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Integral operators in a complex setting

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Abstract Abel et al. recently defined the generalization of some exponential-type operators depending on a parameter $\sigma > 0$. In this article, we consider the summation-integral type operator based on semi-exponential Baskakov operators and with weights based on the semi-exponential Szász operators. In section 2, we study the approximation properties in the real domain, and in section 3, we make the convergence estimation in a complex setting.

Keywords semi-exponential Baskakov operators · compact disk · exact order of approximation · quantitative asymptotic formula.

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1 Introduction

Since the introduction of exponential-type operators four decades ago (see [21]), no new exponential operators have been presented or explored by the researchers, despite various extensions being offered and studied (for eg. [1], [5]). Tyliba and Wachnicki [23] introduced the use of parameter $\sigma > 0$ to generalize exponential-type operators and created semi-exponential equivalents of

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the two well-known operators Szász-Mirakyan and Gauss-Weierstrass operators. Because of their complex nature, limited study has been done on these operators. Herzog [20] extended the studies and introduced semi exponential equivalent of Post-Widder operators. Very recently, Abel et al [2] proposed some other such operators. The semi-exponential Szász-Mirakyan operators introduced in [2] can be defined by

$$\left(S_n^\sigma g\right)(u)=\sum_{j=0}^{\infty} s_{n,j}^\sigma(u)\, g\left(\frac{j}{n}\right), \tag{1}$$

where

$$s_{n,j}^\sigma(u)=e^{-(\sigma+n)u}\frac{((\sigma+n)\,u)^j}{j!}.$$

The kernel of these operators satisfy the following differential equation:

$$u(D+\sigma)s_{n,j}^\sigma(u)=(j-nu)s_{n,j}^\sigma(u)\,, D\equiv\frac{d}{du}.$$

The semi-exponential Baskakov operators proposed in [2] are given by

$$\left(V_n^\sigma g\right)(u)=\sum_{j=0}^{\infty} b_{n,j}^\sigma(u)\, g\left(\frac{j}{n}\right), \tag{2}$$

where

$$b_{n,j}^\sigma(u)=\sum_{i+\ell=j}\frac{(n+i)_\ell}{j!}\binom{j}{i}\sigma^i\frac{u^j}{(1+u)^{n+j}}e^{-\sigma u}.$$

The kernel $b_{n,j}^\sigma(u)$ of semi-exponential Baskakov operators satisfy the differential equation:

$$u(1+u)(D+\sigma)b_{n,j}^\sigma(u)=(j-nu)b_{n,j}^\sigma(u)\,, D\equiv\frac{d}{du}.$$

We have $i=0$ and $\ell=j$ in the specific case $\sigma=0$, resulting in the Baskakov operators.

We now define the summation-integral type operator based on the semi-exponential Baskakov operators and with weights based on the semi-exponential Szász-Mirakyan operators as follows:

$$\left(G_n^\sigma g\right)(u)=(\sigma+n)\sum_{j=0}^{\infty} b_{n,j}^\sigma(u)\int_0^\infty s_{n,j}^\sigma(\nu)g(\nu)d\nu, \tag{3}$$

where $s_{n,j}^\sigma(\nu)$ and $b_{n,j}^\sigma(u)$ are as defined in (1) and (2) respectively.

2 Convergence estimates in real domain

Lemma 1 *For $k \in \mathbb{N} \cup \{0\}$, if we denote $\Theta^{\sigma}_{n,k}(u) = (G^{\sigma}_n e_k)(u)$, then*

$$\Theta^{\sigma}_{n,k+1}(u) = \frac{u(1+u)}{\sigma+n}[\Theta^{\sigma}_{n,k}(u)]' + \frac{k+1+nu+\sigma u(1+u)}{\sigma+n}\Theta^{\sigma}_{n,k}(u),$$

where $e_k := e_k(t) = t^k$.

Proof By definition

$$\begin{aligned}
&u(1+u)(D+\sigma)\Theta^{\sigma}_{n,k}(u)\\
&= (\sigma+n)\sum_{j=0}^{\infty}(j-nu)b^{\sigma}_{n,j}(u)\int_0^{\infty} s^{\sigma}_{n,j}(\nu)\nu^k d\nu\\
&= (\sigma+n)\sum_{j=0}^{\infty} b^{\sigma}_{n,j}(u)\int_0^{\infty}[j-(\sigma+n)\nu]s^{\sigma}_{n,j}(\nu)\nu^k d\nu + (\sigma+n)\Theta^{\sigma}_{n,k+1}(u) - nu\Theta^{\sigma}_{n,k}(u)\\
&= (\sigma+n)\sum_{j=0}^{\infty} b^{\sigma}_{n,j}(u)\int_0^{\infty}[s^{\sigma}_{n,j}(\nu)]'\nu^{k+1} d\nu + (\sigma+n)\Theta^{\sigma}_{n,k+1}(u) - nu\Theta^{\sigma}_{n,k}(u)\\
&= -(k+1)\Theta^{\sigma}_{n,k}(u) + (\sigma+n)\Theta^{\sigma}_{n,k+1}(u) - nu\Theta^{\sigma}_{n,k}(u),
\end{aligned}$$

implying

$$(\sigma+n)\Theta^{\sigma}_{n,k+1}(u) = u(1+u)(D+\sigma)\Theta^{\sigma}_{n,k}(u) + (k+1+nu)\Theta^{\sigma}_{n,k}(u).$$

Remark 1 Using Lemma 1, we conclude that for certain constants $\wp_i, i = 0,1,2,..., \wp_i \neq 0$

$$\begin{aligned}
\left(G^{\sigma}_n \sum_{i\geq 0}\wp_i e_i\right)(u) &= \wp_0 + \wp_1\left(u + \frac{\sigma u^2+1}{\sigma+n}\right)\\
&\quad +\wp_2\left(u^2 + \frac{\sigma^2u^4}{(\sigma+n)^2} + \frac{2\sigma(\sigma+n+1)u^3}{(\sigma+n)^2} + \frac{(n+6\sigma)u^2}{(\sigma+n)^2} + \frac{4u}{(\sigma+n)} + \frac{2}{(\sigma+n)^2}\right)\\
&\quad +....
\end{aligned}$$

