 31, 32, 33, 34, 44, 47]). These mathematical models can provide us with good observations of the mechanism of the transmission of the HBV; consequently, appropriate and effective strategies for preventing and controlling the hepatitis B can be suggested.

We start this work by considering a recognized hepatitis B epidemic model, which was proposed by Khan et al. in [27]. This model is constructed based on some suitable hypotheses of the hepatitis B virus spreading and

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is given by

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \frac{\lambda S(t)I(t)}{1+\gamma I(t)} - \big(\mu_0+\nu\big)S(t),\\
\frac{dI(t)}{dt} &= \frac{\lambda S(t)I(t)}{1+\gamma I(t)} - \big(\mu_0+\mu_1+\beta\big)I(t),\\
\frac{dR(t)}{dt} &= \beta I(t) + \nu S(t) - \mu_0 R(t).
\end{aligned}
\tag{1}
$$

In the model

- the entire population is divided into three classes: susceptible (S) class, infected (I) class and recovered (R) class;
- Λ is the birth rate;
- λ is the transmission rate of hepatitis B virus;
- μ_0 and μ_1 are the natural and disease induced death rates, respectively;
- β is the recovery rate;
- ν and γ are the vaccination and saturation rates, respectively;

We refer the readers to [28] for more details and qualitative dynamical properties of the model (1).

Although the model (1) provides a good mathematical model for transmission dynamics of the HBV with several applications in real-life, it can be improved by using fractional-order derivatives, which have the ability to describe the memory effect on population dynamic models (see, for instance, [1, 2, 8, 13, 18, 35, 42, 46, 51]). In recent years, many fractional-order models have been proposed and analyzed because of its accuracy compared to integer-order (ODE) models [4, 19, 25, 48, 51]. Motivated and inspired by the above reason, Hoang and Egbelowo in [23] proposed a fractional-order hepatitis B epidemic model, which is described by the following system of fractional differential equations

$$
\begin{aligned}
{}^{C}D^{\alpha}_{0+}S(t) &= \Lambda^\alpha - \frac{\lambda^\alpha S(t)I(t)}{1+\gamma^\alpha I(t)} - \big(\mu_0^\alpha+\nu^\alpha\big)S(t),\\
{}^{C}D^{\alpha}_{0+}I(t) &= \frac{\lambda^\alpha S(t)I(t)}{1+\gamma^\alpha I(t)} - \big(\mu_0^\alpha+\mu_1^\alpha+\beta^\alpha\big)I(t),\\
{}^{C}D^{\alpha}_{0+}R(t) &= \beta^\alpha I(t) + \nu^\alpha S(t) - \mu_0^\alpha R(t),
\end{aligned}
\tag{2}
$$

where ${}^{C}D^{\alpha}_{0+}f(t)$ with $\alpha \in (0,1)$ stands for the Caputo fractional derivative of the function $f(t)$ [8, 13, 35, 46]. Note that the derivation of the model (2) can be explained in terms of memory effect on population dynamics by using the approach used in [19].

In [23], a threshold quantity for the model (2) was defined by

$$
\mathcal{R}_0^\alpha := \frac{\lambda^\alpha \Lambda^\alpha}{\big(\mu_0^\alpha+\nu^\alpha\big)\big(\mu_0^\alpha+\mu_1^\alpha+\beta^\alpha\big)}.
\tag{3}
$$

15 Jan 2023 00:45:19 PST
220623-Hoang Version 2 - Submitted to Rocky Mountain J. Math.

Also, it was proved that: The model (2) always possesses a disease free equilibrium (DFE) point $E_0 = (S_0, I_0, R_0)$ for all values of the parameters, whereas, a disease endemic equilibrium (DEE) point $E_* = (S_*, I_*, R_*)$ exists if and only if $\mathcal{R}_0^\alpha > 1$, where

$$S_0 = \frac{\Lambda^\alpha}{\mu_0^\alpha + \nu^\alpha}, \qquad I_0 = 0, \qquad R_0 = \frac{\nu^\alpha \Lambda^\alpha}{\mu_0^\alpha\big(\mu_0^\alpha + \nu^\alpha\big)}, \tag{4}$$

and

$$\begin{aligned} I_* &= \frac{\Lambda^\alpha \lambda^\alpha - \big(\mu_0^\alpha + \nu^\alpha\big)\big(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\big)}{\big(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\big)\Big[\lambda^\alpha + \gamma^\alpha\big(\mu_0^\alpha + \nu^\alpha\big)\Big]}, \\ S_* &= \frac{(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha)(1 + \gamma^\alpha I^*)}{\lambda^\alpha}, \\ R_* &= \frac{\beta^\alpha I^* + \nu^\alpha S_*}{\mu_0^\alpha}. \end{aligned} \tag{5}$$

The local asymptotic stability of E_0 and E_* was established as follows (see Propositions 2 and 3 in [23]):

(i) The DFE point E_0 is locally asymptotically stable if $\mathcal{R}_0^\alpha < 1$.

(ii) The DEE point E_* is locally asymptotically stable if $\mathcal{R}_0^\alpha > 1$.

It is clear that the stability analysis performed in [23] lacks the global asymptotic stability (GAS) of the fractional-order model (2). Motivated by this, in the present work we establish the complete GAS of the model (2) by using a simple approach, which is based on extensions of the Lyapunov stability theory and the fractional Barbalat's lemma (see [18, 42, 50, 51]) in combination with some nonstandard techniques for fractional dynamical systems. Here, we first use general Volterra-type Lyapunov functions with undetermined coefficients as potential candidates. Then, nonstandard techniques of mathematical analysis are used to prove that the time derivatives of the proposed Lyapunov functions are globally positive define. Finally, the fractional Barbalat's lemma is applied to show the convergence of solutions of the fractional-order model. Consequently, the GAS of DEE and DFE points is analyzed rigorously.

It is worth noting that the analysis of GAS of integer-order and fractional-order dynamical systems is very important but not simple in general. The Lyapunov stability theory and its extensions can be considered as one of the most successful approaches to this problem [1, 2, 18, 39, 42, 43]. However, this approach requires suitable Lyapunov functions but there is no general technique for constructing Lyapunov functions for dynamical systems. Although many researchers have successfully constructed Lyapunov functions for important differential equation models (see, for example, [1, 2, 18, 36, 37, 38, 42, 51]), the construction of Lyapunov functions for the fractional-order model (2) is not a trivial problem. However, by the present approach, we obtain the complete GAS of the fractional-order model. Moreover, this approach can be extended to study stability properties of general fractional dynamical systems. It should be emphasized that our approach is differently from the one used in [25].

