LINEAR MAPS ON NEST ALGEBRAS THAT ARE COMMUTANT PRESERVERS, DOUBLE COMMUTANT PRESERVERS OR LOCAL LIE CENTRALIZERS

HOGER GHAHRAMANI

ABSTRACT. Let $\mathcal{N}$ be a non-trivial nest on a Hilbert space H, $\mathcal{T}(\mathcal{N})$ be the associated nest algebra, and ϕ be a linear map on $\mathcal{T}(\mathcal{N})$. We prove, among other results, that ϕ preserves the commutant or the double commutant if and only if there is a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$. In fact, we show a more general result than this. Also, we show that ϕ is a local Lie centralizer if and only if there exists a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ vanishing on each commutator such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$.

1. Introduction, and statements of the results

An active research topic in mathematics is the linear preserver problems. By a linear preserver we mean a linear map of an algebra $\mathcal{A}$ into itself which preserves certain functions, subsets, relations or certain properties of some elements in $\mathcal{A}$. Linear preserver problems deal with the characterization of such maps. The reader is referred to [2, Chapter 7] and [1, 3, 9, 15] for an account of the topic and a list of references. Let $\mathcal{A}$ be a topological algebra, and $a \in \mathcal{A}$. We denote by $alg(a)$ the closure of the set of all polynomials in a. The *commutant* $\{a\}'$ of a is the following set: $\{a\}' = \{x \in \mathcal{A} : ax = xa\}$ and the *double commutant* $\{a\}''$ is the set of all elements of $\mathcal{A}$ commuting with any element in $\{a\}'$. Let X be a real or complex Banach space, and $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. In [9, Theorem 1], it is shown that $alg(\phi(T)) = alg(T)$ for every $T \in \mathcal{B}(X)$ if and only if $\{T\}' = \{\phi(T)\}'$ for every $T \in \mathcal{B}(X)$ if and only if $\{\phi(T)\}'' = \{T\}''$ for every $T \in \mathcal{B}(X)$ if and only if there is a scalar $\lambda \neq 0$ and a linear functional f on $\mathcal{B}(X)$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{B}(X)$. Let H be a real or complex Hilbert space. A *nest* $\mathcal{N}$ on H is a chain (with respect to inclusion) of closed subspaces of H such that $\{0\}$ and H lies in $\mathcal{N}$, and $\mathcal{N}$ is closed under the operations of taking arbitrary intersections and closed linear spans of its elements. The associated *nest algebra* $\mathcal{T}(\mathcal{N})$ is the algebra of all operators $T \in \mathcal{B}(H)$ which leave each element $N \in \mathcal{N}$

MSC(2020): 47L35, 47B49, 46B07, 15A86.

Keywords: Nest algebra, commutant preserver, double commutant preserver, local Lie centralizer.

11 Jul 2023 10:59:18 PDT
220623-Ghahramani Version 2 - Submitted to Rocky Mountain J. Math.

invariant. When $\mathcal{N} \neq \{\{0\}, H\}$, we say that the nest $\mathcal{N}$ is non-trivial (see [5]). Characterization of linear maps that are commutant preservers or double commutant preservers on nest algebras is one of the main topics of this article. In fact, we have the following result in this case.

Theorem 1.1. *Let $\mathcal{N}$ be an arbitrary non-trivial nest on a real or complex Hilbert space H, and $\mathcal{T}(\mathcal{N})$ be the associated nest algebra. Suppose that $\phi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is a linear map. The following conditions are equivalent:*
- *(i) $alg(\phi(T)) = alg(T)$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(ii) $\{T\}' = \{\phi(T)\}'$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(iii) $\{\phi(T)\}'' = \{T\}''$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(iv) there is a scalar $\lambda \neq 0$ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$.*

To prove Theorem 1.1, we prove the following more general result.