Furthermore, if we denote $\mu^{\sigma}_{n,j}(u) = (G^{\sigma}_n(\nu-u)^j)(u)$, then by basic computations, for some non-zero constants $\varpi_j, j = 0,1,2,...$, the central moments satisfy

$$\begin{aligned}
\sum_{j\geq 0}\varpi_j\mu^{\sigma}_{n,j}(u) &= \varpi_0 + \varpi_1\left(\frac{\sigma u^2+1}{\sigma+n}\right)\\
&\quad +\varpi_2\left(\frac{\sigma^2u^4}{(\sigma+n)^2} + \frac{2\sigma u^3}{(\sigma+n)^2} + \frac{(n+6\beta)u^2}{(\sigma+n)^2} + \frac{2u}{(\sigma+n)} + \frac{2}{(\sigma+n)^2}\right) + ...
\end{aligned}$$

Now, we study the quantitative Voronovskaja-kind estimate which have been studied in different forms by various researchers (see for instance [6], [11], [12], [3]). Consider $\widehat{C}[0,\infty)$, the space of real-valued continuous functions $g(u)$ on $[0,\infty)$ such that $\lim_{u\to\infty} g(u)$ exists and is finite. Further, for $\varrho \geq 0$, consider the following weighted modulus of continuity:

$$\widehat{\omega}(g,\varrho) := \sup_{u,\nu\in[0,\infty)} \left\{|g(u)-g(\nu)| : |e^{-u}-e^{-\nu}| \leq \varrho\right\}$$

Theorem 1 *Let $g, g'' \in \widehat{C}[0,\infty)$. Then, for any $u \geq 0$, we have*

$$\begin{aligned}&\left|n\left[(G_n^\sigma g)(u)-g(u)\right]-(\sigma u^2+1)g'(u)-\frac{u(u+2)}{2}g''(u)\right|\\ &\leq |\alpha_n^\sigma(u)|\cdot|g'(u)|+|\gamma_n^\sigma(u)|\cdot|g''(u)|+2\cdot\left[2\gamma_n^\sigma(u)+u(u+2)+\delta_n^\sigma(u)\right],\end{aligned}$$

where

$$\begin{aligned}\alpha_n^\sigma(u) &= \frac{n(\sigma u^2+1)}{\sigma+n}-(\sigma u^2+1)\\ \gamma_n^\sigma(u) &= \frac{n}{2}\cdot\left(\frac{\sigma^2u^4}{(\sigma+n)^2}+\frac{2\sigma u^3}{(\sigma+n)^2}+\frac{(n+6\beta)u^2}{(\sigma+n)^2}+\frac{2u}{(\sigma+n)}+\frac{2}{(\sigma+n)^2}\right)-\frac{u(u+2)}{2}\\ \delta_n^\sigma(u) &= \sqrt{n^2\left(G_n^\sigma(e^{-u}-e^{-\nu})^4\right)(u)}\sqrt{n^2\left(G_n^\sigma(\nu-u)^4\right)(u)}.\end{aligned}$$

Proof Using Taylor's expansion of function g at the point $u\in[0,\infty)$ gives us the following:

$$\begin{aligned}&\left|n\left[(G_n^\sigma g)(u)-g(u)\right]-(\sigma u^2+1)g'(u)-\frac{u(u+2)}{2}g''(u)\right|\\ &\leq |\alpha_n^\sigma(u)|\cdot|g'(u)|+|\gamma_n^\sigma(u)|\cdot|g''(u)|+\left|n\left(G_n^\sigma\left(h(\nu,u)(\nu-u)^2\right)\right)(u)\right|,\end{aligned}$$

where $h(\nu,u)=\frac{g''(\xi)-g''(u)}{2}$ and ξ lies between u and ν. Using the same type of technique as applied in [4], we can write

$$|h(\nu,u)| \leq \left(1+\frac{(e^{-u}-e^{-\nu})^2}{\varrho^2}\right)\widehat{\omega}(g'',\varrho), \quad \varrho>0.$$

On applying the Cauchy-Schwarz inequality and choosing $\varrho = 1/\sqrt{n}$, we get

$$n\left(G_n^\sigma\left|h(\nu,u)\right|(\nu-u)^2\right)(u) \leq 2\left[2\gamma_n^\sigma(u)+u(u+2)+\delta_n^\sigma(u)\right]\widehat{\omega}\left(g'',\frac{1}{\sqrt{n}}\right).$$

Thus we get the desired result.

Corollary 1 *Let $g, g'' \in \widehat{C}_[0,\infty)$. Then*

$$\lim_{n\to\infty} n\left[(G_n^\sigma g)(u)-g(u)\right] = (\sigma u^2+1)g'(u)+\frac{u(u+2)}{2}g''(u).$$

Proof follows by Theorem 1, we skip the details.

Theorem 2 *For $g \in C_B[0,\infty)$ (the class of all continuous and bounded function on $[0,\infty)$) and for all $u \in [0,\infty)$, there exists a constant $\mathcal{C} > 0$, such that*

$$|(G_n^\sigma g)(u) - g(u)| \leq \omega\left(g, \frac{\sigma u^2+1}{\sigma+n}\right) + \mathcal{C}\omega_2\left(g, \frac{\left[2\beta u^3(\sigma u+1) + u^2(n+8\sigma) + 2u(\sigma+n) + 3\right]^{1/2}}{(\sigma+n)}\right).$$

The proof of the theorem follows using Remark 1 and proceeding along the lines of [10, Th. 5], we omit the details.

For $s \geq 4$, if $B^s[0,\infty) = \{g : \frac{|g(u)|}{(1+u^s)} \leq a_g, \forall u \in [0,\infty)\}$, where a_g is an absolute constant depends only on g. Let $C^s[0,\infty) = C[0,\infty) \cap B^s[0,\infty)$. For each $g \in C^s[0,\infty)$, the weighted modulus of continuity (see[22]) is defined as

$$\Omega(g,\delta) = \sup_{|h|<\delta, u\in R^+} \frac{|g(u+h) - g(u)|}{(1+h^s)(1+u^s)}.$$

Also, $C^{*,s}[0,\infty)$ denotes the subspace of continuous functions $g \in B^s[0,\infty)$ for which $\lim_{u\to\infty} |g(u)|(1+u^s)^{-1} < \infty$. We consider the norm by

$$||g||_s = \sup_{0\leq u<\infty} \frac{|g(u)|}{(1+u^s)}.$$

Theorem 3 *If $g \in C^{*,s}[0,\infty)$, then we have*

$$\lim_{n\to\infty} ||(G_n^\sigma g) - g||_s = 0.$$

Proof Following the weighted Korovkin's theorem due to Gadjiev [13], if $g \in C^{*,s}[0,\infty)$ satisfies

$$\lim_{n\to\infty} ||(G_n^\sigma e_j) - e_j||_s = 0, \quad j = 0,1,2,$$

then we have

$$\lim_{n\to\infty} ||(G_n^\sigma g) - g||_s = 0.$$

Using Remark 1, obviously the result holds for $j = 0$. Next

$$\begin{aligned} &\lim_{n\to\infty} ||G_n^\sigma e_1) - e_1||_s \\ &= \lim_{n\to\infty} \left[\frac{\sigma}{(\sigma+n)} \sup_{0\leq u<\infty} \left(\frac{u^2}{1+u^s}\right) + \frac{1}{\sigma+n} \sup_{0\leq u<\infty} \left(\frac{1}{1+u^s}\right)\right] \\ &= 0. \end{aligned}$$

