ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

MAXIMAL ORDER GROUP ACTIONS ON RIEMANN SURFACES OF GENUS $1+3p$

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ABSTRACT. A natural problem is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g. Let $N(g)$ (respectively $M(g)$) be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ preserving the orientation (respectively possibly reversing the orientation) of the surface.

Let $g = 1+3p$ for a large prime p. It has been established that if p is congruent to $1 \pmod 6$, then $N(g) = M(g) = 24(g-1)$. Suppose p is congruent to $5 \pmod 6$. We prove that if p is also congruent modulo 25 to 1, 6, 11 or 16, then $N(g) = 8(g+11)$ and $M(g) = 16(g+11)$; otherwise $N(g) = 8(g+1)$ and $M(g) = 16(g+1)$.

1. Introduction.

A finite group G can be represented as a group of automorphisms of a compact Riemann surface. In other words, G acts on a Riemann surface. The group actions were required, in most of the classical work, to preserve the orientation of the Riemann surface. It is possible, of course, to allow a group action to reverse the orientation of the surface.

Among the most interesting group actions for a particular value of the genus g are those such that the orders of the groups are "large" relative to the genus g. A natural problem, then, is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g.

First, let $N(g)$ be the largest order of a group of orientation preserving automorphisms of a Riemann surface of genus $g \geq 2$. Also, let $M(g)$ be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ (possibly reversing the orientation of the surface). Clearly, $N(g) \leq M(g)$.

Suppose the group G acts on the Riemann surface X of genus $g \geq 2$ (possibly reversing the orientation of X). Let G^+ be the subgroup of G consisting of the orientation preserving automorphisms. Then $|G^+| \leq N(g)$ and

$$|G| \leq 2|G^+| \leq 2N(g). \tag{1}$$

Consequently, if $|G| = M(g)$, we obtain the basic inequalities comparing $N(g)$ and $M(g)$,

$$N(g) \leq M(g) \leq 2N(g). \tag{2}$$

The classical upper bound of Hurwitz shows that, for all $g \geq 2$,

$$N(g) \leq 84(g-1) \text{ and } \mathrm{M(g)} \leq 168(\mathrm{g}-1). \tag{3}$$

The lower bounds for both parameters have also been established. For all $g \geq 2$,

$$N(g) \geq 8(g+1) \text{ and } \mathrm{M(g)} \geq 16(\mathrm{g}+1). \tag{4}$$

2020 *Mathematics Subject Classification.* Primary: 57M60; Secondary: 20F38, 20H10.
Key words and phrases. Riemann surface, genus, group action, NEC group.

The lower bound for $N(g)$ was established independently by Accola [1] and Maclachlan [13]. The lower bound for $M(g)$ was obtained by constructing, for each $g \geq 2$, a group of order $16(g+1)$ that acts on a Riemann surface of genus g [22, Th. 1.1]. Singerman noted this in ([21, p. 24]). Each of the four bounds in (3) and (4) is the best possible, that is, there are infinitely many g such that the bound is attained.

In general, determining $N(g)$ (or $M(g)$) for a particular g or for all g with a particular form is a very difficult problem. The difficulty is related to the form of the integer $g-1$ (which is $-1/2$ times the Euler characteristic of a Riemann surface of genus g). Both $N(g)$ and $M(g)$ have been completely determined for the simplest case, in which $g-1$ is an odd prime. Accola first determined $N(1+p)$ for all odd primes $p > 84$ [2, Th. 7.11, p. 84]. Also important here is the work of Belolipetsky and Jones [4] on orientation preserving actions on compact Riemann surfaces of genus $p+1$ for an odd prime p. Their work yields another determination of $N(1+p)$ for all primes p [4, Th. 2]. The analogous result for the parameter $M(g)$ has also been determined. The main result of [16] is the determination of $M(1+p)$ for all primes p [16, Th. 1].

The next natural step is to determine the parameters $N(g)$ and $M(g)$ in case $g-1$ is a small multiple of a prime p. First, Accola calculated $N(1+2p)$ for all primes p [2, Th. 7.17, p. 93]. In [22, Th. 6.3] it was shown that $N(1+2p) = M(1+2p) = 48p$ for p congruent to $1 \pmod 6$ and $p > (24)^2$. The parameter $M(1+2p)$ has not yet been found for p congruent $5 \pmod 6$.

Our focus here is the next step, finding $N(g)$ and $M(g)$ in case $g-1$ is 3 times a prime p. Some of the work has already been done. Let p be a prime such that $p \equiv 1 \pmod 6$ and $p > (36)^2$, and let $g = 1+3p$. Then for any such g, $M(g) = N(g) = 24(g-1)$ [22, Th. 5.7]. This surprising result shows that there are infinitely many g such that $M(g) = N(g)$; this result was the focus of [22].

Intuitively, one expects $M(g)$ to "often" be equal to $2N(g)$. The families of groups for which the lower bounds in (4) are attained provide examples of groups for which $M(g) = 2N(g)$. But it is certainly possible that $M(g) < 2N(g)$ and even for $M(g) = N(g)$.

In any case, our focus here is to complete the determination of both $N(1+3p)$ and $M(1+3p)$ for a prime p. Our main result is the following.

Theorem 1. *Let $g = 1+3p$ for some prime $p > (36)^2$. If p is congruent to $1 \pmod 6$, then $N(g) = M(g) = 24(g-1)$. Suppose p is congruent to $5 \pmod 6$. If p is also congruent modulo 25 to 1, 6, 11 or 16, then $N(g) = 8(g+11)$ and $M(g) = 16(g+11)$; otherwise $N(g) = 8(g+1)$ and $M(g) = 16(g+1)$.*

Alternately, if p is congruent modulo 150 to 11, 41, 101 or 131 and $p > (36)^2$, then $N(1+3p) = 24p+96 = 8(g+11)$ and $M(g) = 2N(g)$.

2. Background results.

Much of the following background information is taken from [15]; also see [7, Section 2]. We shall assume that all surfaces are compact. Group actions on Riemann surfaces have often been studied using non-euclidean crystallographic (NEC) groups . Let $\mathscr{L}$ denote the group of automorphisms of the open upper half-plane U, and let $\mathscr{L}^+$ denote the subgroup of index 2 consisting of the orientation preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathscr{L}$ (with the quotient space U/Γ compact). If $\Gamma \subseteq \mathscr{L}^+$, then Γ is called a *Fuchsian* group. Otherwise Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathscr{L}^+$ of index 2.