Theorem 1.2. *Let $\mathcal{N}$ be an arbitrary non-trivial nest on a real or complex Hilbert space H, and $\mathcal{T}(\mathcal{N})$ be the associated nest algebra. Suppose that $\phi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is a linear map. The following conditions are equivalent:*
- *(i) $alg(\phi(T)) \subseteq alg(T)$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(ii) $\{T\}' \subseteq \{\phi(T)\}'$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(iii) $\{\phi(T)\}'' \subseteq \{T\}''$ for every $T \in \mathcal{T}(\mathcal{N})$;*
- *(iv) $ST = TS = 0 \Longrightarrow S\phi(T) = \phi(T)S$ for $S, T \in \mathcal{T}(\mathcal{N})$;*
- *(v) there is a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$.*

In general, by routine verifications, it can be seen that each of the conditions (i), (ii) and (iii) of Theorem 1.2 implies condition (iv). But the following example shows that in general, condition (iv) does not give the conditions (i), (ii) and (iii).

Example 1.3. Let $\mathbb{C}$ be the complex field. By the following make $\mathbb{C} \times \mathbb{C}$ into a $\mathbb{C}$-bimodule:

$$a(b, c) := (ab, 0) \quad \text{and} \quad (b, c)a := (0, ca) \quad (a, b, c \in \mathbb{C}).$$

Then

$$\mathcal{A} := \left\{ \begin{pmatrix} a & (b, c) \\ 0 & a \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$$

is an algebra over $\mathbb{C}$, under the usual matrix operations. Define $\phi : \mathcal{A} \to \mathcal{A}$ by

$$\phi(\begin{pmatrix} a & (b, c) \\ 0 & a \end{pmatrix}) := \begin{pmatrix} 0 & (a, a) \\ 0 & 0 \end{pmatrix}.$$

The mapping ϕ is linear. Suppose that $S = \begin{pmatrix} a & (b, c) \\ 0 & a \end{pmatrix}$ and $T = \begin{pmatrix} a' & (b', c') \\ 0 & a' \end{pmatrix}$ are elements in $\mathcal{A}$ with $ST = TS = 0$. Therefore, $aa' = a'a = 0$ and by a simple calculation we have

$$S\phi(T) = \phi(T)S.$$

Let $T := \begin{pmatrix} 1 & (0,0) \\ 0 & 1 \end{pmatrix}$ and we see that

$$\phi(T)T = \begin{pmatrix} 0 & (0,1) \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & (1,0) \\ 0 & 0 \end{pmatrix} = T\phi(T).$$

Let $\mathcal{A}$ be an algebra. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is said to be a *Lie centralizer* if $\phi([a,b]) = [\phi(a), b]$ for any $a, b \in \mathcal{A}$, where $[a,b] = ab - ba$ is the Lie product of a and b in $\mathcal{A}$. The linear map $\phi : \mathcal{A} \to \mathcal{A}$ is called a *local Lie centralizer* if for each $a \in \mathcal{A}$ there is a Lie centralizer $\phi_a : \mathcal{A} \to \mathcal{A}$, depending on a, such that $\phi(a) = \phi_a(a)$. The study of local maps (a function that agrees at each point with some map that has the desired property) represents one of the most active research areas in operator theory. The interest in these types of maps has been aroused by two lines of research. One is the study of Hochschild cohomology for various operator algebras. In this regard, Kadison has proved in [10] that any local derivation on a von Neumann algebra is actually a derivation. Another is the study of the reflexivity of the space of linear maps from an algebra to itself. Also in this regard Larson in [12] has asked which algebras have a reflexive derivation space and Larson and Sourour [13] have studied local derivations and local automorphisms on $\mathcal{B}(X)$, where X is a Banach space. Then other local maps such as local Jordan derivations, local Lie derivations and local centralizers have also been considered and extensive studies have been done on local maps. The reader is referred to [4, 6, 7, 11, 14] and references therein for an account of the topic. Motivated by these developments, in the present article, we study the local Lie centralizers on nest algebras and we have the following result which will be proved by using the previous results.