As noted before, the model (2) is a generalization of the integer-order HBV model (1). In [27], Khan et al. proved the GAS of the DFE point but failed to conclude the GAS of the DEE point of the model (1). Since the GAS of the fractional-order model (2) can imply the GAS of the integer-order model (1), we also obtain the GAS of the model (1) from the stability analysis of the model (2). Although Hoang and Egbelowo in [24] provided a proof of the GAS of DEE point of the model (1) based on the Bendixson-Dulac criterion and the Poincare-Bendixson theory, this approach is only appropriate for two-dimensional dynamical systems governed by ODEs and not applicable for the model (2) in particular and for fractional dynamical systems in general. This means that the present approach is more general and efficient.

The plan of this work is as follows:

Section 2 provides some important basic definitions and preliminaries. The complete GAS of the fractional-order model (2) is studied in Section 3. Numerical experiments are conducted in Section 4. Some remarks and discussions are presented in the last section.

2. Preliminaries

We first recall from [8, 13, 35, 46] the definitions of the C*aputo fractional derivatives* and their properties.

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The *Riemann-Liouville fractional integrals*$I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ are defined by (see [35, Section 2.1])

$$(I^{\alpha}_{a+}f)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > a; \Re(\alpha) > 0)$$

and

$$(I^{\alpha}_{b-}f)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x < b; \Re(\alpha) > 0),$$

respectively. The above integrals are called the left-sided and the right-sided fractional integrals.

The *Riemann-Liouville fractional derivatives* $D^{\alpha}_{a+}y$ and $D^{\alpha}_{b-}y$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ are given by

$$(D^{\alpha}_{a+}y)(x) = \left(\frac{d}{dx}\right)^n (I^{n-\alpha}_{a+}y)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}} \quad (n = [\Re(\alpha)] + 1; x > a)$$

and

$$(D^{\alpha}_{b-}y)(x) = \left(-\frac{d}{dx}\right)^n (I^{n-\alpha}_{b-}y)(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{y(t)dt}{(x-t)^{\alpha-n+1}} \quad (n = [\Re(\alpha)] + 1; x < b),$$

respectively, where $[\Re(\alpha)]$ means the integer part of $\Re(\alpha)$.

The *Caputo fractional derivatives* of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ on $[a, b]$ are defined via the above Riemann-Liouville fractional derivatives by

$$({}^C D^{\alpha}_{a+}y)(x) := \left(D^{\alpha}_{a+}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x)$$

and

$$({}^C D^{\alpha}_{b-}y)(x) := \left(D^{\alpha}_{b-}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{k!}(b-t)^k\right]\right)(x),$$

respectively, where

$$n = [\Re(\alpha)] + 1 \quad \text{for} \quad \alpha \notin \mathbb{N}_0; \quad n = \alpha \quad \text{for} \quad \alpha \in \mathbb{N}_0.$$

These derivatives are called the *left-sided and right-sided Caputo fractional derivatives* of order α.

Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$ and $y(x)$ be a function belonging to the space $AC[a,b]$ of absolutely continuous functions on $[a,b]$. As a direct consequence of Theorem 2.1 in [35], the *left-sided and right-sided Caputo fractional derivatives* of order $0 < \alpha < 1$ are given by (see [35, Section 2.4])

$$({}^C D^{\alpha}_{a+} y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{y'(\tau)d\tau}{(x-\tau)^{\alpha}}$$

and

$$({}^C D^{\alpha}_{b-} y)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{y'(\tau)d\tau}{(x-t)^{\alpha}},$$

respectively. In particular, when $\alpha = 0$ and $\alpha = 1$ we have

$$({}^C D^{0}_{a+} y)(x) = ({}^C D^{0}_{b-} y)(x) = y(x)$$

and

$$({}^C D^{1}_{a+} y)(x) = ({}^C D^{1}_{b-} y)(x) = y'(x),$$

respectively.

Remark 1. *The notation* ${}^C_a D^{\alpha}_t f(t)$ *is also often used to denote the left-sided Caputo fractional derivative of a function* $f(t)$ *(see, for instance, [2, 18, 42, 51]).*

Property 1. *(Linearity property [13]). Let* $f(t), g(t) : [a,b] \to \mathbb{R}$ *be such that* ${}^C_a D^{\alpha}_t f(t)$ *and* ${}^C_a D^{\alpha}_t g(t)$ *exist everywhere and let* $c_1, c_2 \in \mathbb{R}$*. Then,* ${}^C_a D^{\alpha}_t (c_1 f(t) + c_2 g(t))$ *exists everywhere, hence*

$${}^C_a D^{\alpha}_t \big(c_1 f(t) + c_2 g(t)\big) = c_1 {}^C_a D^{\alpha}_t f(t) + c_2 {}^C_a D^{\alpha}_t g(t).$$

Lemma 1. *(Generalized mean value theorem [45]) Suppose that* $w \in C[a,b]$ *and* ${}^C_a D^{\alpha}_t w(t) \in C[a,b]$ *for* $0 < \alpha \leq 1$*, then we have*

$$w(t) = w(a) + \frac{1}{\Gamma(\alpha)} {}^C_a D^{\alpha}_t w(\xi)(t-a)^{\alpha},$$

with $a \leq \xi \leq t$*, for all* $t \in (a,b]$*.*

Theorem 1. *([13]) Assume that* $f \in C^1[a,b]$ *is such that* ${}^C_a D^{\alpha}_t f(t) \geq 0$ *for all* $t \in [a,b]$ *and all* $\alpha \in (\alpha_0, 1)$ *with some* $\alpha_0 \in (0,1)$*. Then,* f *is monotone increasing. Similarly, if* ${}^C_a D^{\alpha}_t f(t) \leq 0$ *for all* t *and* α *mentioned above, then* f *is monotone decreasing.*

Consider a general dynamical systems governed by the Caputo fractional differential equations of the form

$${}^C_{t_0} D^{\alpha}_t y(t) = f(t,y), \quad y(t_0) = y_0, \quad \alpha \in (0,1). \tag{6}$$

Definition 1. *([42]). The constant* y^* *is an equilibrium point of the Caputo fractional dynamical system* (6) *if and only if* $f(t, y^*) = 0$*.*

We now present some concepts of stability for the system (6) (see [1, 13, 18, 39, 41, 42, 43]).

