GENERALIZATION OF HERMITE-HADAMARD-MERCER AND TRAPEZOID FORMULA TYPE INEQUALITIES INVOLVING BETA FUNCTION

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ABSTRACT. In this work, we prove a generalized version of Hermite-Hadamard-Mercer type inequalities using the Beta function. Moreover, we prove some new trapezoidal type inequalities involving Beta functions for differentiable convex functions. The main advantage of these inequalities is that these can be converted into similar classical integral inequalities, Riemann-Liouville fractional integral inequalities and k-Riemann-Liouville fractional integral inequalities. Finally, we give applications to special means of real numbers for newly established inequalities.

1. Introduction

In both pure and applied mathematics, the concept of convex functions is extremely important. Convex functions have a wide range of applications in fields such as finance, economics, and engineering.

Definition 1. *[21] A function $F : [\pi_1, \pi_2] \to \mathbb{R}$ is called convex for all $\varkappa, \gamma \in [\pi_1, \pi_2]$ and $\tau \in [0,1]$ if*

$$F(\gamma + \tau(\varkappa - \gamma)) \leq \tau F(\varkappa) + (1-\tau) F(\gamma). \tag{1.1}$$

Function F is said to be a strictly convex if the inequality in (1.1) is strict for F, while F is said to be a concave function if $-F$ is convex.

Many important inequalities hold for convex functions, including Jensen, Jensen-Mercer, Hermite-Hadamard, and support line inequalities. The traditional Jensen's inequality is one of the most well-known inequalities, as stated below:

Theorem 1. *[12, 13] For a convex function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds:*

$$F\left(\sum_{i=1}^{n} w_i \varkappa_i\right) \leq \sum_{i=1}^{n} w_i F(\varkappa_i), \tag{1.2}$$

where $\varkappa_i \in [\pi_1, \pi_2]$, $w_i \in [0,1]$ and $\sum_{i=1}^{\infty} w_i = 1$.

In 2003, Mercer presented a variant of Jensen's inequality known as Jensen-Mercer inequality in [16].

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Theorem 2. *For a convex function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds:*

$$F\left(\pi_1 + \pi_2 - \sum_{i=1}^{n} w_i \varkappa_i\right) \leq F(\pi_1) + F(\pi_2) - \sum_{i=1}^{n} w_i F(\varkappa_i), \tag{1.3}$$

where $\varkappa_i \in [\pi_1, \pi_2]$, $w_i \in [0,1]$ and $\sum_{i=1}^{\infty} w_i = 1$.

The Hermite–Hadamard inequality is the first result given between convex functions and integrals. This inequality was introduced by Hermite [8] in 1883 which was later proved by Hadamard [9] in 1893. This inequality has the following mathematical form:

$$F\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\varkappa)\, d\varkappa \leq \frac{F(\pi_1) + F(\pi_2)}{2}, \tag{1.4}$$

where F is a convex function. This inequality also holds in reverse direction for concave functions.

This inequality has many advantages, especially in approximation theory, it is used a lot. Because of its great applications, mathematicians started working on it and came up with many new results. For example, Dragomir and Agarwal [6] found the boundaries of the trapezoidal formula by taking the difference of the middle part and the right part of this inequality and used differentiable convexity in the whole process. Later, Kirmaci [15] gave the boundaries of the midpoint formula, which were formed from the same inequality, he took the difference of the middle part from the left part and he also derived his results by using differentiable convexity. Qi and Xi [22] took the difference of the middle part of this inequality with the average of the left and right parts to establish a new inequality that we know as Bullen's inequality.

Recently, Kian and Moslehian [14] presented the following variant of Hermite-Hadamard inequality (1.4) using the inequality (1.3) and it is known as Hermite-Hadamard-Mercer inequality.

Theorem 3. *For a convex function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds:*

$$\begin{aligned} F\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) &\leq \frac{1}{\gamma - \varkappa} \int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} F(w)\, dw \\ &\leq \frac{F(\pi_1 + \pi_2 - \gamma) + F(\pi_1 + \pi_2 - \varkappa)}{2} \\ &\leq F(\pi_1) + F(\pi_2) - \frac{F(\varkappa) + F(\gamma)}{2} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} & F\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) \\ \leq\ & F(\pi_1) + F(\pi_2) - \int_{\varkappa}^{\gamma} F(w)\, dw \leq F(\pi_1) + F(\pi_2) - F\left(\frac{\varkappa + \gamma}{2}\right), \end{aligned} \tag{1.6}$$

where $\varkappa, \gamma \in [\pi_1, \pi_2]$.

Fractional integral operators have been used to expand the Hermite–Hadamard inequality. The Riemann–Liouville fractional operators are defined as:

Definition 2. *[17,20] For an integrable function F on $[\pi_1, \pi_2]$, the left and right Riemann-Liouville fractional integrals are defined as:*

$$J_{\pi_1+}^{\alpha} F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\pi_1}^{\varkappa} (\varkappa - w)^{\alpha - 1} F(w)\, dw,$$

$$J_{\pi_2-}^{\alpha}F(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\pi_2}(w-\varkappa)^{\alpha-1}F(w)\,dw,$$

where Γ *represents the Gamma function and defined as* $\Gamma(\alpha)=\int_0^{\infty}e^{-\tau}\tau^{\alpha-1}d\tau$.