Theorem 1.4. *Let $\mathcal{N}$ be an arbitrary non-trivial nest on a real or complex Hilbert space H, and $\mathcal{T}(\mathcal{N})$ be the associated nest algebra. Suppose that $\phi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is a linear map. The following conditions are equivalent:*

- (i) *ϕ is a Lie centralizer;*
- (ii) *ϕ is a local Lie centralizer;*
- (iii) *there is a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ vanishing on each commutator such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$.*

In the next section we give the proof of the above theorems.

2. **Proofs**

Through this section $\mathbb{F}$ is the real or complex field, H is a Hilbert space over $\mathbb{F}$, $\mathcal{N}$ is a non-trivial nest on H, and $\mathcal{T}(\mathcal{N})$ is the associated nest algebra. If N is a closed subspace of H, then P_N denotes the orthonormal projection onto N. Also, I is the identity operator on H. In order to prove our results we need the following results which are given in [5, Chapter 2].

Lemma 2.1. *$Z(\mathcal{T}(\mathcal{N})) = \mathbb{F}I$, where $Z(\mathcal{T}(\mathcal{N}))$ is the center of $\mathcal{T}(\mathcal{N})$.*

11 Jul 2023 10:59:18 PDT
220623-Ghahramani Version 2 - Submitted to Rocky Mountain J. Math.

Lemma 2.2. *If $\mathcal{N}$ is a non-trivial nest on H, and $N \in \mathcal{N} \setminus \{\{0\}, H\}$, then $P_N \mathcal{N} P_N$ and $(I - P_N)\mathcal{N}(I - P_N)$ are nests and $\mathcal{T}(P_N \mathcal{N} P_N) = P_N \mathcal{T}(\mathcal{N}) P_N$ and $\mathcal{T}((I - P_N)\mathcal{N}(I - P_N)) = (I - P_N)\mathcal{T}(\mathcal{N})(I - P_N)$.*

Lemma 2.3. *Let H be a Hilbert space over $\mathbb{F}$, $\mathcal{N}$ be a non-trivial nest on H, and $N \in \mathcal{N} \setminus \{\{0\}, H\}$. Suppose that $A \in P_N \mathcal{T}(\mathcal{N}) P_N$ and $B \in (I - P_N)\mathcal{T}(\mathcal{N})(I - P_N)$. If $AT = TB$ for any $T \in P_N \mathcal{T}(\mathcal{N})(I - P_N)$, then there exists $\lambda \in \mathbb{F}$ such that $A = \lambda P_N$ and $B = \lambda(I - P_N)$.*

Firstly, we prove the Theorem 1.2.

Proof of Theorem 1.2: (ii)$\Leftrightarrow$(iii), (ii)$\Rightarrow$(iv), and (v)$\Rightarrow$(i) are clear. We only prove the following cases.

(i)$\Rightarrow$(ii): Let $T \in \mathcal{T}(\mathcal{N})$. It follows from $alg(\phi(T)) \subseteq alg(T)$ that $\phi(T) = \lim_{n\to\infty} p_n(T)$ (in norm), where each $p_n(T)$ is a polynomial in T. Assume that $S \in \mathcal{T}(\mathcal{N})$ and $ST = TS$. So

$$S\phi(T) = \lim_{n\to\infty} Sp_n(T) = \lim_{n\to\infty} p_n(T)S = \phi(T)S.$$

(iv)$\Rightarrow$(v): We choose a non-trivial element $N \in \mathcal{N}$ and set $P_1 = P_N$ and $P_2 = P_{N^\perp} = I - P_N$. Let $\mathcal{T}_{ij} = P_i \mathcal{T}(\mathcal{N}) P_j$, $1 \le i, j \le 2$. Then $\mathcal{T}_{21} = 0$ and we can write $\mathcal{T}(\mathcal{N}) = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$, which is the Pierce decompositions of $\mathcal{T}(\mathcal{N})$. For an element $T_{ij} \in \mathcal{T}(\mathcal{N})$, we always mean that $T_{ij} \in \mathcal{T}_{ij}$. We complete the proof by checking some steps.