Finally

$$
\begin{aligned}
&\lim_{n\to\infty} ||(G_n^\sigma e_2) - e_2||_s \\
&= \lim_{n\to\infty} \left\| \frac{\sigma^2 u^4 + 2\sigma(\sigma+n+1)u^3 + (n+6\sigma)u^2 + 4(\sigma+n)u + 2}{(\sigma+n)^2} \right\|_s \\
&= \lim_{n\to\infty} \frac{1}{(1+u^s)} \left[\frac{\sigma^2 u^4 + 2\sigma(\sigma+n+1)u^3 + (n+6\sigma)u^2 + 4(\sigma+n)u + 2}{(\sigma+n)^2} \right] \\
&= 0.
\end{aligned}
$$

This completes the proof of theorem.

3 Convergence of hybrid operator in complex setting

Overconvergence, or the extension of estimation properties from the real domain to the complex domain, occurs in the complex domain. S. Gal [14] did praiseworthy work on complex operators, presenting results on the overconvergence of various complex operators. In the recent years, several types of complex integral operators have been explored, we refer to some of the works in this direction due to Agarwal and Gupta [8], Gal and Gupta [15], Gal and Iancu [17] and Gal et al [16]. Throughout the paper, we shall denote $e_j := e_j(z) = z^j$ for $j \in \mathbb{N} \cup \{0\}$. Thus, the hybrid operator (3) in complex setting takes the following form:

$$
(G_n^\sigma g)(e_1) = (\sigma+n) \sum_{j=0}^{\infty} b_{n,j}^\sigma (e_1) \int_0^\infty s_{n,j}^\sigma(\nu) g(\nu) d\nu,
$$

where

$$
b_{n,j}^\sigma (e_1) = \sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i \frac{e_j}{(1+e_1)^{n+j}} e^{-\sigma e_1}.
$$

Let $R > 1$ and denote $\mathfrak{D}_R = \{e_1 \in \mathbb{C} : |e_1| < R\}$. Since the definition of the operator G_n^σ includes the value of function in the interval $[0, \infty)$. Consider a class of functions given by:

$\mho_R := \{g : [R, +\infty) \cup \overline{\mathfrak{D}}_R \to \mathbb{C} \mid g$ is continuous in $(R, +\infty) \cup \overline{\mathfrak{D}}_R$ and analytic in $\mathfrak{D}_R\}$.

Obviously, $g(e_1) = \sum_{k=0}^{\infty} \wp_k e_k$, for all $e_1 \in \mathfrak{D}_R$ and $g \in \mho_R$.

Lemma 2 *Assume a function $g : [R, +\infty) \bigcup \overline{\mathfrak{D}}_R$ having the following properties:*

1. *analytic in $\mathfrak{D}_R$ i.e. $g(e_1) = \sum_{k=0}^{\infty} \wp_k e_k$; $e_1 \in \mathfrak{D}_R$*

2. $|g(u)| \leq \mathcal{C}e^{\mathcal{B}u}$ *for all* $u \in [R, +\infty)$*, where* $\mathcal{B}$ *and* $\mathcal{C}$ *are absolute constants, then* $\forall\ |e_1| \leq \varsigma$ *with* $Re(e_1) \geq 0$*,* $n > (\mathcal{B} + h\sigma - \sigma)/(1-h)$*, where* $h = \sqrt{\varsigma^2/(1+\varsigma^2)}$ *and* $1 \leq \varsigma < R$*, we have*

$$(G_n^\sigma g)(e_1) = \sum_{k=0}^{\infty} \wp_k (G_n^\sigma e_k)(e_1).$$

Proof For any natural m, consider

$$g_m(e_1) = \sum_{j=0}^{m} \wp_j e_j \text{ if } |e_1| \leq \varsigma \text{ and } g_m(u) = g(u) \text{ if } u \in (\varsigma, +\infty).$$

Since $|g_m(e_1)| \leq \sum_{j=0}^{\infty} |\wp_j| \varsigma^j := \mathcal{C}_\varsigma$, $\forall\ |e_1| \leq \varsigma$ and $g \in C[\varsigma, R]$ implying $|g_m(u)| \leq \mathcal{C}_{\varsigma,R} e^{\mathcal{B}u}$, $\forall\ u \in [0, +\infty)$. This follows that for $|e_1| \leq \varsigma$ under the condition $Re(e_1) \geq 0$, one has

$$\begin{aligned}
&|(G_n^\sigma g_m)(e_1)| \\
&\leq \mathcal{C}_{\varsigma,R}|(1+e_1)^{-n}| \left(\sum_{j=0}^{\infty} \sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i |e^{-\sigma e_1}| \left(\frac{|e_1|}{|1+e_1|} \right)^j (\sigma+n) \int_0^\infty e^{-(\sigma+n)\nu} \cdot \frac{(\sigma+n)^j}{j!} \nu^j e^{B\nu} d\nu \right) \\
&= \mathcal{C}_{\varsigma,R}|(1+e_1)^{-n}| \sum_{j=0}^{\infty} \sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i |e^{-\sigma e_1}| \left(\frac{|e_1|}{|1+e_1|} \right)^j \cdot \frac{(\sigma+n)^{j+1}}{(\sigma+n-\mathcal{B})^{j+1}} < \infty \\
&\leq \mathcal{C}_{\varsigma,R}|(1+e_1)^{-n}| \sum_{j=0}^{\infty} \sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i |e^{-\sigma e_1}| h^j \cdot \frac{(\sigma+n)^{j+1}}{(\sigma+n-B)^{j+1}},
\end{aligned}$$

where $h = \sqrt{\frac{\varsigma^2}{1+\varsigma^2}} < 1$, taking into account that for $e_1 = u + iv$ with $u \geq 0$ we have

$$\left(\frac{|e_1|}{|1+e_1|} \right)^2 = \frac{u^2+v^2}{1+2u+(u^2+v^2)} \leq \frac{u^2+v^2}{1+(u^2+v^2)} \leq \frac{\varsigma^2}{1+\varsigma^2}.$$

If

$$c_j = \sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i |e^{-\sigma e_1}| h^j \cdot \frac{(\sigma+n)^{j+1}}{(\sigma+n-\mathcal{B})^{j+1}},$$

we have by ratio test

$$\frac{c_{j+1}}{c_j} = \frac{\sum_{i+\ell=j+1} \frac{(n+i)_\ell}{(j+1)!} \binom{j+1}{i} \sigma^i}{\sum_{i+\ell=j} \frac{(n+i)_\ell}{j!} \binom{j}{i} \sigma^i} \cdot \frac{h(\sigma+n)}{\sigma+n-\mathcal{B}} \leq 1, \quad \forall j \geq j_0$$

Therefore $(G_n^\sigma g_m)(e_1)$ is well-defined for $n > \frac{\mathcal{B}+h\sigma-\sigma}{1-h}$.