Definition 3. *[18] For an integrable function* F *over* $[\pi_1,\pi_2]$*, the left and right k-Riemann-Liouville fractional integrals are defined as:*

$$\begin{aligned} J_{\pi_1+}^{\alpha,k}F(\varkappa) &= \frac{1}{k\Gamma_k(\alpha)}\int_{\pi_1}^{\varkappa}(\varkappa-w)^{\frac{\alpha}{k}-1}F(w)\,dw,\\ J_{\pi_2-}^{\alpha,k}F(\varkappa) &= \frac{1}{k\Gamma_k(\alpha)}\int_{\varkappa}^{\pi_2}(w-\varkappa)^{\frac{\alpha}{k}-1}F(w)\,dw,\end{aligned}$$

where $\Gamma_k(\alpha)$ *is k-Gamma functions and defined as* $\Gamma_k(\alpha)=\int_0^{\infty}e^{-\frac{\tau^k}{k}}\tau^{\alpha-1}d\tau$.

Recently, Sarikaya and Ata presented the generalized variant of Hermite-Hadamard inequality (1.4) using the notions of Beta functions.

Theorem 4. *[24] For a convex functions* $F:[\pi_1,\pi_2]\to\mathbb{R}$*, the following inequality holds:*

$$\begin{aligned} F\left(\frac{\pi_1+\pi_2}{2}\right)\beta(m,n) &\le \frac{1}{2(\pi_2-\pi_1)^{m+n-1}}\int_{\pi_1}^{\pi_2}\Xi(w)F(w)\,dw\\ &\le \frac{F(\pi_1)+F(\pi_2)}{2}\beta(m,n),\end{aligned} \tag{1.7}$$

where $\beta(m,n)=\int_0^1\tau^{m-1}(1-\tau)^{n-1}$ *is a Beta function and*

$$\Xi(w)=(\pi_2-w)^{m-1}(w-\pi_1)^{n-1}+(\pi_2-w)^{n-1}(w-\pi_1)^{m-1}.$$

The fundamental benefit of inequality (1.7) is that it can be turned into classical Hermite-Hadamard inequality (1.4), fractional Hermite-Hadamard inequality [23] and k-fractional Hermite-Hadamard inequality [10] without having to prove each one separately.

Many studies have been conducted in the topic of integral inequality, particularly in Hermite-Hadamard inequality, using various integral operators. For example, the authors of [6] demonstrated various Hermite-Hadamard type inequalities for convex functions and applied them to numerical integration formulas. In [15] Kirmaci established Hermite-Hadamard type inequalities for differentiable convex functions. The authors of [23] used Riemann-Liouville fractional integrals to prove Hermite-Hadamard type inequalities for convex functions. The authors of [5] established Hermite-Hadamard type inequalities for (α,m)-logarithmically convex functions. Riemann-Liouville fractional integrals were used by Awan et al. [3] to refine fractional Hermite-Hadamard type inequalities for strongly convex functions. Using generalized fractional integrals, the authors showed some new Hermite-Hadamard type inequalities for harmonically convex functions in [27].

On the other hand, different authors have offered distinct versions of Hermite-Hadamard-Mercer type inequalities. For example, in [1], fractional Hermite-Hadamard-Mercer type inequalities are demonstrated using the Jensen-Mercer inequalities. Chu et al. [4] showed some new fractional estimates of the Hermite-Hadamard inequality for differentiable convex functions. Using the k-Caputo fractional derivative, the authors demonstrated Hermite-Hadamard type inequalities for convex functions in [28]. In [11], İşcan demonstrated a weighted variant of the Hermite-Hadamard-Mercer inequality. In [26], the authors showed Hermite-Hadamard-Mercer inequality for Harmonically convex functions.

Inspired by recent research, we establish a generalized version of Hermite-Hadamard-Mercer inequalities using the Beta function. The fundamental benefit of these inequalities is that these can be turned into classical Hermite-Hadamard-Mercer inequalities (1.6) and (1.5), fractional Hermite-Hadamard-Mercer inequalities [19] and k-fractional Hermite-Hadamard-Mercer inequalities [2, 25] without having to prove each one separately.

2. Hermite-Hadamard-Mercer Inequalities

In this section, we prove some new Hermite-Hadamard-Mercer type inequalities involving the Beta function.

Theorem 5. *For a convex function $F:[\pi_1,\pi_2]\to\mathbb{R}$, the following inequality holds:*

$$\begin{aligned} &F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \\ \leq\ & [F(\pi_1)+F(\pi_2)]\beta(m,n)-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\varkappa}^{\gamma}\Lambda(w)F(w)\,dw \\ \leq\ & [F(\pi_1)+F(\pi_2)]\beta(m,n)-\beta(m,n)F\left(\frac{\varkappa+\gamma}{2}\right), \end{aligned} \tag{2.1}$$

where $\beta(m,n)$ is a Beta function,

$$\Lambda(w)=(\gamma-w)^{m-1}(w-\varkappa)^{n-1}+(\gamma-w)^{n-1}(w-\varkappa)^{m-1}$$

and $\varkappa,\gamma\in[\pi_1,\pi_2]$.

