On the dynamics of a family of planar differential systems with two limit cycles explicitly given

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Abstract

For a given family of planar differential systems it is a very diffcult problem to determine its limit cycles. In this paper we give a family of planar polynomial differential systems of degree 7 with two limit cycles explicitly given, one of them is an algebraic limit cycles and the other one is a non-algebraic limit cycles. Concrete examples exhibiting the applicability of our results are introduced.

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1 Introduction

We consider here a class of autonomous two dimensional dynamical systems of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = P_n(x,y), \\ \\ y' = \dfrac{dy}{dt} = Q_n(x,y), \end{cases} \tag{1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are real polynomials in the variables x and y. The degree of the system (1) is the maximum of the degrees of the polynomials P and Q. A limit cycle of system (1) is an isolated periodic solution in the set of all periodic solutions of system (1) see [2, 6], and it is said to be algebraic if it is contained in the zero level set of a polynomial function [3, 4]. In 1900 Hilbert

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[8] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$, this has been one of the main problems in the qualitative theory of planar differential equations in the 20th century [5, 9].

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} P_n(x,y) + \frac{\partial H(x,y)}{\partial y} Q_n(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant.

Since for such vector fields the notion of integrability is based on the existence of a first integral [4, 6], the following question arises: Given the polynomial differential systems (1), how to recognize if this polynomial differential systems has a first Integral? and how to compute it when it exists?

A curve $U(x,y) = 0$, where $U(x,y)$ is a polynomial with real coefficients, is an invariant algebraic curve of system (1) if and only if there exists a polynomial $K = K(x,y)$ of degree at most $n-1$ satisfying

$$\frac{\partial U(x,y)}{\partial x} P_n(x,y) + \frac{\partial U(x,y)}{\partial y} Q_n(x,y) = K(x,y)\, U(x,y). \tag{2}$$

The polynomial $K(x,y)$ is called the cofactor of $U(x,y) = 0$, if the cofactor is identically zero, then $U(x,y)$ is a polynomial first integral for system (1). If U is real, the curve $U(x,y) = 0$ is an invariant under the flow of differential system (1) and the set $\left\{(x,y) \in \mathbb{R}^2,\ U(x,y) = 0\right\}$ is formed by orbits of system (1).

In the work of J. Giné and M. Grau [7], they gave families of systems of the form (1) with explicit limit cycles, there are systems of their family with algebraic limit cycles and other systems with non-algebraic ones, and they gave an example for degree 9 of coexistence of an algebraic limit cycle with a non-algebraic one, this it is the first result for the coexistence of algebraic and non-algebraic limit cycles. Another result goes back to A. Bendjeddou and R. Cheurfa [1] for degree 5. In this work, we obtain a new result of coexistence of limit cycles for an another class of systems of degree 7.

In this paper we introduce an explicit expression of invariant algebraic curves, then we proved that these systems are integrable and we introduce an explicit expression of a first integral of a multi-parameter polynomial differential systems planar of degree seven of the form

$$\begin{cases} x' = xP_2(x,y) + P_5(x,y) - xP_6(x,y), \\ y' = yP_2(x,y) - Q_5(x,y) - yP_6(x,y), \end{cases} \tag{3}$$

where

$$\begin{array}{l} P_2(x,y) = \lambda x^2 + (m-b)\,xy + \lambda y^2, \\ P_5(x,y) = \left(x^2+y^2\right)\left(ax^3 + bx^2y + cxy^2 + my^3\right), \\ P_6(x,y) = \left(x^2+y^2\right)^2\left((a+\lambda)\,x^2 + (c+\lambda)\,y^2\right), \\ Q_5(x,y) = \left(x^2+y^2\right)\left(mx^3 - ax^2y + (2m-b)\,xy^2 - cy^3\right), \end{array}$$

in which a, b, c and λ are real constants and $m \in \mathbb{R} - \{0\}$.

Moreover, we determine sufficient conditions for a polynomial differential system to possess two limit cycles : one of them is algebraic and the other one is shown to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our results are introduced.