Associated with the NEC group Γ is its *signature*, which has the form

$$(p;\pm;[m_1,\cdots,m_t];\{(n_{1,1},\cdots,n_{1,s_1}),\cdots,(n_{k,1},\cdots,n_{k,s_k})\}). \tag{5}$$

The quotient space U/Γ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and non-orientable otherwise. Associated with the signature (5) is a presentation for the NEC group Γ; see [20, p.234]. Further, the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature. This is shown in [20, p.235], where $\mu(\Gamma)$ is given in terms of the topological genus of the quotient surface U/Γ and the periods and link periods of Γ.

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. Let X be a Riemann surface of genus $g \geq 2$. Then X can be represented as U/K where K is a Fuchsian surface group with $\mu(K) = 4\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Riemann surface X. Then there are an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$ and thus the group of automorphisms G is isomorphic to Γ/K.

If Δ is a subgroup of finite index in Γ, then $[\Gamma : \Delta] = \mu(\Delta)/\mu(\Gamma)$. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{6}$$

The simpler, classical case is that G acts on X preserving orientation. This is the case if and only if Γ is a Fuchsian group and G is generated by elements a_i, b_i for $1 \leq i \leq h$ and x_j of order m_j for $1 \leq j \leq k$ with relation $x_1 \cdots x_k[a_1,b_1]\cdots[a_h,b_h] = 1$. Then the application of (6) yields the classical Riemann-Hurwitz equation

$$2g-2 = |G|\left(2h-2+\sum_{j=1}^{k}\left(1-\frac{1}{m_j}\right)\right). \tag{7}$$

The group G acts reversing the orientation of X in case Γ is a proper NEC group. Then it is necessary to check that the surface group K does not contain orientation-reversing elements, or equivalently, the image $\alpha(\Gamma^+)$ has index two in G [19, Th. 1, p. 52]. If this condition holds, then we will say that G has a particular partial presentation *with the Singerman subgroup condition*. The Riemann-Hurwitz equation in this case is more complicated and is in [7, p. 274], for instance. In this case, though, $|G| = 2|G^+|$ and (7) can be employed to calculate the relationship between the genus g and $|G|$.

Let Γ be a proper NEC group. Then Γ has a canonical Fuchsian subgroup Γ^+ of index 2. Further, the quotient group Γ^+/K acts on X preserving orientation. For a particular Fuchsian group Λ, however, there may be more than one type of NEC group Δ such that Δ^+ is isomorphic to Λ; see [21].

Next we quickly survey the Fuchsian groups with relatively small non-euclidean area. We use the notation of [15]. First, an (ℓ,m,n) triangle group is a Fuchsian group Λ with signature

$$(0;+;[\ell,m,n];\{\}), \text{ where } 1/\ell+1/m+1/n<1.$$

If the group G is a quotient of Λ by a surface group, then G has a presentation of the form

$$X^\ell = Y^m = (XY)^n = 1. \tag{8}$$

We will say that G has partial presentation $T(\ell,m,n)$.

There are two types of NEC groups with a triangle group as canonical Fuchsian subgroup. We are interested in the full (or extended) (ℓ,m,n) triangle group is an NEC group Γ with signature

$$(0;+;[\,];\{(\ell,m,n)\}),\text{ where } 1/\ell+1/m+1/n<1.$$

If G is a quotient of Γ (by a surface group), then G has a presentation of the form

$$A^2=B^2=C^2=(AB)^\ell=(BC)^m=(CA)^n=1, \tag{9}$$

and, further, the subgroup generated by AB and BC (the image of Γ^+) has index 2. The partial presentation (9) will be denoted $FT(\ell,m,n)$.

An (ℓ,m,n,t) quadrilateral group is a Fuchsian group Λ with signature

$$(0;+;[\ell,m,n,t];\{\,\}),\text{ where } 1/\ell+1/m+1/n+1/t<2.$$

A quotient group G of Λ has a presentation of the form

$$X^\ell=Y^m=Z^n=(XYZ)^t=1 \tag{10}$$

We will denote this partial presentation $Q(\ell,m,n,t)$.

3. The General Approach.

Let p be an odd prime, and let $g=1+3p$. Let X be a Riemann surface X of genus $g\geq 2$, and let the group G act on X preserving orientation. Then, regardless of whether p is congruent to 1 or 5 modulo 6, we know that if $p\geq(36)^2$, then

$$|G|\leq 24(g-1) \tag{11}$$

[22, Th. 5.6]. We also have the basic lower bound $N(g)\geq 8(g+1)$ for all $g\geq 2$.

Here we will be concerned with primes congruent to 5 modulo 6 and orientation preserving actions such that

$$24(g-1)\geq|G|>8(g+1). \tag{12}$$

Most of the work here is showing that, except for four special congruence classes of primes, there are no group actions satisfying (12) (as long as p is not small). Our general approach is to represent $X=U/K$ and $G=\Gamma/K$, where Γ is a Fuchsian group and K a surface group and then consider two cases, depending upon whether or not $|G|$ is divisible by the prime p.

Corresponding to (12) is a restriction on the non-euclidean area of the Fuchsian group Γ and the types of partial presentations that Γ can have. The area restriction is

$$\frac{1}{12}\leq\mu(\Gamma)/2\pi<\frac{1}{4}\left(1-\frac{2}{g+1}\right). \tag{13}$$

A careful check of the signatures gives the following. Here we have added the specific Riemann-Hurwitz equation for each case. For example, if G has the partial presentation $T(2,4,\lambda)$, then $\mu(\Gamma)/2\pi=(\lambda-4)/4\lambda$. Then using (7) gives $8(g-1)=|G|(\lambda-4)/\lambda$.

Theorem A. *Let G be a group that acts on a Riemann surface of genus $g\geq 2$ preserving the orientation of the surface. If $24(g-1)\geq|G|>8(g+1)$, then G has one of the following partial presentations. The application of the Riemann-Hurwitz equation is included for each case.*

1. $T(2,3,\lambda)$, $12(g-1)=|G|(\lambda-6)/\lambda$ where $\lambda\geq 12$,
2. $T(2,4,\lambda)$, $8(g-1)=|G|(\lambda-4)/\lambda$ where $6\leq\lambda<2(g+1)$,
3. $T(2,5,\lambda)$, $20(g-1)=|G|(3\lambda-10)/\lambda$ where $5\leq\lambda<20$,

4. $T(2,6,\lambda)$, $6(g-1) = |G|(\lambda-3)/\lambda$ *where* $6 \leq \lambda < 12$,
5. $T(2,7,\lambda)$, $28(g-1) = |G|(5\lambda-14)/\lambda$ *where* $7 \leq \lambda \leq 9$,
6. $T(3,3,\lambda)$, $6(g-1) = |G|(\lambda-3)/\lambda$ *where* $4 \leq \lambda < 12$,
7. $T(3,4,\lambda)$, $24(g-1) = |G|(5\lambda-12)/\lambda$ *where* $\lambda = 4,5$,
8. $Q(2,2,2,3)$, $12(g-1) = |G|$.