Step 1. $P_1\phi(T_{11})P_2 = 0$ and $P_1\phi(T_{22})P_2 = 0$ for all $T_{11} \in \mathcal{T}_{11}$ and $T_{22} \in \mathcal{T}_{22}$.

Let $T_{11} \in \mathcal{T}_{11}$. We have $P_2T_{11} = T_{11}P_2 = 0$ and hence $P_2\phi(T_{11}) = \phi(T_{11})P_2$. So $P_1\phi(T_{11})P_2 = 0$. Similarly, it follows from $P_1T_{22} = T_{22}P_1 = 0$ that $P_1\phi(T_{22})P_2 = 0$ for all $T_{22} \in \mathcal{T}_{22}$.

Step 2. For each $T_{11} \in \mathcal{T}_{11}$ and $T_{22} \in \mathcal{T}_{22}$ there are unique scalars $\alpha, \beta \in \mathbb{F}$, respectively, such that $P_2\phi(T_{11})P_2 = \alpha P_2$ and $P_1\phi(T_{22})P_1 = \beta P_1$.

Let $T_{11} \in \mathcal{T}_{11}$ be fixed. For any $T_{22} \in \mathcal{T}_{22}$ we have $T_{22}T_{11} = T_{11}T_{22} = 0$, which implies $T_{22}\phi(T_{11}) = \phi(T_{11})T_{22}$. Hence $T_{22}\phi(T_{11})P_2 = P_2\phi(T_{11})T_{22}$, and so $P_2\phi(T_{11})P_2 \in Z(\mathcal{T}_{22})$. By Lemma 2.2, $\mathcal{T}_{22} = \mathcal{T}(P_2\mathcal{N}P_2)$ and from Lemma 2.1 it follows that $P_2\phi(T_{11})P_2 = \alpha P_2$ for some $\alpha \in \mathbb{F}$ (P_2 is the unity of $\mathcal{T}_{22}$). Similarly, we can show that for each $T_{22} \in \mathcal{T}_{22}$ there is a scalar $\beta \in \mathbb{F}$ such that $P_1\phi(T_{22})P_1 = \beta P_1$. The uniqueness of α and β are clear.

Step 3. For each $T_{12} \in \mathcal{T}_{12}$ there is a unique scalar $\gamma \in \mathbb{F}$ such that $\phi(T_{12}) - P_1\phi(T_{12})P_2 = \gamma I$.

For any $T_{12}, S_{12} \in \mathcal{T}_{12}$ we have $S_{12}T_{12} = T_{12}S_{12} = 0$. This implies that $S_{12}\phi(T_{12}) = \phi(T_{12})S_{12}$. Multiplying this identity by P_1 from the left and by

P_2 from the right, we get

$$S_{12}\phi(T_{12})P_2 = P_1\phi(T_{12})S_{12}$$

for all $T_{12}, S_{12} \in \mathcal{T}_{12}$. It follows from Lemma 2.3 that $P_1\phi(T_{12})P_1 = \gamma P_1$ and $P_2\phi(T_{12})P_2 = \gamma P_2$ for some $\gamma \in \mathbb{F}$. Hence

$$\phi(T_{12}) - P_1\phi(T_{12})P_2 = P_1\phi(T_{12})P_1 + P_2\phi(T_{12})P_2 = \gamma I$$

for $T_{12} \in \mathcal{T}_{12}$. The uniqueness of γ is clear.