We define the trigonometric functions
$f(\theta) = \lambda + \frac{1}{2}(m-b)\sin 2\theta$, $g(\theta) = \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}(a-c)\cos 2\theta + \frac{1}{2}(b-m)\sin 2\theta$,
$k(\theta) = -\frac{1}{2}a - \frac{1}{2}c - \lambda + \frac{1}{2}(c-a)\cos 2\theta$,
$A(\theta) = \dfrac{2}{m}\displaystyle\int_0^\theta (2k(s) + g(s))\,ds$ and $B(\theta) = \dfrac{2}{m}\displaystyle\int_0^\theta (k(w)\exp(-A(w)))\,dw$.

2 Main result

Our main results are contained in the following Theorem.

Theorem 1 *Consider a multi-parameter differential systems* (3)*, then the following statements hold.*

1) The origin $O(0,0)$ is the unique critical point at finite distance.

2) The curve $U(x,y) = x^2 + y^2 - 1$ is an invariant algebraic curve of system (3)*.*

3) If $m \in \mathbb{R} - \{0\}$, the system (3) *has the first integral, given in Cartesian coordinates by*

$$H(x,y) = \frac{\left(1-x^2-y^2\right)B\left(\arctan\frac{y}{x}\right) + \exp\left(-A\left(\arctan\frac{y}{x}\right)\right)}{x^2+y^2-1}.$$

4) The system (3) *has an explicit limit cycle, given in Cartesian coordinates by $(\Gamma_1) : x^2 + y^2 - 1 = 0$.*

5) If

$$\begin{array}{l} m(a+c+4\lambda) + |am - cm| + \left|m^2 - bm\right| < 0, \ b - m \neq 0, \\ m(a+c+2\lambda) + |cm - am| < 0 \ \text{ and } m \in \mathbb{R} - \{0\}. \end{array} \tag{4}$$

Then system (3) *has non-algebraic limit cycle (Γ_2), explicitly given in polar coordinates (r,θ), by the equation*

$$r(\theta, r_*) = \sqrt{\frac{B(2\pi) - B(\theta) + B(\theta)\exp(-A(2\pi)) + (-1+\exp(-A(2\pi)))\exp(-A(\theta))}{B(2\pi) - B(\theta) + B(\theta)(\exp A(2\pi))}}.$$

Moreover, the algebraic limit cycle (Γ_1) lies inside the non-algebraic limit cycle (Γ_2).

Proof.

Proof of statement (1).

By definition, $A(x_*, y_*) \in \mathbb{R}^2$ is critical point of system (3) if,

$$\begin{cases} x_* P_2(x_*, y_*) + P_5(x_*, y_*) - x_* P_6(x_*, y_*) = 0, \\ y_* P_2(x_*, y_*) - Q_5(x_*, y_*) - y_* P_6(x_*, y_*) = 0, \end{cases}$$

that is to say $x_*^6 + y_*^6 + 3x_*^2 y_*^4 + 3x_*^4 y_*^2 = 0$, then $x_* = 0, y_* = 0$ is the unique solution of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of statement (1) of Theorem 1.

Proof of statement (2).

An computation shows that $U(x,y) = x^2 + y^2 - 1$ satisfy the linear partial differential equation (2), the associated cofactor being

$$K(x,y) = -2\left(x^2+y^2\right)\left((m-b)\,xy + \left((a+\lambda)\,x^2 + (c+\lambda)\,y^2 + \lambda\right)\left(x^2+y^2\right)\right),$$

then the curve $U(x,y) = 0$ is an invariant algebraic curve of system (3) with cofactor $K(x,y)$.

This completes the proof of statement (2) of Theorem 1.

Proof of statement (3), (4) and (5) of Theorem 1.

In order to prove our results (3) , (4) and (5) we write the polynomial differential system (3) in polar coordinates (r,θ) , defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system becomes

$$\begin{cases} r' = \dfrac{dr}{dt} = f(\theta)\,r^3 + g(\theta)\,r^5 + k(\theta)\,r^7, \\ \theta' = \dfrac{d\theta}{dt} = -mr^4. \end{cases} \tag{5}$$

Suppose that $m \in \mathbb{R} - \{0\}$.