Denoting

$$g_{m,k}(e_1) = \wp_k e_k \text{ if } |e_1| \leq \varsigma \text{ and } g_{m,k}(u) = \frac{g(u)}{m+1} \text{ if } u \in (\varsigma, \infty),$$

By the linearity property, we have

$$(G_n^\sigma g_m)(e_1) = \sum_{k=0}^{m} \wp_k (G_n^\sigma e_k)(e_1),$$

it is enough to show that $\lim_{m\to\infty} (G_n^\sigma g_m)(e_1) = (G_n^\sigma g)(e_1)$. However, this follows immediately from the following obvious relations:

i. $\lim_{m\to\infty} \|g_m - g\|_\varsigma = 0$

ii. $\|g_m - g\|_{B[0,+\infty)} \leq \|g_m - g\|_\varsigma$

iii. $|(G_n^\sigma g_m)(e_1) - (G_n^\sigma g)(e_1)| \leq \mathcal{M}_{\varsigma,n,\sigma} \|g_m - g\|_\varsigma$,

where

$$\mathcal{M}_{\varsigma,n,\sigma} = (\sigma+n) \sum_{j=0}^{\infty} \sum_{i+l=j} \frac{(n+i)_l}{j!} \binom{j}{i} \sigma^i \frac{\varsigma^j}{(1+\varsigma)^{n+j}} \int_0^\infty \left| s_{n,j}^\sigma(\nu) \right| |d\nu|,$$

and $\|\cdot\|_{B[0,+\infty)}$ denotes the uniform norm on $CB(\mathbb{R}^+)$.

Theorem 4 (***Upper bound***) *For $R \in (3, +\infty)$, if a function $g \in \mho_R$ satisfies the following properties:*

1. there exist $\mathcal{M} > 0$ and $\Lambda \in (\frac{1}{R}, 1)$, with $|\wp_k| \leq \frac{\mathcal{M}\cdot\Lambda^k}{\Gamma(2k+\sigma\rho+1)}$, for all $k = 0, 1, ...,$ (which implies $|g(e_1)| \leq \mathcal{M}e^{\Lambda|e_1|}$ for all $e_1 \in \mathfrak{D}_R$)

2. $|g(u)| \leq \mathcal{C}e^{\mathcal{B}u}$, for all $u \in [R, +\infty)$,

Further if $1 \leq \varsigma < \varsigma + 1 < \frac{1}{\Lambda}$ and $h = \sqrt{\varsigma^2/(1+\varsigma^2)}$, then for all $|e_1| \leq \varsigma$ with $Re(e_1) \geq 0$, $n \in \mathbb{N}$ and $n > \frac{\mathcal{B}+h\sigma-\sigma}{1-h}$, we have

$$|(G_n^\sigma g)(e_1) - g(e_1)| \leq \frac{\mathcal{C}_{\varsigma,\Lambda}}{\sigma+n},$$

where $\mathcal{C}_{\varsigma,\Lambda} = \mathcal{M}\sum_{k=1}^{\infty}((\varsigma+1)\Lambda)^k < \infty$.

Proof Using the recurrence relation of Lemma 1, we get

$$\Theta_{n,k}^\sigma(e_1) = \frac{e_1+e_2}{\sigma+n}[\Theta_{n,k-1}^\sigma(e_1)]' + \frac{k+ne_1+\sigma(e_1+e_2)}{\sigma+n}\Theta_{n,k-1}^\sigma(e_1).$$

From this, we immediately have

$$\begin{aligned}\Theta_{n,k}^\sigma(e_1) - e_k = {} & \frac{e_1+e_2}{\sigma+n}[\Theta_{n,k-1}^\sigma(e_1) - e_{k-1}]' + \frac{k+ne_1+\sigma(e_1+e_2)}{\sigma+n}[\Theta_{n,k-1}^\sigma(e_1) - e_{k-1}] \\ & + \frac{(2k-1)+(k-1)e_1+\sigma e_2}{\sigma+n} e_{k-1}.\end{aligned} \tag{4}$$

Now for $1 \leq \varsigma < R$, if we denote the norm-$\|g\|_\varsigma := \max_{|e_1| \leq \varsigma} |g(e_1)|$ in $C(\overline{\mathfrak{D}}_\varsigma)$, where $\overline{\mathfrak{D}}_\varsigma = \{e_1 \in \mathbb{C} : |e_1| \leq \varsigma\}$, then by applying the Bernstein's inequality i.e. $|P'_k(e_1)| \leq \frac{k}{\varsigma}\|P_k\|_\varsigma$, $\forall\ |e_1| \leq \varsigma$, where $P_k(e_1)$ is a polynomial of degree $\leq k$ and using (4), we obtain

$$\begin{aligned}
&\left\|\Theta^{\sigma}_{n,k} - e_k\right\|_\varsigma \\
&\leq \frac{\varsigma(1+\varsigma)}{\sigma+n}\|\Theta^{\sigma}_{n,k-1} - e_{k-1}\|_\varsigma \frac{k-1}{\varsigma} + \frac{n\varsigma + k + \sigma\varsigma(1+\varsigma)}{\sigma+n}\|\Theta^{\sigma}_{n,k-1} - e_{k-1}\|_\varsigma \\
&\quad + \frac{(2k-1) + (k-1)\varsigma + \sigma\varsigma^2}{\sigma+n}\varsigma^{k-1},
\end{aligned}$$

which implies

$$\begin{aligned}
&\left\|\Theta^{\sigma}_{n,k} - e_k\right\|_\varsigma \\
&\leq \left(\varsigma + \frac{(2k-1) + \varsigma(k-1) + \sigma\varsigma^2}{\sigma+n}\right) \cdot \|\Theta^{\sigma}_{n,k-1} - e_{k-1}\|_\varsigma + \frac{(2k-1) + (k-1)\varsigma + \sigma\varsigma^2}{\sigma+n}\varsigma^{k-1} \\
&\leq \rho\left(1 + \frac{(2k-1+\sigma\rho)}{\sigma+n}\right) \cdot \|\Theta^{\sigma}_{n,k-1} - e_{k-1}\|_\varsigma + \frac{(2k-1+\sigma\rho)}{\sigma+n}\rho^k,
\end{aligned}$$

where $\rho := \varsigma + 1$.