Now let p be an odd prime number and $g = 1+3p$. Let X be a Riemann surface of genus $g \geq 2$, and let G act on X preserving orientation. If G satisfies the inequality (12), then G has one of the partial presentations in Theorem A. For each of the partial presentations in Theorem A, then Riemann-Hurwitz formulas give $|G|$ in terms of λ and p. For example, if G has partial presentation $T(2,4,\lambda)$, then $|G| = 24p\lambda/(\lambda-4)$. In addition, as long as $(\lambda-4)/6 < p$, then $|G|$ satisfies inequality (12).

Next, as long as the value of λ is bounded above, applying the Riemann-Hurwitz equation in a straightforward way shows that $|G|$ is a multiple of p for large enough values of p. It is also clear that p^2 does not divide $|G|$. In cases (3) - (7) in Theorem A, the prime p needs to be larger than 47 in order to guarantee that p divides the order of G. The exceptional cases $T(2,3,\lambda)$ and $T(2,4,\lambda)$ where λ does not have an upper bound must be treated separately. In summary, we have the following.

Lemma 1. *Let p be an odd prime with $p > 47$, and let $g = 1+3p$. Let G act on a surface of genus g preserving orientation such that $|G|$ satisfies the inequality (12). If G has one of the partial presentations (3) - (8) in Theorem A, then p divides $|G|$.*

4. $T(2,3,\lambda)$ groups.

Assume p is a prime, and let $g = 3p+1$. Here it is not necessary to assume $p \equiv 5 \pmod 6$, but we need to assume that p is not small in order to apply the following useful result of Accola [1, Lemma 5, p. 402]. We use the argument from the proof of [22, Lemma 5.1].

Accola's Lemma. *Let G be a non-abelian group with partial presentation $T(2,3,\lambda)$. If G has order $\mu\lambda$, then $\lambda \leq \mu^2$.*

Lemma 2. *Let p be an odd prime, and let $g = 1+3p$. Let G act on a surface of genus g preserving orientation having partial presentation $T(2,3,\lambda)$, with $\lambda \geq 12$. If the prime $p > (36)^2$, then p divides $|G|$.*

Proof. By Theorem A 1), $|G| = 36p\lambda/(\lambda-6)$ so that $72p\lambda = |G|(\lambda-6)$. Now by Euclid's Lemma, either p divides $|G|$ or p divides $(\lambda-6)$.

Assume that p divides $(\lambda-6)$ and write $\lambda-6 = mp$ for some integer $m \geq 1$. Now $\lambda = mp+6 > p > (36)^2$ (by assumption). But on the other hand, $|G| = 36p\lambda/mp = 36\lambda/m$. Then the group of orientation preserving automorphisms G is a $T(2,3,\lambda)$ group of order $\mu\lambda$, where $\mu = 36/m \leq 36$. Now by Accola's Lemma, $p < \lambda \leq \mu^2 \leq (36)^2$, an obvious contradiction. Thus, if G is a $T(2,3,\lambda)$ group (and $p > (36)^2$), then p divides $|G|$. □

Hence, assuming $p > (36)^2$ guarantees that p divides $|G|$ in case G has partial presentation $T(2,3,\lambda)$.

5. $T(2,4,\lambda)$ groups.

We need to examine the structure of a general $T(2,4,\lambda)$ group. Let G be a group with partial presentation $T(2,4,\lambda)$ of order $n=\mu\lambda$. Let $G=\langle a,b\rangle$ with $a^2=b^4=(ab)^\lambda=1$, and set $c=ab$ and $d=ba$. Note that $G=\langle a,c\rangle$ and that d is a conjugate of c. Also μ is the index of $\langle c\rangle$ in G.

Let $J=\langle c\rangle\cap\langle d\rangle$. Then $J=\langle c^k\rangle$ for some k that divides λ. Also J is normal in G, and G/J is a $T(2,4,k)$ group of order μk.

The notation, Z_n is the cyclic group of order n, D_n is the dihedral group of order $2n$ and S_n is the symmetric group on n elements, will be used throughout this section. For $k\le 5$, $T(2,4,k)$ is a full presentation of a well-known finite group. Specifically, $T(2,4,1)\cong Z_2$, $T(2,4,2)\cong D_4$, $T(2,4,3)\cong S_4$ and $T(2,4,5)\cong S_5$. So for $k=1$, it follows that G is an extension of Z_λ by Z_2 and has order 2λ. Using the Riemann-Hurwitz equation from Theorem A, we see that $\lambda=4g$ and therefore, $|G|=8g$. By Theorem A, this group will not have maximal order. So $k>1$ for the groups in which we are interested.

Next, if $k=2$, then $T(2,4,2)\cong D_4$ and so $\mu=4$. Also $T(2,4,3)\cong S_4$, with $\mu=8$ and $T(2,4,5)\cong S_5$, with $\mu=24$.

Since $G=\langle a,c\rangle$, the subgroup $\langle c,d\rangle$ has index one or two in G. Thus there are two cases. Let ℓ be the index of $\langle c,d\rangle$ in G so that ℓ is 1 or 2. Since μ is the index of $\langle c\rangle$ in G, it follows that μ/ℓ is the index of $\langle c\rangle$ in $\langle c,d\rangle$.

Lemma 3. *Let G be a group with partial presentation $T(2,4,\lambda)$ of order $n=\mu\lambda$. Let $k=\lambda/|J|$, where $J=\langle c\rangle\cap\langle d\rangle$ as defined above. Then $\mu/\ell\ge k$.*

Proof. Consider the group $\langle c,d\rangle/J$ of order $\mu k/\ell$. Accola [1, p. 401] has shown this group has k^2 distinct elements of the form $(cJ)^i(dJ)^j$, where i and j are between 0 and $k-1$. So $\mu k/\ell\ge k^2$ and we are done. □

Lemma 4. *Suppose $G=\langle c,d\rangle$. If 4 divides $k\mu$, then $\lambda\le\mu^2$.*

Proof. The following proof comes directly from Accola [1, Lemma 4, p. 401]. Since $G=\langle c,d\rangle$, J is central in G. By Lemma 3, $\mu\ge k$. Now, the transfer map into J is $g\mapsto g^{k\mu}$. Since 4 divides $k\mu$, this map takes both a and b to the identity and so it is the zero map. Hence λ divides $k\mu$ and we are done. □

Now we focus on orientation preserving actions on surfaces of genus $g=1+3p$, where p is an odd prime. We begin by applying the Riemann-Hurwitz equation and Euclid's Lemma, as in the proof of Lemma 2.