Definition 2 (Concepts of stability). *The equilibrium point $y^* = 0$ of the system (6) is said to be*

(i) stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for any $y_0 \in \mathbb{R}^n$ the inequality $\|y_0\| < \delta$ implies that $\|y(t; t_0, y_0)\| < \epsilon$ for $t \geq t_0$;

(ii) local asymptotically stable if it is stable and there exists some $\gamma > 0$ such that $\lim_{t\to\infty} \|y(t)\| = 0$ whenever $\|y_0\| < \gamma$;

(iii) globally asymptotically stable if it is stable and $\lim_{t\to\infty} \|y(t)\| = 0$ for all y_0 satisfying $\|y_0\| < \infty$.

Definition 3. *(Class-$\mathcal{K}$ functions [26]) A continuous function $\alpha : [0, t) \to [0, \infty)$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.*

Lemma 2. *(A relationship between positive define functions and class-$\mathcal{K}$ functions [49]) A function $V(x, t)$ is locally (or globally) positive definite if and only if there exists a class-$\mathcal{K}$ function γ_1 such that $V(0, t) = 0$ and*

$$V(x, t) \geq \gamma_1(\|x\|)$$

$\forall t \geq t_0$ and $\forall x$ belonging to the local space (or the whole space).

Theorem 2. *(Fractional Lyapunov direct method by using the class-$\mathcal{K}$ functions [42]) Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (6). Assume that there exists a Lyapunov function $V(t, y(t))$ and class-$\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ satisfying:*

$$\alpha_1(\|y\|) \leq V(t, y) \leq \alpha_2(\|y\|)$$

and

$${}_{t_0}^{C}D_t^{\beta} y(t) \leq -\alpha_3(\|y\|)$$

where $\beta \in (0, 1)$. Then the system (6) is asymptotically stable.

Theorem 3. *(Lyapunov stability and uniform stability of fractional order systems [18]) Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (6). Let us assume that there exists a continuous Lyapunov function $V(y(t), t)$ and a scalar class-$\mathcal{K}$ function $\gamma_1(.)$ such that, $\forall y \neq 0$*

$$\gamma_1(\|y(t)\|) \leq V(y(t), t)$$

and

$${}_{t_0}^{C}D_t^{\beta} y(t) \leq 0, \quad \text{with} \quad \beta \in (0, 1]$$

then the origin of the system (6) is Lyapunov stable (stable).

If, furthermore, there is a scalar class-$\mathcal{K}$ function $\gamma_2(.)$ satisfying

$$V(y(t), t) \leq \gamma_2(\|y\|)$$

then the origin of the system (6) is Lyapunov uniformly stable (uniformly stable).

Theorem 4. *(Fractional order Barbalat's lemma [50, Theorem 3]) If a scalar function $V(t, y(t))$ is positive semi-definite and the Caputo fractional derivative of $V(t, y(t))$ along the solution $y(t)$ of the system (6) satisfies ${}_{t_0}^{C}D_t^{\alpha} V(t, y(t)) \leq -\varphi(\|y(t)\|)$, where $\varphi(.)$ belongs to class-$\mathcal{K}$, then $y(t) \to 0$ as $t \to +\infty$ if $y_i(t)$ $i = 1, 2, \ldots, n$ are uniformly continuous.*

The following results are very useful in studying stability properties of the system (6)

Corollary 1. *([50, Corollary 3]) If a scalar function $V(t, y(t))$ is positive semi-definite and the Caputo fractional derivative of $V(t, y(t))$ along the solution $y(t)$ of the system (6) satisfies ${}_{t_0}^{C}D_t^{\alpha} V(t, y(t))$ is negative semi-define, then $y(t) \to 0$ as $t \to +\infty$ if $f_i(t, y(t))$ $i = 1, 2, \ldots, n$ for the system (6) are bounded.*

Lemma 3. *(A fractional comparison principle [42, Lemma 6.1]) Let $x(0) = y(0)$ and ${}_{t_0}^{C}D_t^{\beta} x(t) \geq {}_{t_0}^{C}D_t^{\beta} y(t)$, where $\beta \in (0, 1)$. Then $x(t) \geq y(t)$.*

Lemma 4. *([51]). Let $x(t) \in \mathbb{R}^+$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$ {}_{t_0}^{C}D_t^{\alpha}\left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right) {}_{t_0}^{C}D_t^{\alpha} x(t), \quad x^* \in \mathbb{R}^+, \quad \forall \alpha \in (0, 1). \tag{7} $$

3. Stability analysis

In this section, the GAS of the fractional-order model (2) is studied. First, it is important to note that the two first equations of (2) do not depend on R; consequently, we only need to consider the following sub-model:

$$ \begin{aligned} {}^{C}D_{0+}^{\alpha} S &= \Lambda^{\alpha} - \frac{\lambda^{\alpha} S I}{1 + \gamma^{\alpha} I} - \big(\mu_0^{\alpha} + \nu^{\alpha}\big) S, \\ {}^{C}D_{0+}^{\alpha} I &= \frac{\lambda^{\alpha} S I}{1 + \gamma^{\alpha} I} - \big(\mu_0^{\alpha} + \mu_1^{\alpha} + \beta^{\alpha}\big) I. \end{aligned} \tag{8} $$

The model (8) always possesses a DFE point $\widehat{E}_0 = (S_0, I_0)$ for all values of the parameters, meanwhile, it has a DEE point $\widehat{E}_* = (S_*, I_*)$ if and only if $\mathcal{R}_0^{\alpha} > 1$, where (S_0, I_0) and (S_*, I_*) are given by (4) and (5), respectively.