In the following, we prove the result by the principle of mathematical induction with respect to k (n must be arbitrarily fixed), this recurrence implies

$$||\Theta^{\sigma}_{n,k} - e_k||_\varsigma \leq \frac{\Gamma(2k+\sigma\rho+1)}{\sigma+n}\rho^k \quad \text{for all} \quad k \geq 1, n \geq 1.$$

The left side is $\frac{\sigma\varsigma^2+1}{\sigma+n}$ and the right side is $\frac{(\varsigma+1)\Gamma(\sigma\rho+3)}{\sigma+n}$ for $k = 1$. We indeed can claim the required inequality for $k = 1$. The above recurrence relation suggests that if it is true for k, then we have

$$||\Theta^{\sigma}_{n,k+1} - e_{k+1}||_\varsigma \leq \rho\left(1 + \frac{(2k+\sigma\rho+1)}{\sigma+n}\right) \cdot \frac{\Gamma(2k+\sigma\rho+1)}{\sigma+n}\rho^k + \frac{2k+\sigma\rho+1}{\sigma+n}\rho^{k+1}.$$

It remains to show that

$$\left(1 + \frac{(2k+\sigma\rho+1)}{\sigma+n}\right) \cdot \frac{\Gamma(2k+\sigma\rho+1)}{\sigma+n}\rho^{k+1} + \frac{2k+\sigma\rho+1}{\sigma+n}\rho^{k+1} \leq \frac{\Gamma(2k+\sigma\rho+3)}{\sigma+n}\rho^{k+1},$$

or after simplifications, equivalently to

$$\left(1 + \frac{(2k+\sigma\rho+1)}{\sigma+n}\right) \cdot \Gamma(2k+\sigma\rho+1) + (2k+\sigma\rho+1) \leq \Gamma(2k+\sigma\rho+3).$$

It is clearly sufficient if we will show that

$$\Gamma(2k+\sigma\rho+1) + 1 \leq \Gamma(2k+\sigma\rho+2).$$

This last inequality holds true(can be proved by mathematical induction) for all k, $n \in \mathbb{N}$.

From the supposition on g, using Lemma 2 we may write

$$(G_n^\sigma g)(e_1) = \sum_{k=0}^{\infty} \wp_k \Theta_{n,k}^\sigma(e_1), \text{ for all } e_1 \in \mathfrak{D}_R,\ Re(e_1) \geq 0,\ n > \frac{\mathcal{B} + h\sigma - \sigma}{1-h},$$

which immediately from the hypothesis on $\wp_k$ implies for all $|e_1| \leq \varsigma$ with $Re(e_1) \geq 0$ and $n > (\mathcal{B} + h\sigma - \sigma)/(1-h)$,

$$\begin{aligned} |(G_n^\sigma g)(e_1) - g(e_1)| &\leq \sum_{k=1}^{\infty} |\wp_k|.|\Theta_{n,k}^\sigma(e_1) - e_k| \\ &\leq \sum_{k=1}^{\infty} \mathcal{M} \frac{\Lambda^k}{\Gamma(2k+\sigma\rho+1)} \frac{\Gamma(2k+\sigma\rho+1)}{\sigma+n} \rho^k \\ &= \frac{\mathcal{M}}{\sigma+n} \sum_{k=1}^{\infty} (\rho\Lambda)^k = \frac{\mathcal{C}_{\varsigma,\Lambda}}{\sigma+n}, \end{aligned}$$

where $\mathcal{C}_{\varsigma,\Lambda} := \mathcal{M}\sum_{k=1}^{\infty}(\rho\Lambda)^k < \infty$ for all $1 \leq \varsigma \leq \rho < \frac{1}{\Lambda}$, taking into account the uniform convergence of the series $\sum_{k=1}^{\infty} s^k$ in any compact disk contained in the open unit disk.

Theorem 5 (***Voronovskaja result***) *For $R \in (3, +\infty)$, if a function $g \in \mho_R$ satisfies the following properties:*

1. *there exist $\mathcal{M} > 0$ and $\Lambda \in (\frac{1}{R}, 1)$, with $|\wp_k| \leq \frac{\mathcal{M}\cdot\Lambda^k}{\Gamma(2k+\sigma\rho+1)}$, $\forall\ k = 0, 1, ...$*
2. *$|g(u)| \leq \mathcal{C}e^{\mathcal{B}u}$, $\forall\ u \in [R, +\infty)$,*

Also, if $1 \leq \varsigma < \rho < \frac{1}{\Lambda}$, $h = \sqrt{\varsigma^2/(1+\varsigma^2)}$ and $\sigma \geq 2$, then $\forall\ |e_1| \leq \varsigma$ with $Re(e_1) \geq 0$ and $n > \max\left\{\frac{\mathcal{B}+h\sigma-\sigma}{1-h}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\}$, we have

$$\left|(G_n^\sigma g)(e_1) - g(e_1) - \frac{\sigma e_2 + 1}{\sigma+n} g'(e_1) - \frac{e_2 + 2e_1}{2(\sigma+n)} g''(e_1)\right| \leq \frac{\mathcal{C}^\sigma_{\varsigma,\Lambda,\mathcal{M}}(g)}{(\sigma+n)^2},$$

where

$$\mathcal{C}^\sigma_{\varsigma,\Lambda,\mathcal{M}}(g) = \mathcal{M}\sum_{k=2}^{\infty} \frac{(k-1)[\Lambda(\varsigma+1)]^k \mathcal{B}_{k,\varsigma,\sigma}}{\Gamma(2k+\sigma\rho+1)} + \frac{2\mathcal{M}}{[1-\Lambda(\varsigma+1)]\cdot[\Lambda(\varsigma+1)]^\sigma} < \infty,$$

and

$$\begin{aligned} \mathcal{B}_{k,\varsigma,\sigma} &= k^2(\varsigma+2)^2 + k\left(3\beta\varsigma^3 + \varsigma^2(6\beta-3) - 11\varsigma - 8\right) \\ &+2\left(\sigma^2\varsigma^4 - \sigma\varsigma^3 + (1-\sigma)\varsigma^2 + 3\varsigma + 2\right) + \Gamma(2k+\sigma\rho+1). \end{aligned}$$