Lemma 5. *Let p be an odd prime, and let $g=1+3p$. Let G act on a surface of genus g preserving orientation having partial presentation $T(2,4,\lambda)$ with $6\le\lambda<2(g+1)$. Then either p divides $|G|$ or G has one of the four partial presentations $T(2,4,mp+4)$ with $1\le m\le 4$.*

Proof. By Theorem A 2), $|G|=24p\lambda/(\lambda-4)$ so that

$$24p\lambda=|G|(\lambda-4). \tag{14}$$

Now by Euclid's Lemma, either p divides $|G|$ or p divides $(\lambda-4)$.

Assume that p divides $(\lambda-4)$ and write $\lambda-4=mp$ for some integer $m\geq 1$. Now $\lambda=mp+4$, and λ divides $|G|$. Write $|G|=\mu\lambda$. Now we have $|G|=24\lambda p/mp=24\lambda/m$ and $\mu=24/m$. Hence m divides 24 and, since $\lambda<2(g+1)=6p+4$, $m<6$. Thus m is 1, 2, 3, or 4. □

Thus, if p does not divide $|G|$, G has one of four partial presentations. We exhibit these possibilities. It is also clear that if G has one of these partial presentations, then p does not divide $|G|$.

TABLE 1. Partial Presentations of G

m	Lambda	Order	Mu
$m=1$	$\lambda=p+4$	$\lvert G\rvert=24\lambda=8(g+11)$	$\mu=24$
$m=2$	$\lambda=2p+4$	$\lvert G\rvert=12\lambda=8(g+5)$	$\mu=12$
$m=3$	$\lambda=3p+4$	$\lvert G\rvert=8\lambda=8(g+3)$	$\mu=8$
$m=4$	$\lambda=4p+4$	$\lvert G\rvert=6\lambda=8(g+2)$	$\mu=6$

As we shall see, there are group actions of the first type for infinitely many $p\equiv 5 \pmod 6$. There are no actions of the three remaining types at all, as long as p is not small.

One of the four possibilities requires special treatment.

Lemma 6. *Let p be an odd prime, and let $g=1+3p$. Let G act on a surface of genus g preserving orientation having partial presentation $T(2,4,\lambda)$. If G has order 6λ ($\mu=6$), then $\lambda\leq 36$.*

Proof. First $k\leq 3$ is not possible so that $k\geq 4$. By Lemma 3, $6/\ell\geq k$. Hence $\ell\neq 2$. This means $\ell=1$, $G=\langle c,d\rangle$, and k must be 4, 5 or 6. If k is 4 or 6, then 4 divides $k\mu=6k$ and $\lambda\leq 36$ by Lemma 4.

Suppose $k=5$. Then the quotient group G/J would be a non-abelian $T(2,4,5)$ group of order 30. Each of the three non-abelian groups of order 30 is obviously not generated by an involution and an element of order 4. Thus $k\neq 5$ and $\lambda\leq 36$. □

Now we consider the general case in which $G=\langle c,d\rangle$. As in the previous section, we assume that p is not small and apply Lemma 4.

Lemma 7. *Let p be an odd prime, and let $g=1+3p$. Let G act on a surface of genus g preserving orientation. Suppose G has partial presentation $T(2,4,\lambda)$, with $6\leq\lambda<2(g+1)$. Suppose $G=\langle c,d\rangle$. If the prime $p>(24)^2$, then p divides $|G|$.*

Proof. As in the proof of Lemma 5, if p does not divide $|G|$, then $\lambda-4=mp$ for where m is 1,2,3 or 4 and $\mu=24/m$. Then $\lambda=mp+4>p>(24)^2$ (by assumption). Assume $m\neq 4$. Then by Accola's Lemma 4, $p<\lambda\leq\mu^2\leq(24)^2$, an obvious contradiction. Finally, Lemma 6 immediately rules out the case with $m=4$ and $\mu=6$. Hence p must divide $|G|$. □

Thus, if G is a $T(2,4,\lambda)$ group with $G=\langle c,d\rangle$ (and $p>(24)^2$), then p divides $|G|$.

We still must consider the case in which $G\neq\langle c,d\rangle$. Lemma 5 still applies so that either p divides $|G|$ or G has one of four partial presentations. We focus on these partial presentations.

Lemma 8. *Assume that G is a $(2,4,\lambda)$ group of order $n=\mu\lambda$ with $\mu>4$. Let $a,b\in G$ with $o(a)=2$ and $o(b)=4$ and let $c=ab$ and $d=ba$. Suppose that $G\neq\langle c,d\rangle$. Then there is a number k which divides λ satisfying $2\leq k\leq\mu/2$.*

Proof. Since $G=\langle a,c\rangle$, we have that $N=\langle c,d\rangle$ has index 2 in G. Next define $J=\langle c\rangle\cap\langle d\rangle$. Since conjugation by a interchanges c and d, the subgroup J is normal in G. Define $k=\lambda/|J|$ and so k divides λ. Now let $\bar{c}$ and $\bar{d}$ be the image of c and d in G/J. Since $\bar{c}^m\bar{d}^n$ for $m,n=0,1,\cdots(k-1)$ are distinct elements in N/J, we have that $k^2\le k\mu/2$ and $k\le\mu/2$. □

At this point, we assume that the prime p is congruent to 5 modulo 6 and that p is not small.

Lemma 9. *Let p be a prime satisfying $p\equiv 5 \pmod 6$ with $p>(24)^2$. Let $g=1+3p$ and let G act on a surface of genus g preserving orientation with partial presentation $T(2,4,mp+4)$ with $1\le m\le 4$. Then $m=1$ and $p+4$ is divisible by 5 but not divisible by 25. Further, the group G contains a cyclic normal subgroup J of odd order with $G/J\cong S_5$.*

Proof. Since G acts with one of the four partial presentations, p does not divide $|G|$. Since $p>(24)^2$, then we must have $G\neq\langle c,d\rangle$ by Lemma 7. Now, as in the proof of Lemma 8, G contains a cyclic normal subgroup J of order λ/k for some integer k. Notice that the quotient group G/J is a $(2,4,k)$ group.