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = -\frac{f(\theta)}{m}\frac{1}{r} - \frac{g(\theta)}{m} r - \frac{k(\theta)}{m} r^3. \tag{6}$$

Via the change of variables $\rho = r^2$, this equation (6) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = -\frac{2k(\theta)}{m}\rho^2 - \frac{2g(\theta)}{m}\rho - \frac{2f(\theta)}{m}. \tag{7}$$

We have $f(\theta) + g(\theta) + k(\theta) = 0$ for $\theta \in \mathbb{R}$, this equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $y = \rho - 1$ we obtain the Bernoulli equation

$$\frac{dy}{d\theta} = \frac{-4k(\theta) - 2g(\theta)}{m} y - \frac{2k(\theta)}{m} y^2. \tag{8}$$

We note that $y=0$ is solution for (8).

Assume now that $y\neq 0$. Then, doing the change of variables $z=\dfrac{1}{y}$ we obtain the linear equation

$$\frac{dz}{d\theta}=\frac{2g(\theta)+4k(\theta)}{m}z+\frac{2k(\theta)}{m}. \tag{9}$$

The general solution of linear equation (9) is

$$z(\theta)=(\alpha+B(\theta))\exp A(\theta),$$

where $\alpha\in\mathbb{R}$, $A(\theta)=\dfrac{2}{m}\displaystyle\int_0^\theta(2k(s)+g(s))\,ds$ and

$B(\theta)=\dfrac{2}{m}\displaystyle\int_0^\theta(k(w)\exp(-A(w)))\,dw.$

Then the general solution of equation (8) is

$$y(\theta)=0,\quad y(\theta)=\frac{\exp(-A(\theta))}{\alpha+B(\theta)},\ \text{where } \alpha\in\mathbb{R}.$$

Then the general solution of equation (7) is

$$\rho(\theta)=1,\qquad \rho(\theta)=\frac{\alpha+B(\theta)+\exp(-A(\theta))}{\alpha+B(\theta)},\ \text{where } \alpha\in\mathbb{R}.$$

Consequently, the general solution of (6) is

$$r(\theta,\alpha)=1,\qquad r(\theta,\alpha)=\sqrt{\frac{\alpha+B(\theta)+\exp(-A(\theta))}{\alpha+B(\theta)}},\ \text{where }\ \alpha\in\mathbb{R}.$$

From these solution we obtain a first integral in the variables (x,y) of the form

$$H(x,y)=\frac{\left(1-x^2-y^2\right)B\left(\arctan\frac{y}{x}\right)+\exp\left(-A\left(\arctan\frac{y}{x}\right)\right)}{x^2+y^2-1}.$$

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (3) is Liouville integrable.

Hence, statement (3) of Theorem 1 is proved.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (3), in Cartesian coordinates are written as

$$\begin{aligned} x^2+y^2 &= 1,\\ x^2+y^2 &= \frac{h-B\left(\arctan\frac{y}{x}\right)+\exp\left(-A\left(\arctan\frac{y}{x}\right)\right)}{h-B\left(\arctan\frac{y}{x}\right)}, \end{aligned}$$

where $h\in\mathbb{R}$.

Notice that system (3) has a periodic orbit if and only if equation (6) has a strictly positive 2π periodic solution. This, moreover, is equivalent to the existence of a solution of (6) that fulfills $r(0,r_*) = r(2\pi,r_*)$ and $r(\theta,r_*) > 0$ for any θ in $[0,2\pi]$, this r_* is the intersection of the periodic orbit with the OX_+ axis.