Proof By using Lemma 2, we can write $(G_n^\sigma g)(e_1) = \sum_{k=0}^{\infty} \wp_k \Theta_{n,k}^\sigma(e_1)$. Also

$$\begin{aligned}\frac{\sigma e_2+1}{\sigma+n}g'(e_1)+\frac{e_2+2e_1}{2(\sigma+n)}g''(e_1) &= \frac{\sigma e_2+1}{\sigma+n}\sum_{k=1}^{\infty}k\wp_k e_{k-1}+\frac{(e_1+2)}{2(\sigma+n)}\sum_{k=2}^{\infty}k(k-1)\wp_k e_{k-1}\\ &= \frac{1}{2(\sigma+n)}\sum_{k=1}^{\infty}k\wp_k\left[2(\sigma e_2+1)+(e_1+2)(k-1)\right]e_{k-1}.\end{aligned}$$

Thus

$$\begin{aligned}&\left|\left(G_n^\sigma g\right)(e_1)-g(e_1)-\frac{\sigma e_2+1}{\sigma+n}g'(e_1)-\frac{e_2+2e_1}{2(\sigma+n)}g''(e_1)\right|\\ &\leq \sum_{k=1}^{\infty}|\wp_k|\left|\Theta_{n,k}^\sigma(e_1)-e_k-\frac{\left[2k(\sigma e_2+1)+k(k-1)(e_1+2)\right]e_{k-1}}{2(\sigma+n)}\right|.\end{aligned}$$

By Lemma 1, for $k=1,2,...$, we have

$$\Theta_{n,k}^\sigma(e_1)=\frac{e_1+e_2}{\sigma+n}[\Theta_{n,k-1}^\sigma(e_1)]'+\frac{k+(\sigma+n)e_1+\sigma e_2}{\sigma+n}\Theta_{n,k-1}^\sigma(e_1).$$

If we denote

$$\Xi_{k,n}^\sigma(e_1)=\Theta_{n,k}^\sigma(e_1)-e_k-\frac{\left[2k(\sigma e_2+1)+k(k-1)(e_1+2)\right]e_{k-1}}{2(\sigma+n)},$$

then the polynomial $\Xi_{k,n}^\sigma(e_1) \leq deg\ k$ and subsequently we obtain

$$\Xi_{k,n}^\sigma(e_1)=\frac{e_1+e_2}{\sigma+n}[\Xi_{k-1,n}^\sigma(e_1)]'+\frac{k+(\sigma+n)e_1+\sigma e_2}{\sigma+n}\Xi_{k-1,n}^\sigma(e_1)+\Upsilon_{k,n,\sigma}(e_1),$$

where

$$\begin{aligned}\Upsilon_{k,n,\sigma}(e_1)=\frac{(k-1)e_{k-2}}{2(\sigma+n)^2}\Big[&k^2(e_1+2)^2+k(3\sigma e_3\\ &+e_2(6\sigma-3)-11e_1-8)+2(\sigma^2e_4-\sigma e_3+(1-\sigma)e_2+3e_1+2)\Big],\end{aligned}$$

for all $k\geq 2$.

Utilizing estimate from the proof of Theorem 4 with $1\leq\varsigma<R$, we get

$$|\Theta_{n,k}^\sigma(e_1)-e_k|\leq\frac{\Gamma(2k+\sigma\rho+1)}{\sigma+n}(\varsigma+1)^k.$$

For $\sigma\geq 2$, it follows that

$$\left|\Xi_{k,n}^\sigma(e_1)\right|\leq\frac{\varsigma^2}{\sigma+n}\left(1+\frac{1}{\varsigma}\right)|[\Xi_{k-1,n}^\sigma(e_1)]'|+\left(\frac{k+(n+\sigma)\varsigma+\sigma\varsigma^2}{\sigma+n}\right)\left|\Xi_{k-1,n}^\sigma(e_1)\right|+|\Upsilon_{k,n,\sigma}(e_1)|.$$

Now we estimate $|[\Xi_{k-1,n}^\sigma(e_1)]'|$ for $k\geq 2$. First

$$|[\Xi_{k-1,n}^\sigma(e_1)]'|\leq\frac{k-1}{\varsigma}||\Xi_{k-1,n}^\sigma||_\varsigma$$

$$\leq \frac{k-1}{\varsigma}\left[||\Theta^{\sigma}_{n,k-1}(e_1)-e_{k-1}||_{\varsigma}+\left|\left|\frac{[2(\sigma e_2+1)(k-1)+(k-1)(k-2)(e_1+2)]\,e_{k-2}}{2(\sigma+n)}\right|\right|_{\varsigma}\right]$$

$$\leq \frac{k-1}{\varsigma}\left[\frac{(\varsigma+1)^{k-1}\Gamma(2k+\sigma\rho-1)}{\sigma+n}+\frac{\varsigma^{k-2}(k-1)\left[2(\sigma\varsigma^2+1)+(k-2)(\varsigma+2)\right]}{\sigma+n}\right]$$

$$\leq \frac{(k-1)(\varsigma+1)^{k-1}}{\varsigma(\sigma+n)}\left[\Gamma(2k+\sigma\rho-1)+2(\sigma\varsigma+1)(k-1)+3(k^2-3k+2)\right]$$

$$\leq \frac{2(k-1)(\varsigma+1)^{k-1}\Gamma(2k+\sigma\rho-1)}{\varsigma(\sigma+n)}.$$

Thus

$$\frac{\varsigma^2}{\sigma+n}\left(1+\frac{1}{\varsigma}\right)|[\Xi^{\sigma}_{k-1,n}(e_1)]'|\leq\frac{(\varsigma+1)^k\Gamma(2k+\sigma\rho+1)}{(\sigma+n)^2}$$

and

$$|\Xi^{\sigma}_{k,n}(e_1)|\leq\frac{(\varsigma+1)^k\Gamma(2k+\sigma\rho+1)}{(\sigma+n)^2}+\left(\frac{k+(n+\sigma)\varsigma+\sigma\varsigma^2}{\sigma+n}\right)|\Xi^{\sigma}_{k-1,n}(e_1)|+|\Upsilon_{k,n,\sigma}(e_1)|,$$

$\forall\ |e_1|\leq\varsigma, k\geq 2$ and $\sigma\geq 2$.