First suppose that $\lambda=4p+4$ so that $|G|=6\lambda$. By the proof of Lemma 6, $G=\langle c,d\rangle$, an obvious contradiction. Hence, it is not possible for G to act on a surface of genus $g=3p+1$ with this partial presentation.

Next, consider $\lambda=3p+4$ and $|G|=8\lambda$. Since λ is odd, so is k. Also by Lemma 8 we have that $k\le 4$. Therefore, $k=3$. Now λ is divisible by 3, by Lemma 8 and this case does not occur.

Now suppose $\lambda=2p+4$ so that $|G|=12\lambda$. Then $k\le 6$. Since $p\equiv 5 \pmod 6$, we see that $\lambda\equiv 2 \pmod 6$ and so 3 and 6 do not divide λ. If $k=2$, then G/J is a $(2,4,2)$ group and hence dihedral of order 8. Thus $|G/J|\neq 24=12k$. Therefore, $k=4$ or $k=5$. However, a search using Magma shows that there are no $(2,4,4)$ groups of order 48 and no $(2,4,5)$ groups of order 60. It follows that $\lambda\neq 2p+4$.

Finally, suppose $\lambda=p+4$, the only remaining possibility. Since λ is odd, so is k. We have $|G|=24\lambda$ so that $k\le 12$. Further, $|G/J|=24k$ and G/J is a $(2,4,k)$ group. Now $k=3$ gives that $G/J\cong S_4$ and $|G/J|=72$. Likewise, if $k=9$, then a MAGMA search shows that there are no $(2,4,9)$ groups of order 216 and for $k=11$, there are no $(2,4,11)$ groups of order 264. Therefore, $k=5$ or $k=7$.

Suppose that $k=7$. It follows that $Q=G/J$ is a $(2,4,7)$ group of order 168. A MAGMA search reveals that Q must be $PSL(2,7)$, the only $(2,4,7)$ group of order 168. Therefore, G is an extension of an odd order cyclic group J by the simple group Q. Since Q must act trivially on the cyclic group, we have a central extension. The equivalence class of central extensions is in one to one correspondence with the second cohomology group $H^2(Q,J)$ [18, Th. 11.4.10]. The Schur multiplier of the group Q is relevant to this central extension (See [18, p. 347]). The simple group $PSL(2,7)$ has Schur Multiplier $M(Q)\cong Z_2$. The Universal Coefficients Theorem [18, Th. 11.4.18] says that

$$H^2(Q,J)\cong Hom(M(Q),J)\times Ext(Q_{ab},J), \tag{15}$$

where $Q_{ab}\cong Q/Q'$ is the abelianization of Q. Thus, the second cohomology group is trivial and so G must be a direct product. This is impossible and $k\neq 7$.

Therefore $k=5$ and G has a cyclic normal subgroup J with G/J is a $(2,4,5)$ group of order 120. A Magma search shows that S_5 is the only such group. Thus $G/J\cong S_5$.

Suppose that 25 divides $p+4$. Now G is an extension of a cyclic group Z_n by S_5, where 5 divides n. Let $\tau : G \to S_5$ be the surjection. There is an element g of order $p+4$ in the group G. Now $\tau(g)$ is an element of order 5 in S_5. Therefore, $\tau(g) \in A_5$. Consider the group $H = \tau^{-1}(A_5)$. So H is a central extension of Z_n by A_5. The group H cannot be a direct product, since the direct product has no element of order 25 and H does have such an element. However, the Schur Multiplier $M(A_5) \cong Z_2$ and by the Universal Coefficients Theorem (15), the second cohomology group is trivial. Therefore, H must be the direct product and we have a contradiction. Thus, 25 does not divide $p+4$. □

Now we construct a family of groups with partial presentation $T(2,4,p+4)$ and order $24(p+4)$ that act on a surface of genus $g = 3p+1$, preserving its orientation.

Lemma 10. *Let p be a prime satisfying $p \equiv 5 \pmod 6$. Suppose $\lambda = p+4$ is divisible by 5 but not divisible by 25. Let $G_\lambda = Z_{\lambda/5} \times_\phi S_5$ be the semidirect product of $Z_{\lambda/5}$ and the symmetric group S_5, with the action ϕ being inversion. Then G_λ is a $(2,4,p+4)$ group of order $8(g+11)$ that acts on a surface of genus $g = 1+3p$ preserving orientation. Consequently, for such a value of g, $N(g) \geq 8(g+11)$.*

Proof. Let

$$G_\lambda = \langle a,b,c | a^2 = b^5 = (ab)^4 = [a,b^2]^2 = c^{(\lambda/5)} = [b,c] = (ca)^2 = 1 \rangle. \tag{16}$$

First, note that $S_5 \cong \langle a,b \rangle$ and $\langle c \rangle$ is a normal subgroup of G_λ. Now $G_\lambda = Z_{\lambda/5} \times_\phi S_5$ with the action being inversion. Let $x = ca$ and $y = ab$. Next, $xy = cb$ has order λ. Since $(xy)^5 = c^5$ and 5 does not divide the order of c, we see that $G = \langle x,y \rangle$ and so G is a $(2,4,p+4)$ group of order $24\lambda = 8(g+11)$. □

Combining the last two lemmas gives the following.

Theorem 2. *Let p be a prime satisfying $p \equiv 5 \pmod 6$ with $p > (24)^2$, and let $g = 1+3p$. Suppose that G is a $(2,4,mp+4)$ group of order larger than $8(g+1)$. Then if G acts on a surface of genus g preserving orientation, then $m = 1$, $\lambda = p+4$ is divisible by 5 and not by 25. Furthermore, if $p+4$ is divisible by 5 and not by 25, then there exists a group G that is a $(2,4,p+4)$ group of order $8(g+11)$.*

Next, we show that groups G with order greater than $8(g+1)$ and p divides $|G|$ cannot act on a surface of genus $g = 3p+1$ preserving its orientation.

6. p divides $|G|$.

Let p be an odd prime with $p \equiv 5 \pmod 6$, and let $g = 1+3p$. Let G act on a surface X of genus g preserving orientation such that $|G|$ satisfies the inequality (12). Now we assume that p divides $|G|$ and $p > 72$. We show that in this case, none of the partial presentations in Theorem A are possible. We let the Sylow p-subgroup act on X and follow the approach in [22, Section 5].