It follows from Step 2 and linearity of ϕ that there exist linear maps $f_1 : \mathcal{T}_{11} \to \mathbb{F}$ and $f_2 : \mathcal{T}_{22} \to \mathbb{F}$ such that

$$P_2\phi(T_{11})P_2 = f_1(T_{11})P_2 \quad \text{and} \quad P_1\phi(T_{22})P_1 = f_2(T_{22})P_1$$

for all $T_{11} \in \mathcal{T}_{11}$ and $T_{22} \in \mathcal{T}_{22}$. Now, defne two linear maps $\varphi : \mathcal{T}_{11} \to \mathcal{T}_{11}$ and $\psi : \mathcal{T}_{22} \to \mathcal{T}_{22}$ by

$$\varphi(T_{11}) = P_1\phi(T_{11})P_1 - f_1(T_{11})P_1 \quad \text{and} \quad \psi(T_{22}) = P_2\phi(T_{22})P_2 - f_2(T_{22})P_2.$$

Step 4. For each $T_{ij} \in \mathcal{T}_{ij}$, $1 \leq i, j \leq 2$, we have

$$P_1\phi(T_{11}T_{12})P_2 = T_{11}\phi(T_{12})P_2 = \varphi(T_{11})T_{12}$$

and

$$P_1\phi(T_{12}T_{22})P_2 = P_1\phi(T_{12})T_{22} = T_{12}\psi(T_{22}).$$

For any $T_{11} \in \mathcal{T}_{11}$ and $T_{12} \in \mathcal{T}_{12}$, let $S = T_{12} - P_2$ and $T = T_{11} + T_{11}T_{12}$. Since $ST = TS = 0$, it follows that $(T_{12} - P_2)\phi(T_{11} + T_{11}T_{12}) = \phi(T_{11} + T_{11}T_{12})(T_{12} - P_2)$. Multiplying this identity by P_1 from the left and by P_2 from the right and using Steps 1 and 3, we get

$$P_1\phi(T_{11}T_{12})P_2 = P_1\phi(T_{11})T_{12} - T_{12}\phi(T_{11})P_2.$$

Then by definitions of f_1 and φ,

$$P_1\phi(T_{11}T_{12})P_2 = P_1\phi(T_{11})T_{12} - f_1(T_{11})T_{12} = \varphi(T_{11})T_{12}.$$

Taking $T_{11} = P_1$ in above identity, we arrive at

$$P_1\phi(T_{12})P_2 = \varphi(P_1)T_{12}.$$

So for any $T_{22} \in \mathcal{T}_{22}$ and $T_{12} \in \mathcal{T}_{12}$ we have

$$P_1\phi(T_{12}T_{22})P_2 = \varphi(P_1)T_{12}T_{22} = P_1\phi(T_{12})T_{22}.$$

For any $T_{22} \in \mathcal{T}_{22}$ and $T_{12} \in \mathcal{T}_{12}$, let $C = P_1 + T_{12}$ and $D = T_{12}T_{22} - T_{22}$. So $CD = DC = 0$ and hence $(P_1 + T_{12})\phi(T_{12}T_{22} - T_{22}) = \phi(T_{12}T_{22} - T_{22})(P_1 + T_{12})$. Multiplying this identity by P_1 from the left and by P_2 from the right and using Steps 1 and 3, we get

$$P_1\phi(T_{12}T_{22})P_2 = T_{12}\phi(T_{22})P_2 - P_1\phi(T_{22})T_{12}.$$

Now, by definitions of f_2 and ψ we have

$$P_1\phi(T_{12}T_{22})P_2 = T_{12}\phi(T_{22})P_2 - T_{12}f_2(T_{22})P_2 = T_{12}\psi(T_{22}).$$

It follows from the above identity that for any $T_{11} \in \mathcal{T}_{11}$ and $T_{12} \in \mathcal{T}_{12}$,

$$P_1\phi(T_{11}T_{12})P_2 = T_{11}T_{12}\psi(P_2) = T_{11}\phi(T_{12})P_2.$$

Step 5. For each $T_{11} \in \mathcal{T}_{11}$ and $T_{22} \in \mathcal{T}_{22}$ we have

$$\varphi(T_{11}) = \varphi(P_1)T_{11} = T_{11}\varphi(P_1) \quad \text{and} \quad \psi(T_{22}) = \psi(P_2)T_{22} = T_{22}\psi(P_2).$$