For $k\leq n$ and $|e_1|\leq\varsigma$, using $\sigma\varsigma^2+k\leq n$, we immediately have

$$|\Xi^{\sigma}_{k,n}(e_1)|\leq\frac{(\varsigma+1)^k\Gamma(2k+\sigma\rho+1)}{(\sigma+n)^2}+(\varsigma+1)|\Xi^{\sigma}_{k-1,n}(e_1)|+|\Upsilon_{k,n,\sigma}(e_1)|,$$

where

$$|\Upsilon_{k,n,\sigma}(e_1)|\leq\frac{(\varsigma+1)^k}{(\sigma+n)^2}\Lambda_{k,\varsigma,\sigma},$$

and $\Lambda_{k,\varsigma,\sigma}$ is defined as

$$\begin{aligned}\Lambda_{k,\varsigma,\sigma}:=&\ k^2(\varsigma+2)^2+k(3\sigma\varsigma^3+\varsigma^2(6\sigma-3)\\&-11\varsigma-8)+2\left(\sigma^2\varsigma^4-\sigma\varsigma^3+(1-\sigma)\varsigma^2+3\varsigma+2\right).\end{aligned}$$

Thus for $k\leq n+1$, we get

$$|\Xi^{\sigma}_{k,n}(e_1)|\leq(\varsigma+1)|\Xi^{\sigma}_{k-1,n}(e_1)|+\frac{(\varsigma+1)^k}{(\sigma+n)^2}\mathcal{B}_{k,\varsigma,\sigma},$$

where

$$\mathcal{B}_{k,\varsigma,\sigma}:=\Lambda_{k,\varsigma,\sigma}+\Gamma(2k+\sigma\rho+1).$$

But $\Xi^{\sigma}_{0,n}(e_1)=\Xi^{\sigma}_{1,n}(e_1)=0$, for any $e_1\in\mathbb{C}$ and thus, by expressing the last inequality for $2\leq k\leq n+1$, we can easily derive step by step the following

$$|\Xi^{\sigma}_{k,n}(e_1)|\leq\frac{(\varsigma+1)^k}{(\sigma+n)^2}\sum_{j=2}^{k}\mathcal{B}_{j,\varsigma,\sigma}\leq\frac{(k-1)(\varsigma+1)^k}{(\sigma+n)^2}\mathcal{B}_{k,\varsigma,\sigma}.$$

It follows that

$$\left|\left(G_n^\sigma g\right)(e_1)-g(e_1)-\frac{\sigma e_2+1}{(\sigma+n)}g'(e_1)-\frac{e_2+2e_1}{2(\sigma+n)}g''(e_1)\right|$$

$$\leq\sum_{k=2}^{n+1}|\wp_k|\cdot|\Xi_{k,n}^\sigma(e_1)|+\sum_{k\geq n+2}|\wp_k|\cdot|\Xi_{k,n}^\sigma(e_1)|$$

$$\leq\frac{1}{(\sigma+n)^2}\sum_{k\geq2}|\wp_k|(k-1)(\varsigma+1)^k\mathcal{B}_{k,\varsigma,\sigma}+\sum_{k\geq n+2}|\wp_k|\cdot\left[|\Theta_{n,k}^\sigma(e_1)-e_k|+\left|\frac{[2k(\sigma e_2+k)+k(k-1)e_1]\,e_{k-1}}{2(\sigma+n)}\right|\right]$$

$$\leq\sum_{k\geq2}\frac{|\wp_k|(k-1)(\varsigma+1)^k\mathcal{B}_{k,\varsigma,\sigma}}{(\sigma+n)^2}+\sum_{k\geq n+2}|\wp_k|\cdot\left[\frac{(\varsigma+1)^k\Gamma(2k+\sigma\rho+1)}{\sigma+n}+\frac{[2k(\sigma\varsigma^2+k)+k(k-1)\varsigma]\,\varsigma^{k-1}}{2(\sigma+n)}\right]$$

$$\leq\sum_{k\geq2}\frac{|\wp_k|(k-1)(\varsigma+1)^k\mathcal{B}_{k,\varsigma,\sigma}}{(\sigma+n)^2}+\sum_{k\geq n+2}\frac{2|\wp_k|(\varsigma+1)^k\Gamma(2k+\sigma\rho+1)}{\sigma+n}$$

$$\leq\sum_{k\geq2}\frac{|\wp_k|(k-1)(\varsigma+1)^k\mathcal{B}_{k,\varsigma,\sigma}}{(\sigma+n)^2}+\sum_{k\geq n+2}\frac{2\mathcal{M}[\Lambda(\varsigma+1)]^k}{\sigma+n}$$

$$\leq\frac{\mathcal{M}}{(\sigma+n)^2}\sum_{k\geq2}\frac{[\Lambda(\varsigma+1)]^k(k-1)\mathcal{B}_{k,\varsigma,\sigma}}{\Gamma(2k+\sigma\rho+1)}+\frac{2\mathcal{M}[\Lambda(\varsigma+1)]^{n+2}}{(\sigma+n)(1-\Lambda(\varsigma+1))}$$

$$\leq\frac{\mathcal{M}}{(\sigma+n)^2}\sum_{k\geq2}\frac{[\Lambda(\varsigma+1)]^k(k-1)\mathcal{B}_{k,\varsigma,\sigma}}{\Gamma(2k+\sigma\rho+1)}+\frac{2\mathcal{M}}{(\sigma+n)^2[1-\Lambda(\varsigma+1)]\cdot[\Lambda(\varsigma+1)]^\sigma},$$

for all $\sigma+n>\frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)}$, where for $\Lambda(\varsigma+1)<1$, we certainly have a convergent series $\sum_{k=2}^{\infty}|\wp_k|(k-1)(\varsigma+1)^k\mathcal{B}_{k,\varsigma,\sigma}$, and we utilised the inequality (which can be readily demonstrated by mathematical induction)

$$\lambda^{\sigma+n+2}\leq\lambda^{\sigma+n+1}\leq\frac{1}{\sigma+n},\ \text{for all } \sigma+n>\frac{\lambda}{1-\lambda},$$

applied for $\lambda=\Lambda(\varsigma+1)<1$.

Theorem 6 (***Exact order***) *Under the same assumption of Theorem 5, suppose a polynomial g has non-zero degree, then for all $1\leq\varsigma<\varsigma+2<R$, we have*

$$\|(G_n^\sigma g)-g\|_{\varsigma^+}\sim\frac{1}{\sigma+n},\ \textit{for all } n>\max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)},\frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)}-\sigma\right\},$$

where $\|g\|_{\varsigma^+}=\sup\{|g(e_1)|:|e_1|\leq\varsigma,\ Re(e_1)\in[0,\infty)\}$.