The solution $r(\theta,r_0)$ of the differential equation (6) such that $r(0,r_0) = r_0$ is

$$r(\theta,r_0) = \sqrt{\frac{1+(B(\theta)+\exp(-A(\theta)))\left(r_0^2-1\right)}{1+B(\theta)\left(r_0^2-1\right)}}.$$

A periodic solution of system (3) must satisfy the condition $r(2\pi,r_0) = r(0,r_0)$, which leads to two values $r_0 = 1$ and $r_0 = r_*$ and the r_* is given by

$$r_* = \sqrt{\frac{-1+B(2\pi)+\exp(-A(2\pi))}{B(2\pi)}}.$$

After the substitution of the value $r_0 = 1$ into $r(\theta,r_0)$ we obtain

$$r(\theta,1) = 1 > 0, \text{ for all } \theta \in [0,2\pi].$$

Moreover, we compute

$$\frac{d}{dr_0} r(\theta,r_0)\Big|_{\substack{r_0=1\\ \theta=2\pi}} = \exp(-A(2\pi)) > 1.$$

Because, according to the conditions (4), we have $A(\theta) < 0$, for all $\theta \in [0,2\pi]$, hence $A(2\pi) < 0$. This is a stable and hyperbolic limit cycle for the differential systems (3) for more details see $[5,$ section $1.6]$.

We note this limit cycle by (Γ_1), in Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, this limit cycle is given by: $(\Gamma_1) : x^2 + y^2 - 1 = 0$, this is the algebraic limit cycle for the differential systems (3).

Hence, statement (4) of Theorem 1 is proved.

Suppose now that $m(a+c+2\lambda) + |cm - am| < 0$, $m(a+c+4\lambda) + |am - cm| + \left|m^2 - bm\right| < 0$ and $m \in \mathbb{R} - \{0\}$.

The second solution of $r(2\pi,r_0) = r(0,r_0)$ is $r_0 = r_*$, given by

$$r_* = \sqrt{\frac{-1+B(2\pi)+\exp(-A(2\pi))}{B(2\pi)}},$$

where

$$A(2\pi) = \int_0^{2\pi} \left(\frac{a+c+4\lambda+(m-b)(\sin 2s)+(a-c)(\cos 2s)}{m}\right) ds,$$

$$B(2\pi) = \int_0^{2\pi} \left(\frac{-a-c-2\lambda+(c-a)\cos 2w}{m} \exp(-A(w))\right) dw.$$

According to the conditions (4), we have $A(2\pi)<0$ and $B(2\pi)>0$, hence $r_*>0$.

After the substitution of these value r_* into $r(\theta,r_0)$ we obtain

$$r(\theta,r_*)=\sqrt{\frac{B(2\pi)-B(\theta)+B(\theta)\exp(-A(2\pi))+(-1+\exp(-A(2\pi)))\exp(-A(\theta))}{B(2\pi)-B(\theta)+B(\theta)\exp(A(2\pi))}}.$$

In what follows it is proved that $r(\theta,r_*)>0$. Indeed

$$B(\theta)=\int_0^\theta\left(\frac{-a-c-2\lambda+(c-a)\cos 2w}{m}\exp(-A(w))\right)dw,$$

$$\begin{aligned}B(2\pi)-B(\theta) &= \int_0^{2\pi}\left(\frac{2k(w)}{m}\exp(-A(w))\right)dw-\int_0^{\theta}\left(\frac{2k(w)}{m}\exp(-A(w))\right)dw\\ &= \int_\theta^{2\pi}\left(\frac{-a-c-2\lambda+(c-a)\cos 2w}{m}\exp(-A(w))\right)dw.\end{aligned}$$

According to the conditions (4), we have $A(\theta)<0$ and $B(2\pi)-B(\theta)>0$ for all $\theta\in[0,2\pi]$, hence

$$r(\theta,r_*)>0 \text{ for all } \theta\in[0,2\pi].$$

Moreover, we compute

$$\frac{d}{dr_0}r(\theta,r_0)\Big|_{\substack{r_0=r_*\\ \theta=2\pi}}=\exp(A(2\pi))<1.$$

Because, $A(\theta)<0$, for all $\theta\in[0,\pi]$, hence $A(2\pi)<0$.

This is an unstable and hyperbolic limit cycle for the differential systems (3), we note it by (Γ_2).