Lemma 5. *Let $\big(S(0), I(0)\big)^T \in \mathbb{R}_2^+$ be an initial data for the initial value problem (8) and $\big(S(t), I(t)\big)^T$ be the corresponding solution. Then, $S(t), I(t) \geq 0$ for all $t > 0$. Furthermore, we have the following estimates*

$$ \begin{aligned} &\limsup_{t\to\infty} S(t) \leq \frac{\Lambda^{\alpha}}{\mu_0^{\alpha} + \nu^{\alpha}}, \\ &\liminf_{t\to\infty} \big[S(t) + I(t)\big] \geq \eta_1, \\ &\limsup_{t\to\infty} \big[S(t) + I(t)\big] \leq \eta_2, \end{aligned} \tag{9} $$

where

$$ \eta_1 := \frac{\Lambda^{\alpha}}{\max\big\{\mu_0^{\alpha} + \nu^{\alpha},\ \mu_0^{\alpha} + \mu_1^{\alpha} + \beta^{\alpha}\big\}}, \quad \eta_2 := \frac{\Lambda^{\alpha}}{\min\big\{\mu_0^{\alpha} + \nu^{\alpha},\ \mu_0^{\alpha} + \mu_1^{\alpha} + \beta^{\alpha}\big\}}. \tag{10} $$

Proof. First, from (8) we have

$$ {}^{C}D_{0+}^{\alpha} S\Big|_{S=0} = \Lambda^{\alpha} > 0, \quad {}^{C}D_{0+}^{\alpha} I\Big|_{I=0} = 0, $$

15 Jan 2023 00:45:19 PST
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for all $S, I \geq 0$. Let $\big(S(0),\, I(0)\big)^T$ be any initial data belonging to $\mathbb{R}_2^+$. Then, the corresponding solution $\big(S(t),\, I(t)\big)^T$ cannot escape from the hyperplanes of $S = 0$ and $I = 0$, and on each hyperplane the vector field is tangent to that hyperplane or points toward the interior of $\mathbb{R}_+^2$. This means that $\big(S(t),\, I(t)\big)^T \in \mathbb{R}_+^2$.

From the first equation of (8) we obtain

$$^{C}D_{0+}^{\alpha} S \leq \Lambda^\alpha - \big(\mu_0^\alpha + \nu^\alpha\big) S.$$

Consider an auxiliary equation

$$^{C}D_{0+}^{\alpha} z = \Lambda^\alpha - \big(\mu_0^\alpha + \nu^\alpha\big) z, \quad z(0) \geq S(0).$$

This equation has a unique positive equilibrium point $z_* = \Lambda^\alpha/(\mu_0^\alpha + \nu^\alpha)$. It is easy to show z_* is globally asymptotically stable. Hence, $\lim_{t\to\infty} z(t) = z_*$. Combining this with Lemma 3 (the fractional comparison principle) we obtain

$$\limsup_{t\to\infty} S(t) \leq \limsup_{t\to\infty} z(t) = z_* = \frac{\Lambda^\alpha}{\mu_0^\alpha + \nu^\alpha}.$$

Similarly, by adding side-by-side the first and second equations of (8), we have

$$\begin{aligned}
&\Lambda^\alpha - \max\big\{\mu_0^\alpha + \nu^\alpha,\ \mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\big\}(S + I) \\
&\leq {}^{C}D_{0+}^{\alpha}(S + I) \\
&\leq \Lambda^\alpha - \min\big\{\mu_0^\alpha + \nu^\alpha,\ \mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\big\}(S + I),
\end{aligned}$$

which follows the last two estimates of (9). The proof is complete. □

Remark 2. *If $S(0) = 0$, then the first equation of (8) implies that*

$$^{C}D_{0+}^{\alpha} S\big|_{t=0} = \Lambda^\alpha > 0.$$

So, there exists a number $t_* > 0$ such that $S(t_*) > 0$. Therefore, without loss of generality, it is sufficient to consider $S(0) > 0$. Similarly, if $I(0) = 0$ then $I(t) \equiv 0$ is a unique solution of the second equation of (8). In this case, it is easy to verify that $\lim_{t\to\infty} S(t) = S_0$. Combining the above observations with Lemma 5, it suffices to study dynamical properties of the model (8) on a feasible set given by

$$\Omega^* = \left\{(S, I)\middle| S, I > 0,\ S \leq \frac{\Lambda^\alpha}{\mu_0^\alpha + \nu^\alpha},\ \eta_1 \leq S + I \leq \eta_2\right\}. \tag{11}$$

Before establishing the GAS of (8), we need the following auxiliary result.

Lemma 6. *Consider the function*

$$f(S) = S - S_0 \ln\left(\frac{S}{S_0}\right) - S_0, \quad S, S_0 \in (0, \eta]. \tag{12}$$

Then, we have

$$f(S) \geq \frac{S_0}{\eta^2}(S - S_0)^2 \quad \textit{for all} \quad S, S_0 \in (0, \eta].$$

Proof. Using the Taylor's formula for the function f, we obtain

$$f(S) = f(S_0) + f'(S_0)(S - S_0) + f''(\xi_S)(S - S_0)^2,$$

where ξ_S is a point between S and S_0. Due to the fact that

$$f(S_0) = 0, \quad f'(S_0) = 0, \quad f''(S) = \frac{S_0}{S^2} \geq \frac{S_0}{\eta^2},$$

we have

$$f(S) \geq \frac{S_0}{\eta^2}(S - S_0)^2.$$

The proof is complete. □

In the following theorems, we will use the l_2 norm, i.e., if $y = (y_1, y_2)^T$ is any vector in $\mathbb{R}^2$, then the norm of y is given by

$$\|y\| = \sqrt{y_1^2 + y_2^2}.$$

Theorem 5. *The DFE point $\widehat{E}_0$ of the model* (8) *is globally asymptotically stable whenever $\mathcal{R}_0^\alpha < 1$.*

Proof. First, we rewrite (8) in the form

$$\begin{aligned} {}^C D_{0+}^\alpha S &= S\left[-\frac{(\mu_0^\alpha + \nu^\alpha)(S - S_0)}{S} - \frac{\lambda^\alpha I}{1 + \gamma^\alpha I}\right], \\ {}^C D_{0+}^\alpha I &= \frac{\lambda^\alpha S I}{1 + \gamma^\alpha I} - (\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha) I. \end{aligned} \tag{13}$$

Next, consider a Lyapunov function $V : \Omega^* \to \mathbb{R}_+$ defined by

$$V(S, I) = \left[S - S_0 \ln\left(\frac{S}{S_0}\right) - S_0\right] + I. \tag{14}$$

Thanks to Lemma 6, we have

$$S - S_0 \ln\left(\frac{S}{S_0}\right) - S_0 \geq \frac{S_0}{\eta_2^2}(S - S_0)^2 \quad \text{for all} \quad (S, I) \in \Omega^*.$$