From Step 4 we have $P_1\phi(T_{12})P_2 = \varphi(P_1)T_{12}$ for all $T_{12} \in \mathcal{T}_{12}$. Hence $P_1\phi(T_{11}T_{12})P_2 = \varphi(P_1)T_{11}T_{12}$ and $T_{11}\phi(T_{12})P_2 = T_{11}\varphi(P_1)T_{12}$ for all $T_{11} \in \mathcal{T}_{11}$ and $T_{12} \in \mathcal{T}_{12}$. Therefore

$$(\varphi(T_{11}) - \varphi(P_1)T_{11})T_{12} = (\varphi(T_{11}) - T_{11}\varphi(P_1))T_{12} = 0$$

for all $T_{11} \in \mathcal{T}_{11}$ and $T_{12} \in \mathcal{T}_{12}$. Since $N \in \mathcal{N}$ is non-trivial, it follows from Lemma 2.3 that $\varphi(T_{11}) = \varphi(P_1)T_{11} = T_{11}\varphi(P_1)$ for all $T_{11} \in \mathcal{T}_{11}$. Similarly, we can prove that $\psi(T_{22}) = \psi(P_2)T_{22} = T_{22}\psi(P_2)$ for all $T_{22} \in \mathcal{T}_{22}$.

Step 6. There exists a scalar $\lambda \in \mathbb{F}$ such that

$$\varphi(T_{11}) + P_1\phi(T_{12})P_2 + \psi(T_{22}) = \lambda T$$

for all $T = T_{11} + T_{12} + T_{22} \in \mathcal{T}(\mathcal{N})$.

Define the linear map $\sigma : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ by $\sigma(T) = \varphi(T_{11}) + P_1\phi(T_{12})P_2 + \psi(T_{22})$ where $T = T_{11} + T_{12} + T_{22} \in \mathcal{T}(\mathcal{N})$. It follows from Steps 4 and 5 that

$$\begin{aligned} T\sigma(I) &= (T_{11} + T_{12} + T_{22})(\sigma(P_1) + \sigma(P_2)) \\ &= (T_{11} + T_{12} + T_{22})(\varphi(P_1) + \psi(P_2)) \\ &= T_{11}\varphi(P_1) + T_{12}\psi(P_2) + T_{22}\psi(P_2) \\ &= \varphi(T_{11}) + P_1\phi(T_{12})P_2 + \psi(T_{22}) \\ &= \sigma(T) \end{aligned}$$

for all $T = T_{11} + T_{12} + T_{22} \in \mathcal{T}(\mathcal{N})$. By using a similar methods as above we can show that $\sigma(T) = \sigma(I)T$ for all $T \in \mathcal{T}(\mathcal{N})$. So $\sigma(I) \in Z(\mathcal{T}(\mathcal{N})) = \mathbb{F}I$. Therefore $\sigma(I) = \lambda I$ for some $\lambda \in \mathbb{F}$ and

$$\varphi(T_{11}) + P_1\phi(T_{12})P_2 + \psi(T_{22}) = \sigma(T) = \lambda T$$

for all $T = T_{11} + T_{12} + T_{22} \in \mathcal{T}(\mathcal{N})$.

Step 7. This is the final step in the proof that (iv) implies (v).

11 Jul 2023 10:59:18 PDT
220623-Ghahramani Version 2 - Submitted to Rocky Mountain J. Math.

By Step 3 and linearity of ϕ there exist a linear map $f_3 : \mathcal{T}_{12} \to \mathbb{F}$ such that

$$P_1\phi(T_{12})P_1 + P_2\phi(T_{12})P_2 = f_3(T_{12})I$$

for all $T_{12} \in \mathcal{T}_{12}$. Define the linear functional $f : \mathcal{T}(\mathcal{N}) \to \mathbb{F}$ by