Lemma 11. *The Sylow p-subgroup of G is a cyclic normal subgroup in G isomorphic to Z_p.*

Proof. We have $|G| \leq 24(g-1) = 24 \cdot 3p = 72p$. Obviously, p^2 does not divide $|G|$, we are done. □

Now let the Sylow p-subgroup S act on X with $Y = X/S$ the quotient space, γ the genus of Y and $\pi : X \to Y$ the quotient map. For a detailed proof of the following, see [22, Lemma 5.3].

Lemma 12. *The quotient map π is unramified, and the quotient space $Y = X/S$ has genus $\gamma = 4$. Further, the quotient group $Q = G/S$ is a group of orientation-preserving automorphisms of Y with $40 < |Q| \leq 72$.*

Orientation preserving group actions on Riemann surfaces of genus 4 are well understood. These group actions were considered in determining the groups of strong symmetric genus 4 [14, Table 1]. The groups with order larger than 36 are groups of reflexible regular maps [14, Lemma 1]. There are three possibilities for the quotient group Q here, and they are presented in Table 2. The regular maps of genus 4 were first classified by Garbe [10, p. 53]. These maps also appear in [6, Table 1]. In Table 2, we give the group number in the MAGMA small groups library. Map symbols are from [6].

TABLE 2. Group Actions on Surfaces of Genus 4

Group	Order	Library Number	Partial Presentation	Map Symbol	G/G′
$Z_3 \times S_4$	72	42	$T(2,3,12)$	R4.1	Z_6
$(2,4,6;2)$	72	40	$T(2,4,6)$	R4.3	$(Z_2)^3$
A_5	60	5	$T(2,5,5)$	R4.6	1

The group G is an extension of Z_p by Q. Since $|Q|$ is relatively prime to p, the group G is a semidirect product, by the Schur-Zassenhaus Lemma.

Lemma 13. $G \cong Z_p \times_\phi Q$.

The following is important here. The proof is an exercise using the definition of semidirect product.

Lemma 14. *Let H be the semidirect product $K \times_\phi Q$, and let $L = kernel(\phi)$. Then L is normal in the big group H.*

For each of the possibilities for Q, we show that G cannot have the relevant partial presentation.

First suppose there were such a group G of order $72p$ with partial presentation $T(2,3,12)$. Let Δ be a Fuchsian group with signature $(0;+;[2,3,12];\{\})$ and presentation

$$X^2 = Y^3 = (XY)^{12} = 1. \tag{17}$$

Then $G \cong \Delta/K$ and is generated by two elements of orders 2 and 3. Let $\alpha : \Delta \longrightarrow G$ be the quotient map.

We have $G \cong Z_p \times_\phi Q$, where $Q \cong Z_3 \times S_4$. Let $L = kernel(\phi)$. Since $\phi : Q \to Aut(Z_p) \cong Z_{p-1}$, Q/L is cyclic. It follows that $Q' \subset L \subset Q$. Now the commutator quotient group $Q/Q' \cong Z_6$. Thus L must have index 1, 2, 3 or 6 in Q, and L is normal in G by Lemma 14. Let $T = G/L$ and let $\rho : G \longrightarrow T$ be the quotient map of G onto T. Also let $\theta = \rho \circ \alpha$ be the composition of α and ρ so that $\theta : \Delta \longrightarrow T$ maps Δ onto T. We eliminate all the possibilities for the quotient group T.

The following preliminary results will be helpful. Let Δ have presentation (17).

Lemma 15. *The only nontrivial odd order quotient of Δ is Z_3.*

Proof. Let $\beta : \Delta \longrightarrow W$ be a homomorphism of Δ onto the nontrivial odd-order group W. If J is an involution in Δ, then $\beta(J) = 1$. In particular, $\beta(X) = 1$ and hence $W = \langle \beta(X), \beta(Y) \rangle = \langle \beta(Y) \rangle \cong Z_3$. □

Lemma 16. *Let p be an odd prime, $p > 3$. Then D_p is not a quotient of Δ.*

Proof. Write $D \cong D_p$, and assume that $\beta : \Delta \longrightarrow D$ be a homomorphism of Δ onto D. Then $D = \langle \beta(X), \beta(Y) \rangle$ so that $\beta(X)$ and $\beta(Y)$ must be non-identity elements of D. But D has no elements of order 3 so that $\beta(Y) = 1$. Hence $D \cong D_p$ is not a quotient of Δ. □

Now we consider the possible indices of L in Q. First suppose $L = Q$ so that $G \cong Z_p \times Q$. Then G and hence Δ would have Z_p as a quotient which is not possible by Lemma 15.

Next assume $[Q : L] = 2$ so that the quotient group $T = G/L$ has order $2p$. Then T is isomorphic to either Z_{2p} or the dihedral group D_p. Suppose $T = Z_{2p}$. Then T and hence Δ would have Z_p as a quotient, which is not possible by Lemma 15. But D_p is not a quotient either, by Lemma 16.

Suppose $[Q : L] = 3$ so that the quotient group G/L has odd order $3p$. This is not possible by Lemma 15.

Finally, suppose $[Q : L] = 6$. Then the quotient group G/L has order $6p$, and there are four possibilities for the group G/L, since 3 does not divide $p-1$. (There are two additional groups of order $6p$ if 3 divides $p-1$.) There are the cyclic group Z_{6p}, the dihedral group D_{3p}, and the direct products $Z_3 \times D_p$ and $Z_p \times D_3$.

We have to consider the four possibilities for the quotient group $T = G/L$. First suppose $T = Z_{6p}$. Then T and hence Δ would have Z_p as a quotient, which is not possible by Lemma 15.

Assume next that $T \cong D_{3p}$. Then T has a characteristic subgroup V of order 3 with $T/V \cong D_p$. This is not possible by Lemma 16. Lemma 16 also eliminates the direct product $Z_3 \times D_p$ which has D_p as a quotient, and Lemma 15 eliminates the direct product $Z_p \times D_3$, which has a nontrivial odd order quotient.

In summary, there is no group of order $72p$ with partial presentation $T(2,3,12)$.

Next suppose there were such a group G of order $72p$ with partial presentation $T(2,4,6)$. Let Γ be a Fuchsian group with signature $(0;+;[2,4,6];\{\})$ and presentation

$$X^2 = Y^4 = (XY)^6 = 1. \tag{18}$$

Then $G \cong \Gamma/K$ and is generated by two elements of orders 2 and 4. Let $\alpha : \Gamma \longrightarrow G$ be the quotient map.