This limit cycle is not algebraic, more precisely, in Cartesian coordinates $r^2=x^2+y^2$ and $\theta=\arctan\left(\frac{y}{x}\right)$, the curve (Γ_2) defined by this limit cycle is $(\Gamma_2): L(x,y)=0$ where.

$$L(x,y)=x^2+y^2-1-\frac{e^{-A\left(\arctan\frac{y}{x}\right)}}{\eta+B\left(\arctan\frac{y}{x}\right)}=0,$$

in which $\eta=\frac{1}{r_*^2-1}$.

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $L(x,y)=0$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^n}{(\partial x)^n}L(x,y)=0$, and this is not the case because in the derivative

$$\frac{\partial}{\partial x}L(x,y)=2x+e^{-A\left(\arctan\frac{y}{x}\right)}\frac{\left(\eta+B\left(\arctan\frac{y}{x}\right)\right)\frac{\partial}{\partial x}A\left(\arctan\frac{y}{x}\right)+\frac{\partial}{\partial x}B\left(\arctan\frac{y}{x}\right)}{\left(\eta+B\left(\arctan\frac{y}{x}\right)\right)^2},$$

it appears again the expression $e^{-A\left(\arctan\frac{y}{x}\right)}$, which already appears in $L\left(x,y\right)$, and this expression will appear in the partial derivative at any order, more precisely, we have,

$$e^{-A\left(\arctan\frac{y}{x}\right)}=e^{\frac{m-b}{2m}}e^{\frac{a+c+4\lambda}{m}\arctan\frac{y}{x}}e^{\frac{b-m}{2m}\left(\cos 2\arctan\frac{y}{x}\right)}e^{\frac{a-c}{2m}\left(\sin 2\arctan\frac{y}{x}\right)}.$$

According to the conditions (4), we have $(b-m)\,m\neq 0$, then the non-algebraic expression $e^{\frac{b-m}{2m}\left(\cos 2\arctan\frac{y}{x}\right)}$ appears in the $e^{-A\left(\arctan\frac{y}{x}\right)}$ hence the expression $e^{-A\left(\arctan\frac{y}{x}\right)}$ is not algebraic. Consequently, $L\left(x,y\right)=0$ is not algebraic, therefore the curve (Γ_2): $L\left(x,y\right)=0$ is non-algebraic and the limit cycle will also be non-algebraic.

According to the conditions (4), hence $A\left(2\pi\right)<0$, $B\left(2\pi\right)>0$, we get

$$r_*=\sqrt{1+\frac{-1+\exp\left(-A\left(2\pi\right)\right)}{B\left(2\pi\right)}}>1.$$

We conclude that system (3) has two limit cycles, the algebraic (Γ_1) lies inside the non-algebraic one (Γ_2).

This completes the proof of statement (5) of Theorem 1. ■

3 Examples

The following examples are given to illustrate our results.

Example 1 If we take $a=-14$, $b=2$, $c=-11$, $m=1$ and $\lambda=10$, then system (3) reads

$$\begin{cases} x'=x\left(10x^2-xy+10y^2\right)+x\left(x^2+y^2\right)^2\left(4x^2+y^2\right)+ \\ \left(x^2+y^2\right)\left(-14x^3+2x^2y-11xy^2+y^3\right), \\ \\ y'=y\left(10x^2-xy+10y^2\right)+y\left(x^2+y^2\right)^2\left(4x^2+y^2\right)- \\ \left(x^2+y^2\right)\left(x^3+14x^2y+11y^3\right). \end{cases} \tag{10}$$

This system (10) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1) : $x^2+y^2-1=0$.