On the other hand

$$I \geq \eta_2^{-1} I^2 \quad \text{for all} \quad (S, I) \in \Omega^*.$$

Consequently,

$$V(S, I) \geq \max\left\{\frac{S_0}{\eta_2^2},\ \eta_2^{-1}\right\}\left[(S - S_0)^2 + I^2\right].$$

So, if setting

$$\gamma_1(z) = \max\left\{\frac{S_0}{\eta_2^2},\ \eta_2^{-1}\right\} z^2,$$

then $\gamma_1(.)$ belongs to class-$\mathcal{K}$ functions and the function V given by (14) satisfies

$$V(y) \geq \gamma_1(\|y\|), \quad y := (S - S_0,\ I).$$

Using Property 1, Lemma 4 and (13) we obtain

$$
\begin{aligned}
{}^{C}D^{\alpha}_{0+}V &= {}^{C}D^{\alpha}_{0+}\left[S - S_0 - S_0\ln\left(\frac{S}{S_0}\right)\right] + {}^{C}D^{\alpha}_{0+}I \\
&\le \frac{S - S_0}{S}\left({}^{C}D^{\alpha}_{0+}S\right) + \left({}^{C}D^{\alpha}_{0+}I\right) \\
&= (S - S_0)\left[\frac{-\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)}{S} - \frac{\lambda^\alpha I}{1 + \gamma^\alpha I}\right] + \frac{\lambda^\alpha S I}{1 + \gamma^\alpha I} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)I \\
&= -\frac{\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)^2}{S} + \left[\frac{\lambda^\alpha S_0}{1 + \gamma^\alpha I} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)\right]I \\
&\le -\frac{\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)^2}{S} + \left[\lambda^\alpha S_0 - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)\right]I \\
&= -\frac{\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)^2}{S} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)(\mathcal{R}_0 - 1)I \\
&\le -\frac{\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)^2}{\eta_2} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)(\mathcal{R}_0 - 1)I \\
&\le -\frac{\left(\mu_0^\alpha + \nu^\alpha\right)(S - S_0)^2}{\eta_2} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)(\mathcal{R}_0 - 1)\eta_2^{-1}I^2.
\end{aligned}
\tag{15}
$$

Since $\mathcal{R}_0^\alpha < 1$, the function V defined by (14) satisfies Theorem 3. Consequently, $\widehat{E}_0$ is stable.

Note that $S(t)$ and $I(t)$ are bounded. So, the fractional Barbalat's lemma (Lemma 1) follows that

$$\lim_{t\to\infty}\big(S(t),\, I(t)\big) = \widehat{E}_0 = (S_0, 0).$$

Hence, $\widehat{E}_0$ is globally asymptotically stable. This is the desired conclusion. □

Theorem 6. *If $\mathcal{R}_0^\alpha > 1$, then the DEE point $\widehat{E}_*$ of the model (8) is globally asymptotically stable.*

Proof. Note that the DEE point $\widehat{E}_*$ exists if and only if $\mathcal{R}_0^\alpha > 1$. We transform (8) to the form

$$
\begin{aligned}
{}^{C}D^{\alpha}_{0+}S &= S\left[\frac{\Lambda^\alpha}{S} - \frac{\lambda^\alpha I}{1 + \gamma^\alpha I} - \left(\mu_0^\alpha + \nu^\alpha\right)\right] \\
&= S\left[\left(\frac{\Lambda^\alpha}{S} - \frac{\Lambda^\alpha}{S_*}\right) - \left(\frac{\lambda^\alpha I}{1 + \gamma^\alpha I} - \frac{\lambda^\alpha I_*}{1 + \gamma^\alpha I_*}\right)\right], \\
&= S\left[\frac{\Lambda^\alpha (S_* - S)}{S S_*} - \frac{\lambda^\alpha (I - I_*)}{(1 + \gamma^\alpha I)(1 + \gamma^\alpha I_*)}\right], \\
{}^{C}D^{\alpha}_{0+}I &= I\left[\frac{\lambda^\alpha S}{1 + \gamma^\alpha I} - \left(\mu_0^\alpha + \mu_1^\alpha + \beta^\alpha\right)\right] \\
&= I\left[\frac{\lambda^\alpha S}{1 + \gamma^\alpha I} - \frac{\lambda^\alpha S_*}{1 + \gamma^\alpha I_*}\right] \\
&= I\left[\frac{\lambda^\alpha (S - S_*) + \lambda^\alpha \gamma^\alpha (S I_* - I S_*)}{(1 + \gamma^\alpha I)(1 + \gamma^\alpha I_*)}\right] \\
&= I\left[\frac{\lambda^\alpha (S - S_*) + \lambda^\alpha \gamma^\alpha I_* (S - S_*) + \lambda^\alpha \gamma^\alpha S_* (I_* - I)}{(1 + \gamma^\alpha I)(1 + \gamma^\alpha I_*)}\right].
\end{aligned}
\tag{16}
$$

We consider a Lyapunov function $V^*: \Omega^* \to \mathbb{R}_+$ as follows

$$V^*(S,I) = \tau_1 \left[S - S_* - S_* \ln\left(\frac{S}{S_*}\right)\right] + \tau_2 \left[I - I_* - I_* \ln\left(\frac{I}{I_*}\right)\right], \tag{17}$$

where τ_1 and τ_2 are undetermined positive real numbers.