We have $G \cong Z_p \times_\phi Q$, where the quotient group $Q \cong (2,4,6;2)$ (see [9, p. 142] for a presentation). Let $L = \mathit{kernel}(\phi)$. Since $\phi : Q \to \mathit{Aut}(Z_p) \cong Z_{p-1}$, Q/L is cyclic. It follows that $Q' \subset L \subset Q$. Now a calculation shows that the commutator quotient group $Q/Q' \cong (Z_2)^2$. Thus L must have index 1 or 2 in Q, and L is normal in G by Lemma 14. Let $T = G/L$ and let $\rho : G \longrightarrow T$ be the quotient map of G onto T. Also let $\theta = \rho \circ \alpha$ be the composition of α and ρ so that $\theta : \Gamma \longrightarrow T$ maps Γ onto T. We eliminate all the possibilities for the quotient group T.

The following preliminary results will be helpful. Let Γ have presentation (18).

Lemma 17. *The group Γ has no nontrivial odd order quotients at all.*

Lemma 18. *Let p be an odd prime. Then D_p is not a quotient of Γ.*

Proof. Write $D \cong D_p$, and assume that $\beta : \Gamma \longrightarrow D$ be a homomorphism of Γ onto D. Then $D = \langle \beta(X), \beta(Y) \rangle$ so that $\beta(X)$ and $\beta(Y)$ must be non-identity elements of D. The dihedral group D has reflections and rotations of order p. Then $\beta(X), \beta(Y)$ must be reflections so that the product $\beta(X)\beta(Y)$ is a rotation of order p. But $[\beta(XY)]^6 = 1$. This means $\beta(X) = \beta(Y)$ and D would be abelian. Hence $D \cong D_p$ is not a quotient of Γ. □

Now we consider the two possible indices of L in Q. First suppose $L = Q$ so that $G \cong Z_p \times Q$. Then G and hence Γ would have Z_p as a quotient which is not possible by Lemma 17.

Next assume $[Q : L] = 2$ so that the quotient group $T = G/L$ has order $2p$. Then T is isomorphic to either Z_{2p} or the dihedral group D_p. Suppose $T = Z_{2p}$. Then T and hence Γ would have Z_p as a quotient, which is not possible by Lemma 17. But D_p is not a quotient either, by Lemma 18.

In summary, there is no group of order $72p$ with partial presentation $T(2,4,6)$.

Finally suppose there were such a group G of order $60p$ with partial presentation $T(2,5,5)$. Let Λ be a Fuchsian group with signature $(0;+;[2,5,5];\{\})$ and presentation

$$X^2 = Y^5 = (XY)^5 = 1. \tag{19}$$

Then $G \cong \Lambda/K$ and is generated by two elements of orders 2 and 5. Let $\alpha : \Lambda \longrightarrow G$ be the quotient map.

We have $G \cong Z_p \times_\phi Q$, where the quotient group $Q \cong A_5$. Since A_5 is simple, this means $G \cong Z_p \times A_5$.

Lemma 19. *The only nontrivial odd order quotient of* Λ *is* Z_5.

Proof. Let $\beta : \Lambda \longrightarrow W$ be a homomorphism of Λ onto the nontrivial odd-order group W. If J is an involution in Λ, then $\beta(J) = 1$. In particular, $\beta(X) = 1$ and so $W = \langle \beta(X), \beta(Y) \rangle = \langle \beta(Y) \rangle \cong Z_5$. □

But the group G and hence Λ have Z_p as quotients, with $p > 5$. Thus there is no group of order $60p$ with partial presentation $T(2,5,5)$.

Therefore, in this case, none of the partial presentations in Theorem A are possible, and consequently, $|G| \leq 8(g+1)$. In summary, we have the following.

Lemma 20. *Let p be an odd prime with $p \equiv 5 \pmod 6$, and let $g = 1 + 3p$. Let G act on a surface X of genus g preserving orientation. If p divides $|G|$ and $p > 72$, then $|G| \leq 8(g+1)$.*

Theorem 3. *Let $g = 1 + 3p$ for some prime $p > (36)^2$. Suppose p is congruent to $5 \pmod 6$. If p is also congruent modulo 25 to 1, 6, 11 or 16, then $N(g) = 8(g+11)$; otherwise $N(g) = 8(g+1)$.*

Finally, we check that the maximal order groups that give an orientation preserving action can be extended to a maximal order orientation reversing action.

7. Extensions to Orientation Reversing Actions

Next, we want to determine if G_λ has an extension to a group of order $48\lambda = 16(g+11)$. In order to do this, we need a presentation of G_λ as a $(2,4,\lambda)$ group. In the cases that we are interested in, λ is odd, divisible by 5 and not divisible by 25. Therefore, $\lambda \equiv 5, 15, 35, 45 \pmod{50}$.

For $\lambda \equiv \pm 15 \pmod{50}$, define

$$H_\lambda = \langle x, y | x^2 = y^4 = (xy)^\lambda = y^{-1}(xy)^5 y (xy)^5 = [y, (xy)^{\lambda/5}]^2 = 1 \rangle. \tag{20}$$

For $\lambda \equiv \pm 5 \pmod{50}$, define

$$H_\lambda = \langle x, y | x^2 = y^4 = (xy)^\lambda = y^{-1}(xy)^5 y (xy)^5 = [y, (xy)^{\lambda/5}]^3 = 1 \rangle. \tag{21}$$

Notice that since $(xy)^5$ is inverted by conjugation by y and centralized by (xy), $\langle (xy)^5 \rangle$ is a normal subgroup of H_λ in both cases. Next, modifying the presentations (20) and (21) by setting $(xy)^5 = 1$ and putting them in Magma, we see that the quotient is isomorphic to S_5 and hence G_λ and H_λ have the same order.

Theorem 4. *For $\lambda \equiv 5, 15, 35, 45 \pmod{50}$, $G_\lambda \cong H_\lambda$. A group H^*_λ of order $16(g+11)$ acting on a surface of genus g reversing orientation exists. Consequently, for such a value of g, $M(g) \geq 16(g+11)$.*

Proof. We will use the presentation for H_λ in equations (20) and (21) and for G_λ in (16). Define $\nu : H_\lambda \to G_\lambda$ by $\nu(x) = ca$ and $\nu(y) = ab$. Clearly, x^2, y^4 and $(xy)^\lambda$ are all mapped to the identity by ν. Next, $(xy)^5$ is mapped to c^5. Therefore, ν maps $y^{-1}(xy)^5 y (xy)^5$ to $(b^{-1}a^{-1})c^5(ab)c^5$ which is the identity in G_λ. Now, we need to consider two cases depending on whether H_λ has presentation (20) or (21).