Proof For all $n\in\mathbb{N}$ with $n>\mathcal{B}$ and $|e_1|\leq\varsigma$ with $0\leq Re(e_1)<\infty$, we get

$$(G_n^\sigma g)(e_1)-g(e_1)=\frac{1}{\sigma+n}\bigg[(1+\sigma e_2)g'(e_1)+\frac{(e_2+2e_1)}{2}g''(e_1)$$
$$+\frac{1}{\sigma+n}\cdot(\sigma+n)^2\left((G_n^\sigma g)(e_1)-g(e_1)-\frac{(1+\sigma e_2)}{\sigma+n}g'(e_1)-\frac{(e_2+2e_1)}{2(\sigma+n)}g''(e_1)\right)\bigg]$$

Applying the inequality

$$\|S\|_{\varsigma^+} - \|T\|_{\varsigma^+} \leq |\, \|S\|_{\varsigma^+} - \|T\|_{\varsigma^+} \,| \leq \|S+T\|_{\varsigma^+},$$

we obtain

$$\|(G_n^\sigma g) - g\|_{\varsigma^+} \geq \frac{1}{\sigma+n}\left[\left\|(\sigma e_2+1)g' + \frac{(e_2+2e_1)}{2}g''\right\|_{\varsigma^+} \right.$$
$$\left. -\frac{1}{\sigma+n}\cdot(\sigma+n)^2\left\|(G_n^\sigma g) - g - \frac{1+\sigma e_2}{\sigma+n}g' - \frac{(e_2+2e_1)g''}{2(\sigma+n)}\right\|_{\varsigma^+}\right].$$

Since g is a polynomial of degree ≥ 1 in $\mathfrak{D}_R$, we get $\left\|(1+\sigma e_2)g' + \frac{(e_2+2e_1)}{2}g''\right\|_{\varsigma^+} > 0$. Indeed, supposing the contrary, it follows that

$$(1+\sigma e_2)g'(e_1) + \frac{(e_2+2e_1)}{2}g''(e_1) = 0, \text{ for all } |e_1| \leq \varsigma, Re(e_1) \geq 0. \tag{5}$$

Let us take

$$g(e_1) = \sum_{k=0}^{\infty} \wp_k e_k,$$

where $\wp_k$, $k = 0, 1, ...$ are constants, then

$$g'(e_1) = \sum_{k=1}^{\infty} k\wp_k e_{k-1},\ g''(e_1) = \sum_{k=2}^{\infty} k(k-1)\wp_k e_{k-2}.$$

Substituting these values in (5), we get

$$\wp_1 + 4\wp_2 e_1 + (9\wp_3 + \wp_2 + \wp_1\beta)e_2 + (16\wp_4 + 3\wp_3 + 2\wp_2\beta)e_3 + \cdots = 0.$$

This indicates that $g(e_1) = \wp_0$, which is contrary to the hypothesis. Now using Theorem 5, we have

$$(\sigma+n)^2\left\|(G_n^\sigma g) - g - \frac{(1+\sigma e_2)g'}{\sigma+n} - \frac{(e_2+2e_1)g''}{2(\sigma+n)}\right\|_{\varsigma^+} \leq \mathcal{C}^\sigma_{\varsigma,\Lambda,\mathcal{M}}(g),$$

for all $n > \max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\}$. Thus, there is a number $n_0 \equiv n_0(g,\sigma,\varsigma)$ satisfying $n_0 > \max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\}$. such that $\forall\, n \geq n_0$, we have

$$\left\|(1+\sigma e_2)g' + \frac{(e_2+2e_1)}{2}g''\right\|_{\varsigma^+}$$
$$-\frac{1}{\sigma+n}\cdot(\sigma+n)^2\left\|(G_n^\sigma g) - g - \frac{(1+\sigma e_2)}{\sigma+n}g'(z) - \frac{(e_2+2e_1)g''}{2(\sigma+n)}\right\|_{\varsigma^+}$$
$$\geq \frac{1}{2}\left\|(1+\sigma e_2)g' + \frac{(e_2+2e_1)}{2}g''\right\|_{\varsigma^+},$$

which implies that

$$\|(G_n^\sigma g) - g\|_{\varsigma^+} \geq \frac{1}{2(\sigma+n)} \left\| (1+\sigma e_2)g' + \frac{(e_2+2e_1)}{2} g'' \right\|_{\varsigma^+},$$

for all $n \geq n_0$.
For $\max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\} < n \leq n_0 - 1$, we get $\|(G_n^\sigma g) - g\|_{\varsigma^+} \geq \frac{\mathcal{F}_{\varsigma,n}^\sigma(g)}{\sigma+n}$ with $\mathcal{F}_{\varsigma,n}^\sigma(g) = (\sigma+n)\cdot\|(G_n^\sigma g) - g\|_{\varsigma^+} > 0$ (since $\|(G_n^\sigma g) - g\|_{\varsigma^+} = 0$ is true for a particular n only if the polynomial f has degree ≤ 1, contradicting the hypothesis on g).

As a result, we now have

$$||(G_n^\sigma g) - g||_{\varsigma^+} \geq \frac{\mathcal{H}_\varsigma^\sigma(g)}{\sigma+n},$$

for all $n > \max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\}$, where

$$\mathcal{H}_\varsigma^\sigma(g) = \min_{n_0-1 \geq n > \max\left\{\frac{(\mathcal{B}+h\sigma-\sigma)}{(1-h)}, \frac{\Lambda(\varsigma+1)}{1-\Lambda(\varsigma+1)} - \sigma\right\}} \left\{ \mathcal{F}_{\varsigma,n}^\sigma(g), ..., \mathcal{F}_{\varsigma,n_0-1}^\sigma(g), \frac{1}{2}\left\| (e_0+\sigma e_2)g' + \frac{e_1(e_1+2)}{2} g'' \right\|_{\varsigma^+} \right\},$$

which paired with Theorem 4, establishes the required conclusion.

Remark 2 Various researchers have explored different approximation properties in real case for exponential operators, semi exponential operators and their hybrid variants, we mention some of their works here [7,9,18,19]. Here in the present study, we examined the approximation properties for the summation-integral type operator (3) in the real domain and in the complex domain. Further, some efforts can be made to compare the considered operators (3) in complex domain with the corresponding classical operators in real case.

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