By Lemma 6, we deduce that there is a scalar class-$\mathcal{K}$ function γ_2 such that

$$\gamma_2(\|z\|) \le V^*(z), \quad z := \big(S - S_*,\ I - I_*\big).$$

Using Property 1, Lemma 4 and (16) we have the following estimate

$$\begin{aligned}
{}^C D^\alpha_{0+} V^* &= \tau_1 {}^C D^\alpha_{0+}\left[S - S_* - S_* \ln\left(\frac{S}{S_*}\right)\right] + \tau_2 {}^C D^\alpha_{0+}\left[I - I_* - I_* \ln\left(\frac{I}{I_*}\right)\right] \\
&\le \tau_1 \frac{S - S_*}{S}\left({}^C D^\alpha_{0+} S\right) + \tau_2 \frac{I - I_*}{I}\left({}^C D^\alpha_{0+} I\right) \\
&= \tau_1 (S - S_*)\left[\frac{\Lambda^\alpha (S_* - S)}{S S_*} - \frac{\lambda^\alpha (I - I_*)}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)}\right] \\
&+ \tau_2 (I - I_*)\left[\frac{\lambda^\alpha (S - S_*) + \lambda^\alpha \gamma^\alpha I_* (S - S_*) + \lambda^\alpha \gamma^\alpha S_* (I_* - I)}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)}\right] \\
&= -\tau_1 \frac{\Lambda^\alpha}{S S_*}(S - S_*)^2 - \tau_2 \frac{\lambda^\alpha \gamma^\alpha S_*}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)}(I - I_*)^2 \\
&+ \left[\tau_2 \frac{\lambda^\alpha + \lambda^\alpha \gamma^\alpha I_*}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)} - \tau_1 \frac{\lambda^\alpha}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)}\right](S - S_*)(I - I_*).
\end{aligned} \tag{18}$$

If τ_1 and τ_2 satisfy

$$\tau_2 \frac{\lambda^\alpha + \lambda^\alpha \gamma^\alpha I_*}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)} - \tau_1 \frac{\lambda^\alpha}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)} = 0,$$

or equivalently to

$$\tau_1 = \frac{\lambda^\alpha + \lambda^\alpha \gamma^\alpha I_*}{\lambda^\alpha}\tau_2,$$

then from (18) we have

$$\begin{aligned}
{}^C D^\alpha_{0+} V^* &\le -\tau_1 \frac{\Lambda^\alpha}{S S_*}(S - S_*)^2 - \tau_2 \frac{\lambda^\alpha \gamma^\alpha S_*}{(1+\gamma^\alpha I)(1+\gamma^\alpha I_*)}(I - I_*)^2 \\
&\le \tau_1 \frac{\Lambda^\alpha}{\eta_2 S_*}(S - S_*)^2 - \tau_2 \frac{\lambda^\alpha \gamma^\alpha S_*}{(1+\gamma^\alpha \eta_2)(1+\gamma^\alpha I_*)}(I - I_*)^2.
\end{aligned}$$

This estimate means that the function V defined by (17) satisfies the Theorem 3. This implies that $\widehat{E}_*$ is stable.

On the other hand, the boundedness of $S(t), I(t)$ and the fractional Barbalat lemma (Theorem 4) follow that

$$\lim_{t\to\infty} \big(S(t),\, I(t)\big) = \widehat{E}_* = (S_*,\, I_*).$$

Therefore, the GAS of $\widehat{E}_*$ is shown. The proof is completed. □

Combining Theorems 5 and 6 we obtain the complete GAS of the full model (2).

Theorem 7. *The DFE point of the model* (2) *is globally asymptotically stable whenever* $\mathcal{R}_0^\alpha \le 1$, *whereas, the DEE point is globally asymptotically stable whenever* $\mathcal{R}_0^\alpha > 1$.

15 Jan 2023 00:45:19 PST
220623-Hoang Version 2 - Submitted to Rocky Mountain J. Math.

As an important consequence of Theorems 5 and 6, we obtain the following result on the GAS of the ODE model (1).

Corollary 2. *The DFE point of the ODE model* (1) *is globally asymptotically stable if the basic reproduction number* $\mathcal{R}_0 := \dfrac{\lambda\Lambda}{(\mu_0+\nu)(\mu_0+\mu_1+\beta)} < 1$ *and the DEE point is globally asymptotically stable if* $\mathcal{R}_0 > 1$.

Proof. To prove the GAS of the DFE point, we consider a Lyapunov function of the form:

$$V_1(S,I) = \left[S - S_0 - S_0 \ln\left(\frac{S}{S_0}\right)\right] + I. \tag{19}$$

Meanwhile, the GAS of the DEE point can be obtained by using a Lyapunov function

$$V_2(S,I) = \tau_1\left[S - S_* - S_* \ln\left(\frac{S}{S_*}\right)\right] + \tau_2\left(I - I_* - I_* \ln\frac{I}{I_*}\right), \tag{20}$$

where

$$\tau_1 = \frac{\lambda + \lambda\gamma I_*}{\lambda}\tau_2.$$

After that, the GAS of the ODE model (1) will be obtained by repeating the proofs of Theorems 5 and 6. □

4. Numerical experiments

In this section, we report some numerical examples to support the theoretical results. For this purpose, we consider the fractional-order model (2) with the parameters given in Table 1.

Table 1: The values (per day) of the parameters used in numerical examples.

Case	Λ	γ	ν	β	λ	μ_0	μ_1	α	Source	$\mathcal{R}_0^\alpha$	GAS
1	0.5	0.8	0.004	0.9	0.005	0.001	0.05	0.90	Assumed	0.5198	$E_0 = (59.9213,\ 0,\ 208.6582)$
2	0.5	0.8	0.004	0.9	0.005	0.001	0.05	0.95	Assumed	0.5234	$E_0 = (77.4399,\ 0,\ 289.0158)$
3	0.5	0.8	0.004	0.9	0.005	0.001	0.05	0.99	Assumed	0.5254	$E_0 = (95.0211,\ 0,\ 374.8517)$
4	0.8	0.1	0.005	0.3	0.01	0.001	0.005	0.90	Assumed	3.5432	$E_* = (26.3177,\ 1.5536,\ 375.5126)$
5	0.8	0.1	0.005	0.3	0.01	0.001	0.005	0.95	Assumed	3.9334	$E_* = (30.9603,\ 1.7256,\ 532.0641)$
6	0.8	0.1	0.005	0.3	0.01	0.001	0.005	0.99	Assumed	4.2697	$E_* = (35.2526,\ 1.8652,\ 701.9766)$

In Table 1, the term "GAS" stands for the globally asymptotically stable equilibrium point.

We now apply a simple numerical method, namely the fractional Euler method (see [21, 40]), which uses the step size $h = 10^{-3}$ to solve the model (2). The solutions of the model (2) are depicted in Figures 1-6. In these figures, each blue curve depicts a phase space corresponding to a specific initial data, the green arrows represent the evolution of the model and the red circles refer to the position of the globally asymptotically stable equilibrium points. It is clear that the solutions are stable and converge to the equilibrium points; consequently, the GAS of the model is shown clearly.