$$f(T) = f_1(T_{11}) + f_2(T_{22}) + f_3(T_{12})$$

where $T = T_{11}+T_{12}+T_{22} \in \mathcal{T}(\mathcal{N})$. By using the definitions of f_j $(1 \le j \le 3)$, f, φ, ψ and Steps 1-6 we have

$$\begin{aligned}
\phi(T) &= P_1\phi(T_{11})P_1 + P_2\phi(T_{11})P_2 \\
&\quad + P_1\phi(T_{22})P_1 + P_2\phi(T_{22})P_2 \\
&\quad + P_1\phi(T_{12})P_1 + P_1\phi(T_{12})P_2 + P_2\phi(T_{12})P_2 \\
&= P_1\phi(T_{11})P_1 + f_1(T_{11})P_2 \\
&\quad + f_2(T_{22})P_1 + P_2\phi(T_{22})P_2 \\
&\quad + P_1\phi(T_{12})P_2 + f_3(T_{12})I \\
&= P_1\phi(T_{11})P_1 - f_1(T_{11})P_1 + f_1(T_{11})I \\
&\quad + P_2\phi(T_{22})P_2 - f_2(T_{22})P_2 + f_2(T_{22})I \\
&\quad + P_1\phi(T_{12})P_2 + f_3(T_{12})I \\
&= \varphi(T_{11}) + \psi(T_{22}) + P_1\phi(T_{12})P_2 + f(T)I \\
&= \lambda T + f(T)I
\end{aligned}$$

where $T = T_{11} + T_{12} + T_{22} \in \mathcal{T}(\mathcal{N})$. The proof of theorem is completed. ■

Proof of Theorem 1.1: (ii)⇔(iii) and (iv)⇒(i) are clear. (i)⇒(ii) is proved in a similar way to (i)⇒(ii) of Theorem 1.2. We only show (ii)⇒(iv).

(ii)⇒(iv): It follows from Theorem 1.2 that there is a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$. Suppose that $\lambda = 0$. From hypothesis for any $T \in \mathcal{T}(\mathcal{N})$ we have

$$\{T\}' = \{\phi(T)\}' = \{f(T)I\}' = \mathcal{T}(\mathcal{N}).$$

Hence $\mathcal{T}(\mathcal{N}) = Z(\mathcal{T}(\mathcal{N})) = \mathbb{F}I$. This contradicts the non-triviality of $\mathcal{N}$. Therefore it should $\lambda \neq 0$. The proof of theorem is completed. ■

Proof of Theorem 1.4: (i)⇒(ii) and (iii)⇒(i) are clear. We prove the following cases.

(i)⇒(iii): Let $T \in \mathcal{T}(\mathcal{N})$. For any $S \in \{T\}'$ we have $[T, S] = 0$ and hence $[\phi(T), S] = 0$. So $S \in \{\phi(T)\}'$. Therefore $\{T\}' \subseteq \{\phi(T)\}'$ and by Theorem 1.2 there exists a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$. By the fact that ϕ is a centralizer, for any $T, S \in \mathcal{T}(\mathcal{N})$ we have

$$\begin{aligned}
f([T,S])I &= \phi([T,S]) - \lambda[T,S] \\
&= [\phi(T), S] - \lambda[T,S] \\
&= [\lambda T + f(T)I, S] - \lambda[T,S] = 0.
\end{aligned}$$

So f vanishes on each commutator.

(ii)$\Rightarrow$(iii): Let $T \in \mathcal{T}(\mathcal{N})$. There is a Lie centralizer $\phi_T : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ such that $\phi(T) = \phi_T(T)$. So for any $S \in \{T\}'$ we have