Then system (10) has the first integral, given in Cartesian coordinates by

$$H\left(x,y\right)=\frac{\left(1-x^2-y^2\right)B\left(\arctan\frac{y}{x}\right)+\exp A\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1}.$$

This system (10) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r\left(\theta,r_*\right)=\sqrt{\frac{B\left(2\pi\right)-B\left(\theta\right)+B\left(\theta\right)\exp\left(-A\left(2\pi\right)\right)+\left(-1+\exp\left(-A\left(2\pi\right)\right)\right)\exp\left(-A\left(\theta\right)\right)}{B\left(2\pi\right)-B\left(\theta\right)+B\left(\theta\right)\exp\left(A\left(2\pi\right)\right)}},$$

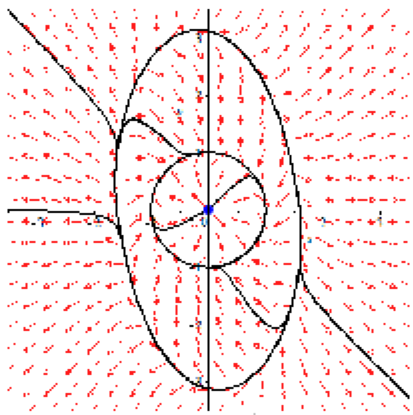
where $\theta \in \mathbb{R}$, $A(\theta) = \frac{1}{2} - 15\theta + \frac{3}{2}\sin 2\theta - \frac{1}{2}\cos 2\theta$ and

$$B(\theta) = \int_0^\theta \left((5 + 3\cos 2w) \exp\left(\frac{-1}{2} + 15w - \frac{3}{2}\sin 2w + \frac{1}{2}\cos 2w \right) \right) dw.$$

Moreover, the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt{\frac{-1 + B(2\pi) + \exp(-A(2\pi))}{B(2\pi)}} = 1.5946$$

We conclude that system (10) has two limit cycles. Since $r_* = 1.5946 > 1$, the algebraic lies inside the non-algebraic one.



Limit cycles of system (10)

Example 2 If we take $a = -11$, $b = 3$, $c = -9$, $m = 2$ and $\lambda = 8$, then system (3) reads

$$\begin{cases} x' = x\left(8x^2 - xy + 8y^2\right) + x\left(x^2 + y^2\right)^2\left(3x^2 + y^2\right) + \\ \left(x^2 + y^2\right)\left(-11x^3 + 3x^2y - 9xy^2 + 2y^3\right), \\ \\ y' = y\left(8x^2 - xy + 8y^2\right) + y\left(x^2 + y^2\right)^2\left(3x^2 + y^2\right) - \\ \left(x^2 + y^2\right)\left(2x^3 + 11x^2y + xy^2 + 9y^3\right). \end{cases} \tag{11}$$

This system (11) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1) : $x^2 + y^2 - 1 = 0$.

Then system (11) has the first integral, given in Cartesian coordinates by

$$H(x, y) = \frac{\left(1 - x^2 - y^2\right) B\left(\arctan \frac{y}{x}\right) + \exp A\left(\arctan \frac{y}{x}\right)}{x^2 + y^2 - 1}.$$

This system (11) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r, θ) is

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$$r(\theta, r_*) = \sqrt{\frac{B(2\pi) - B(\theta) + B(\theta)\exp(-A(2\pi)) + (-1 + \exp(-A(2\pi)))\exp(-A(\theta))}{B(2\pi) - B(\theta) + B(\theta)\exp(A(2\pi))}},$$

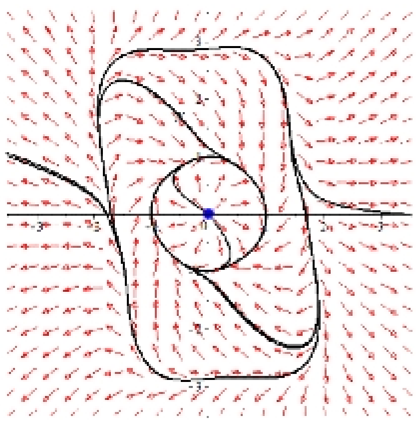
where $\theta \in \mathbb{R}$, $A(\theta) = \frac{1}{4} - 6\theta + \frac{1}{2}\sin 2\theta - \frac{1}{4}\cos 2\theta$ and

$$B(\theta) = \int_0^\theta \left((2 + \cos 2w)\exp\left(\frac{-1}{4} + 6w - \frac{1}{2}\sin 2w + \frac{1}{4}\cos 2w\right)\right) dw.$$

Moreover, the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt{\frac{-1 + B(2\pi) + \exp(-A(2\pi))}{B(2\pi)}} = 1.6808$$

We conclude that system (11) has two limit cycles. Since $r_* = 1.6808 > 1$, the algebraic lies inside the non-algebraic one.