Case 1: Suppose $\lambda \equiv \pm 15 \pmod{50}$. So H_λ has presentation (20). The image of $[y, (xy)^{\lambda/5}]^2$ under ν is the identity and so ν is an isomorphism by Van Dyke's Theorem.

Now suppose that $\phi : H_\lambda \to H_\lambda$ by $\phi(x) = x^{-1} = x$ and $\phi(y) = y^{-1}$. The image of all relators of H_λ under ϕ is the identity. Therefore, ϕ is an isomorphism of order 2 and so the extension H^*_λ exists by Singerman [21, Th. 2]. The group H^*_λ has partial presentation $FT(2, 4, \lambda)$.

Case 2: Suppose $\lambda \equiv \pm 5 \pmod{50}$. So H_λ has presentation (21). The image of $[y, (xy)^{\lambda/5}]^3$ under ν is the identity and so ν is an isomorphism by Van Dyke's Theorem.

Now suppose that $\kappa : H_\lambda \to H_\lambda$ by $\kappa(x) = x^{-1} = x$ and $\kappa(y) = y^{-1}$. As in case 1 all relators map to the identity. Therefore, κ is an isomorphism of order 2 and again the extension H^*_λ exists by Singerman [21, Th. 2].

□

Since $M(g) \geq 16(g+1)$ in all cases, the proof of Theorem 1 is complete.

8. Recent Related Results

We end by mentioning some recent results on related topics. A compact Riemann surface is called *psuedo-real* if it admits anticonformal automorphisms, but none of order 2. In [5], some limitations on the order of the largest group of automorphisms of a psuedo-real surface are obtained. For orientation preserving actions on Riemann surfaces, the paper [3] determines $N(g)$ for $g = qp^m + 1$ where q and p are certain primes. This result gives some information on the asymptotics of $N(g)$. If S is a compact Riemann surface of genus $p+1$ where p is a prime and $G \leq Aut(S)$ of order $\rho(g-1)$ where $\rho \geq 3$, then [12, Th. 1] classifies the groups G that can occur. As a corollary, the authors classify the maps and hypermaps corresponding to the cases in [12, Th. 1]. The paper [17, Th. 1] classifies the surfaces of genus $p-1$ for a prime p which have a group of automorphisms of order $\rho(g+1)$ for some $\rho \geq 1$. Similar problems for complex one-dimensional families were studied in [8], and these results were recently extended to the higher dimensional case in [11].

We would like to thank the referee for several helpful suggestions and for calling our attention to the research in this section.

References

[1] R. Accola, On the number of automorphisms of a closed Riemann surface, Trans. Amer. Math. Soc. 131 (1968), 398 - 407.
[2] R. Accola, Topics in the Theory of Riemann Surfaces, LNM 1595, Springer-Verlag, Berlin, Heidelberg, 1994.
[3] C. Bagiński, and G. Gromadzki, On the orders of largest groups of automorphisms of compact Riemann surfaces, Journal of Pure and Applied Algebra, 225, No. 12(2021), Paper No. 106758, 14 pp.
[4] M. Belolipetsky and G. Jones, Automorphism groups of Riemann surfaces of genus $p+1$, where p is prime, Glasgow Math. J. 47 (2005), 379-393.
[5] E. Bujalance, F.J. Cirre and M.D.E. Conder, Bounds on the orders of groups of automorphisms of a psuedo-real surface of a given genus, J. London Math. Soc. (2) 101 (2020), No. 2, 877 - 906.
[6] M.D.E. Conder and P. Dobcsanyi, Determination of all regular maps of small genus, Journal of Combinatorial Theory (Series B), 81, No. 2, (2001), 224-242.
[7] M.D.E. Conder and T.W. Tucker, The symmetric genus spectrum of finite groups, Ars Math. Contemp., 4 No. 2 (2011), 271 - 289.
[8] A.F. Costa, M. Izquierdo, One-dimensional families of Riemann surfaces of genus g with 4g+4 automorphims, RACSAM 112, 623–631 (2018). https://doi.org/10.1007/s13398-017-0429-0
[9] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, 4th Edition, Springer-Verlag, New York, Heidelberg, Berlin, Tokyo, 1980.
[10] D. Garbe, Uber die regularen Zerlegungen geschlossener orientierbarer Flachen, J. Reine Angew. Math. 237 (1969), 39-55.
[11] M. Izquierdo, S. Reyes - Carocca and A. Rojas, On families of Riemann surfaces with automorphisms, J. Pure Appl. Algebra 225, No. 10 (2021) Paper No. 106704, 21 pp.
[12] M. Izquierdo, G. Jones and S. Reyes - Carocca, Groups of automorphisms of Riemann surfaces and maps of genus p + 1 where p is a prime, Ann. Fenn. Math. 46, No. 2 (2021), 839 - 867.
[13] C. Maclachlan, A bound for the number of automorphisms of a compact Riemann surface, J. London Math. Soc. (2) 44 (1968), 265-272.
[14] C.L. May and J. Zimmerman, Groups of strong symmetric genus 4, Houston J. Math. 31(2005), 21-35.
[15] C.L. May, J. Zimmerman, The groups of symmetric genus $\sigma \leq 8$, Communications in Algebra 36 (2008), 4078 - 4095.
[16] C.L. May, J. Zimmerman, Maximal order group actions on Riemann surfaces of genus $1+p$, to appear.
[17] S. Reyes - Carocca and A. Rojas, On large prime actions on Riemann surfaces, J. Group Theory 25, No. 5 (2022), 887 - 940.
[18] D.J.S. Robinson, A Course in the Theory of Groups, 2nd ed., Graduate Texts in Mathematics 80, Springer-Verlag, New York, 1996.
[19] D. Singerman, Automorphisms of compact non-orientable Riemann surfaces, Glasgow Math. J. 12 (1971), 50-59.
[20] D. Singerman, On the structure of non-Euclidean crystallographic groups, Proc. Cambridge Philos. Soc. 76 (1974), 233-240.
[21] D. Singerman, Symmetries of Riemann surfaces with large automorphism group, Math. Ann. 210 (1974), 17-32.
[22] J. Zimmerman and C. L. May, Maximal order group actions on Riemann surfaces, Ars Math. Contemp. 22 (2022), doi:10.26493/1855-3974.2257.6de.

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