From the numerical examples, we can see the affect of the fractional-order α on the behaviour of the HBV model. Hence, the fractional-order model (2) is more flexible than the ODE one (1) (with $\alpha = 1$) thanks to the appearance α. This may be useful in studying the parameter estimation problem with real data.

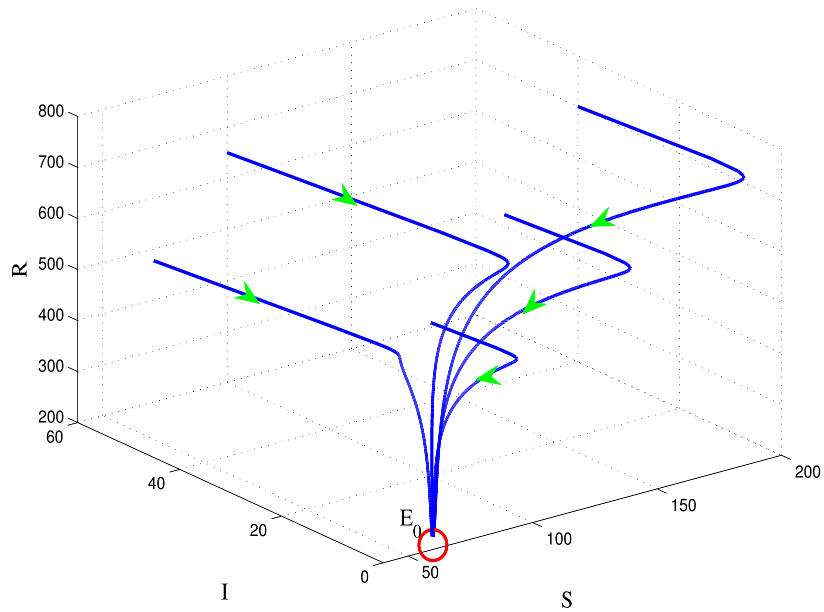

Figure 1: The phase spaces of the model (2) with the parameters given in Case 1.

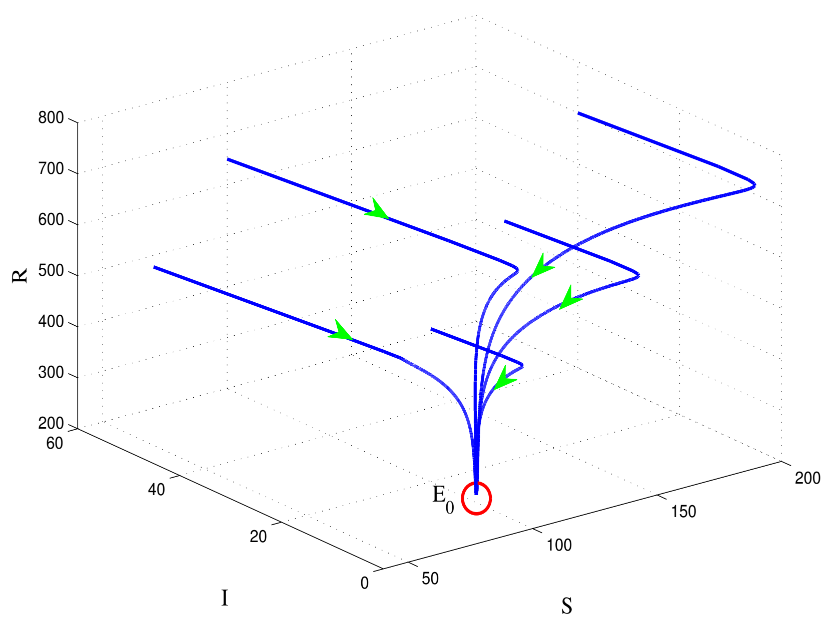

Figure 2: The phase spaces of the model (2) with the parameters given in Case 2.

15 Jan 2023 00:45:19 PST
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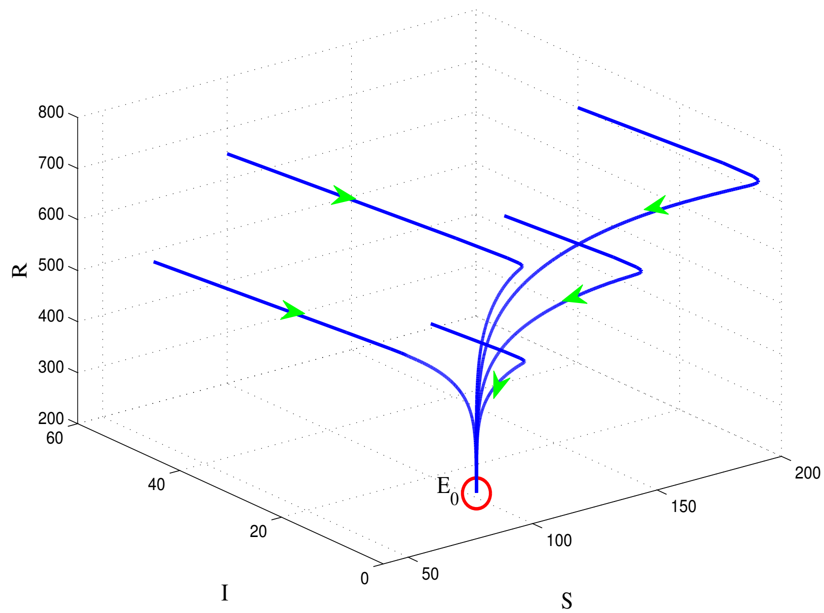

Figure 3: The phase spaces of the model (2) with the parameters given in Case 3.