Limit cycles of system (11)

Example 3 If we take $a = -\frac{7}{2}$, $b = \frac{3}{2}$, $c = -\frac{5}{2}$, $m = 1$ and $\lambda = 2$, then system (3) reads

$$\begin{cases} x' = x\left(2x^2 - \frac{1}{2}xy + 2y^2\right) + \frac{1}{2}x\left(x^2 + y^2\right)^2\left(3x^2 + y^2\right) + \\ \left(x^2 + y^2\right)\left(-\frac{7}{2}x^3 + \frac{3}{2}x^2y - \frac{5}{2}xy^2 + y^3\right), \\ \\ y' = y\left(2x^2 - \frac{1}{2}xy + 2y^2\right) + \frac{1}{2}y\left(x^2 + y^2\right)^2\left(3x^2 + y^2\right) - \\ \left(x^2 + y^2\right)\left(x^3 + \frac{7}{2}x^2y + \frac{1}{2}xy^2 + \frac{5}{2}y^3\right). \end{cases} \tag{12}$$

This system (12) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1) : $x^2 + y^2 - 1 = 0$.

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Then system (12) has the first integral, given in Cartesian coordinates by

$$H(x,y)=\frac{\left(1-x^2-y^2\right)B\left(\arctan\frac{y}{x}\right)+\exp A\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1}.$$

This system (12) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\sqrt{\frac{B(2\pi)-B(\theta)+B(\theta)\exp(-A(2\pi))+(-1+\exp(-A(2\pi)))\exp(-A(\theta))}{B(2\pi)-B(\theta)+B(\theta)\exp(A(2\pi))}},$$

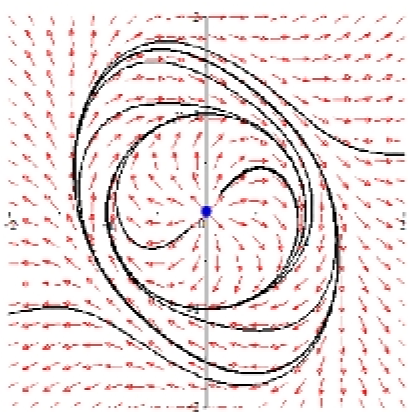
where $\theta\in\mathbb{R}$, $A(\theta)=\frac{1}{4}-2\theta+\frac{1}{2}\sin 2\theta-\frac{1}{4}\cos 2\theta$ and

$$B(\theta)=\int_0^\theta\left((2+\cos 2w)\exp\left(\frac{-1}{4}+2w-\frac{1}{2}\sin 2w+\frac{1}{4}\cos 2w\right)\right)dw.$$

Moreover, the intersection of the limit cycle with the OX_+ axis is the point having

$$r_*=\sqrt{\frac{-1+B(2\pi)+\exp(-A(2\pi))}{B(2\pi)}}=1.296$$

We conclude that system (12) has two limit cycles. Since $r_*=1.296>1$, the algebraic lies inside the non-algebraic one.



Limit cycles of system (12)

Example 4 If we take $a=-10$, $b=4$, $c=-12$, $m=3$ and $\lambda=7$, then system (3) reads

$$\begin{cases} x'=x\left(7x^2-xy+7y^2\right)-\left(x^2+y^2\right)\left(10x^3-4x^2y+12xy^2-3y^3\right)\\ +x\left(x^2+y^2\right)^2\left(3x^2+5y^2\right),\\ y'=y\left(7x^2-xy+7y^2\right)-\left(x^2+y^2\right)\left(3x^3+10x^2y+2xy^2+12y^3\right)\\ +y\left(x^2+y^2\right)^2\left(3x^2+5y^2\right). \end{cases} \quad (13)$$

This system (13) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1) : $x^2+y^2-1=0$.