$$[\phi(T), S] = [\phi_T(T), S] = \phi_T([T, S]) = 0.$$

Hence $\{T\}' \subseteq \{\phi(T)\}'$ and by Theorem 1.2 there is a scalar λ and a linear functional f on $\mathcal{T}(\mathcal{N})$ such that $\phi(T) = \lambda T + f(T)I$ for every $T \in \mathcal{T}(\mathcal{N})$. Suppose that $T, S \in \mathcal{T}(\mathcal{N})$ are arbitrary. There is a Lie centralizer $\phi_{[T,S]}$ on $\mathcal{T}(\mathcal{N})$ such that $\phi([T, S]) = \phi_{[T,S]}([T, S])$. It follows from (i)$\Rightarrow$(iii) that there exists a scalar γ and a linear functional g on $\mathcal{T}(\mathcal{N})$ such that $\phi_{[T,S]}(R) = \gamma R + g(R)I$ for every $R \in \mathcal{T}(\mathcal{N})$. So

$$\begin{aligned}\lambda[T, S] + f([T, S])I &= \phi([T, S]) = \phi_{[T,S]}([T, S]) \\ &= [\phi_{[T,S]}(T), S] = [\gamma T + g(T)I, S] = \gamma[T, S].\end{aligned}$$

So $(\gamma - \lambda)[T, S] = f([T, S])I$. It follows from the Kleinecke–Shirokov Theorem (cf. [8, Problem 230]) that $f([T, S]) = 0$. Thus f vanishes on each commutator. The proof of theorem is completed. ■

Acknowledgment. The author likes to express his sincere thanks to the referee(s) for this paper.

References

[1] H. Benbouziane, Y. Bouramdane, M. Ech-Cherif El Kettani and A. Lahssaini, *Non-linear commutant preservers*, Linear Multilinear Algebra, 66 (2018), 593–601.

[2] M. Brešar, M.A. Chebotar and W.S. Martindale 3rd, *Functional Identities*, Birkhäuser Verlag, 2007.

[3] M. Brešar and P. Šemrl, *Linear preservers on $\mathcal{B}(X)$*, Linear operators (Warsaw, 1994), Banach Center Publ., Vol. 38. Warsaw: Polish Acad. Sci.; 1997, 49–58.

[4] L. Chen and F. Lu, *Local Lie derivations of nest algebras*, Linear Algebra Appl. 475 (2015), 62–72.

[5] K.R. Davision, *Nest Algebras*, Pitman Research Notes in Mathematics Series, vol. 191, Longman Scientific and Technical, Burnt Mill Harlow, Essex, UK, 1988.

[6] H. Ghahramani and S. Sattari, *Characterization of reflexive closure of some operator algebras acting on Hilbert C^*-modules*, Acta Math. Hungar. 157(1) (2019), 158–172.

[7] H. Ghahramani, *Left ideal preserving maps on triangular algebras*, Iran. J. Sci. Technol. Trans. Sci. 44 (2020), 109–118.

[8] P.R. Halmos, *A Hilbert Space Problem Book*, 2nd edition, Springer-Verlag, New York/Heidelberg/Berlin, 1982.

[9] A.A. Jafarian and A.R. Sourour, *Linear maps that preserve the commutant, double commutant or the lattice of invariant subspaces*, Linear Multilinear Algebra, 38 (1994), 117–129.

[10] R. Kadison, *Local derivations*, J. Algebra 130 (1990), 494–509.

[11] E. Katsoulis, *Local maps and the representation theory of operator algebras*, Trans. Amer. Math. 368 (2016), 5377–5397.

[12] D. Larson, *Reflexivity, algebraic reflexivity, and linear interpolation*, Amer. J. Math. 110 (1988), 283–299.

[13] D. Larson and A. Sourour, *Local derivations and local automorphisms*, Proc. Symp. Pure Math. 51 (1990), 187–194.

[14] L. Liu, *On local Lie derivations of generalized matrix algebras*, Banach J. Math. Anal. 14 (2020), 249–268.

[15] L. Molnar, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Springer-Verlag Berlin Heidelberg, 2007.

Department of Mathematics, Faculty of Science, University of Kurdistan, P.O. Box 416, Sanandaj, Kurdistan, Iran.

Email address: h.ghahramani@uok.ac.ir; hoger.ghahramani@yahoo.com