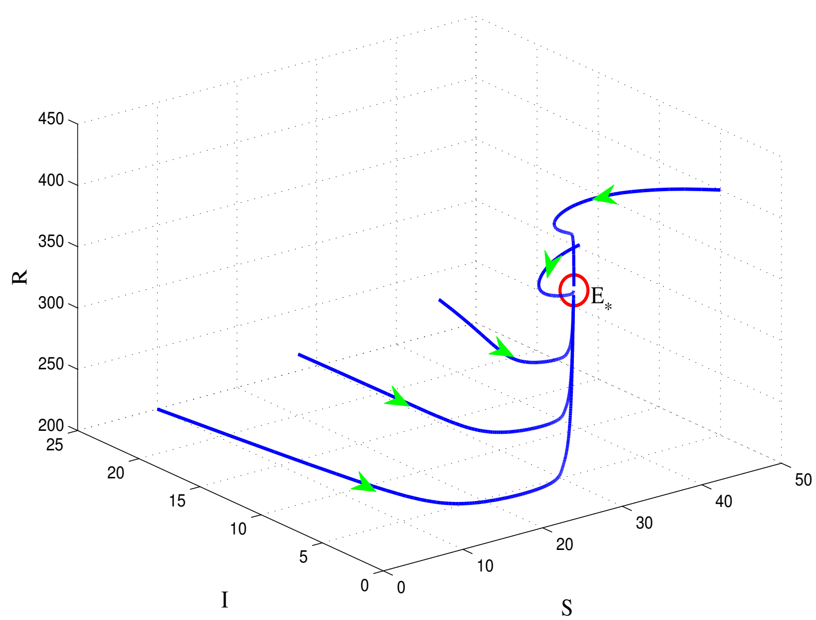

Figure 4: The phase spaces of the model (2) with the parameters given in Case 4.

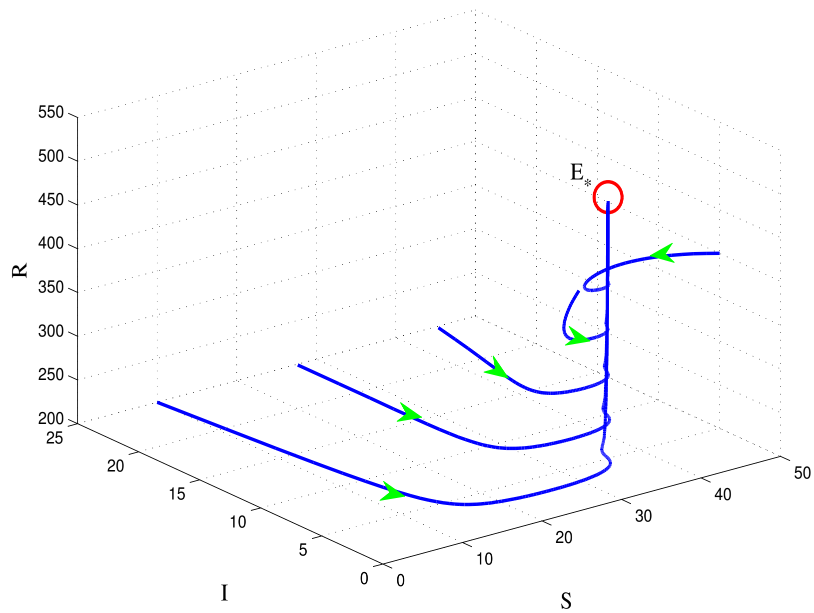

Figure 5: The phase spaces of the model (2) with the parameters given in Case 5.

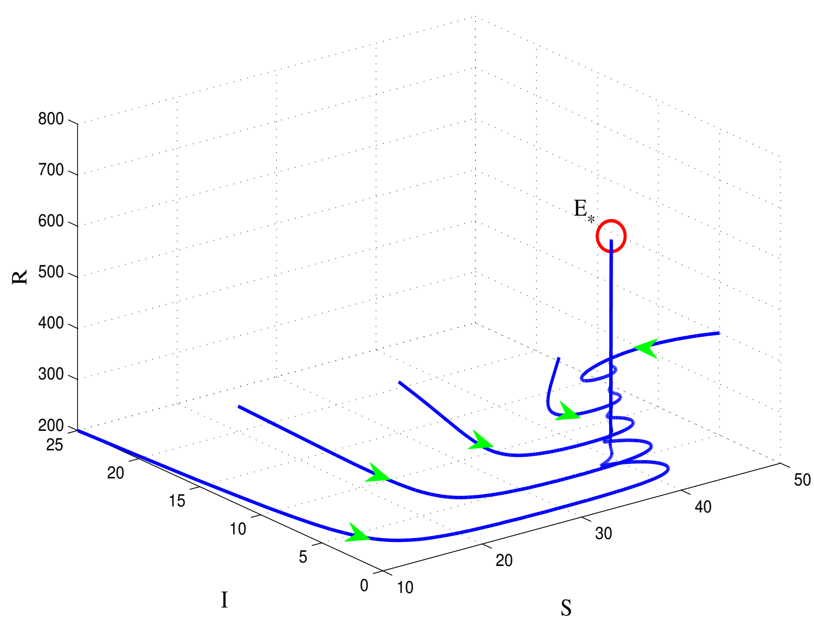

Figure 6: The phase spaces of the model (2) with the parameters given in Case 6.

5. Conclusions and discussions

In this work, we have provided a rigorous mathematical study for the GAS of the fractional-order hepatitis B epidemic model (2). Here, the GAS of the model was established by a simple approach, which is based on extensions of the Lyapunov stability theory and the fractional Barbalat's lemma in combination with some nonstandard techniques for fractional dynamical systems. The main result is that the GAS of the disease free and disease endemic equilibrium points was determined fully. Finally, the theoretical results were supported and illustrated by a set of numerical experiments.

The Lyapunov functions proposed in Theorems 5 and 6 are still appropriate to study the GAS of the ODE model (1). As an important consequence, we also obtain the complete GAS of the model (1). Hence, the obtained results provided an important improvement for the results formulated in [23] and [27].

It is well-known that the Lyapunov stability theory and its extensions can be considered as one of the most powerful and effective approaches to study the asymptotic stability of dynamical systems governed by ordinary and fractional differential equations. Therefore, the present approach in this work can be also suitable for other mathematical models having the same characteristics as the model (2).

It was proved in some previous works that fractional-order derivatives can have certain disadvantages and limitations when modeling real-world phenomena and processes (see, for instance, [6, 7, 13]). However, as emphasized above, the fractional-order model proposed in this work is more flexible than the ODE one (with $\alpha = 1$) thanks to the appearance α. In future works, we will consider disadvantages and limitations of the fractional-order model (2) and how to overcome them.

In the near future, we will extend the approach and results in this work to study stability properties of fractional-order differential equation models arising in real-world applications. Also, dynamics of the model (1) in the context of other fractional derivatives, such as the Riemann-Liouville fractional derivative [13, 46], the Caputo-Fabrizio fractional derivative [9], new fractional derivatives with non-local and non-singular kernel [5], new fractional derivative involving the normalized sinc function without singular kernel [52] and so on will be studied.

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