Then system (13) has the first integral, given in Cartesian coordinates by

$$H(x,y)=\frac{\left(1-x^2-y^2\right)B\left(\arctan\frac{y}{x}\right)+\exp A\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1}.$$

This system (13) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\sqrt{\frac{B(2\pi)-B(\theta)+B(\theta)\exp(-A(2\pi))+(-1+\exp(-A(2\pi)))\exp(-A(\theta))}{B(2\pi)-B(\theta)+B(\theta)\exp(A(2\pi))}},$$

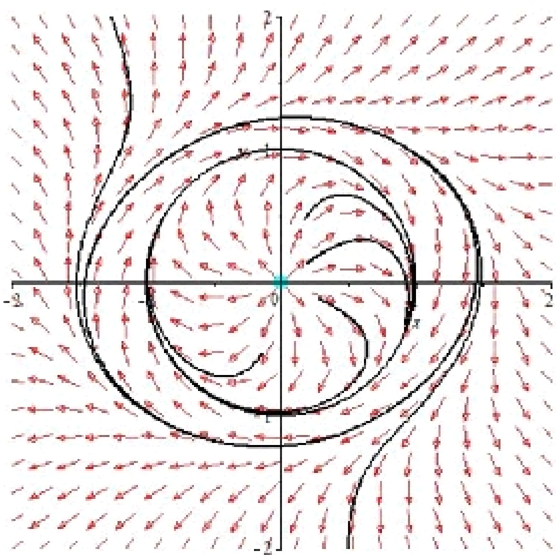
where $\theta\in\mathbb{R}$, $A(\theta)=-\frac{1}{6}+2\theta+\frac{1}{6}\cos 2\theta+\frac{1}{3}\sin 2\theta$ and

$$B(\theta)=\int_0^\theta\left(\left(\frac{8}{3}-\frac{2}{3}\cos 2w\right)\exp\left(\frac{1}{6}-2w-\frac{1}{6}\cos 2w-\frac{1}{3}\sin 2w\right)\right)dw.$$

Moreover, the intersection of the limit cycle with the OX_+ axis is the point having

$$r_*=\sqrt{\frac{-1+B(2\pi)+\exp(-A(2\pi))}{B(2\pi)}}=1.4511$$

We conclude that system (13) has two limit cycles. Since $r_*=1.4511>1$, the algebraic lies inside the non-algebraic one.



Limit cycles of system (13)

References

[1] A. Bendjeddou, R. Cheurfa; *Coexistence of algebraic and non-algebraic limit cycles for quintic polynomial differential systems*, Elect. J. of Diff. Equ, **2017**, no. 71 (2017), 1-7.

[2] R. Benterki, J. Llibre, *Polynomial differential systems with explicit non-algebraic limit cycles*, Elect. J. of Diff. Equ, vol. **2012**, no. 78, pp. 1-6.

[3] R. Boukoucha; *Explicit limit cycles of a family of polynomial differential systems*, Elect. J. of Diff. Equ, vol. **2017**, no. 217, pp. 1-7.

[4] R. Boukoucha, *Explicit expression for a hyperbolic limit cycles of a class of polynomial differential systems,* Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2017, Vol. 23. No 3. pp. 300-307.

[5] F. Dumortier, J. Llibre, J. Artés; *Qualitative Theory of Planar Differential Systems*, (Universitex) Berlin, Springer, 2006.

[6] A. Gasull, H. Giacomini, J. Torregrosa; *Explicit non-algebraic limit cycles for polynomial systems*, J. Comput. Appl. Math, 2007, vol. **200**, pp. 448-457.

[7] J. Giné, M. Grau; *Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations.* Nonlinearity, 2006, vol. **19**, pp. 1939-1950.

[8] D. Hilbert; *Mathematische Probleme, Lecture, Second Internat. Congr. Math.* (Paris, 1900), Nachr. Ges. Wiss. Gttingen Math. Phys. Kl, 1900, pp. 253-297, English transl, Bull. Amer. Math. Soc. 8, 1902, pp. 437-479.

[9] L. Perko; *Differential Equations and Dynamical Systems*, Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.