Proof. From Jensen-Mercer inequality and $u,v\in[\pi_1,\pi_2]$, we have

$$F\left(\pi_1+\pi_2-\frac{u+v}{2}\right)\leq F(\pi_1)+F(\pi_2)-\frac{F(u)+F(v)}{2}. \tag{2.2}$$

Now by setting $u=\tau\varkappa+(1-\tau)\gamma$ and $v=\tau\gamma+(1-\tau)\varkappa$ in (2.2), we have

$$\begin{aligned} &F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \\ \leq\ & F(\pi_1)+F(\pi_2)-\frac{F(\tau\varkappa+(1-\tau)\gamma)+F(\tau\gamma+(1-\tau)\varkappa)}{2}. \end{aligned} \tag{2.3}$$

Integrating to inequality (2.3) with respect to τ over $[0,1]$ after multiplying $\tau^{m-1}(1-\tau)^{n-1}$, we have

$$\begin{aligned} &F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \\ \leq\ & \beta(m,n)[F(\pi_1)+F(\pi_2)] \\ & -\frac{1}{2}\left[\int_0^1\tau^{m-1}(1-\tau)^{n-1}F(\tau\varkappa+(1-\tau)\gamma)\,d\tau\right. \\ & \left.+\int_0^1\tau^{m-1}(1-\tau)^{n-1}F(\tau\gamma+(1-\tau)\varkappa)\,d\tau\right]. \end{aligned} \tag{2.4}$$

Thus, we obtain the first inequality in (2.1) by using the change of variables of integration.

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We use the convexity of F to prove the second inequality in (2.1) and we obtain

$$F\left(\frac{\varkappa+\gamma}{2}\right) \leq \frac{F\left(\tau\varkappa+(1-\tau)\gamma\right)+F\left(\tau\gamma+(1-\tau)\varkappa\right)}{2}. \tag{2.5}$$

Integrating to inequality (2.5) with respect to τ over $[0,1]$ after multiplying $\tau^{m-1}(1-\tau)^{n-1}$, we have

$$\begin{aligned} & F\left(\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \\ \leq\ & \frac{1}{2}\left[\int_0^1 \tau^{m-1}(1-\tau)^{n-1} F\left(\tau\varkappa+(1-\tau)\gamma\right)d\tau \right. \\ & \left. +\int_0^1 \tau^{m-1}(1-\tau)^{n-1} F\left(\tau\gamma+(1-\tau)\varkappa\right)d\tau\right]. \end{aligned} \tag{2.6}$$

By using change of variables of integration, we have the following inequality

$$F\left(\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \leq \frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\varkappa}^{\gamma}\Lambda(w)F(w)\,dw$$

which implies that

$$-F\left(\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \leq -\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\varkappa}^{\gamma}\Lambda(w)F(w)\,dw. \tag{2.7}$$

As a result, we may get the last inequality of (2.1) by adding $F(\pi_1)+F(\pi_2)$ on both sides of the inequality (2.7). The proof is finished. □

Remark 1. *In Theorem 5, if we set $m=n=1$, then we recapture inequality (1.6).*

Remark 2. *In Theorem 5, if we set $m=1$ and $n=\alpha$ or $m=\alpha$ and $n=1$, then we recapture inequality (2.1) in [19, Theorem 2].*

Theorem 6. *For a convex function $F:[\pi_1,\pi_2]\to\mathbb{R}$, the following inequality holds:*

$$\begin{aligned} & F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\beta(m,n) \\ \leq\ & \frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw \\ \leq\ & \frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n) \\ \leq\ & \beta(m,n)\left[F(\pi_1)+F(\pi_2)-\frac{F(\varkappa)+F(\gamma)}{2}\right], \end{aligned} \tag{2.8}$$

where $\beta(m,n)$ is a Beta function and

$$\begin{aligned} \Delta(w) &= (\pi_1+\pi_2-\varkappa-w)^{m-1}(w-(\pi_1+\pi_2-\gamma))^{n-1} \\ &\quad +(\pi_1+\pi_2-\varkappa-w)^{n-1}(w-(\pi_1+\pi_2-\gamma))^{m-1} \end{aligned}$$

and $\varkappa,\gamma\in[\pi_1,\pi_2]$.

Proof. From convexity of F and $u,v\in[\pi_1,\pi_2]$, we have

$$F\left(\pi_1+\pi_2-\frac{u+v}{2}\right)\leq\frac{F\left(\pi_1+\pi_2-u\right)+F\left(\pi_1+\pi_2-v\right)}{2}. \tag{2.9}$$

Now by setting $\pi_1+\pi_2-u=\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)$ and $\pi_1+\pi_2-v=\tau\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)\left(\pi_1+\pi_2-\varkappa\right)$ in inequality (2.9), we have

$$\begin{aligned}&F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\\ \leq\ &\frac{1}{2}\left[F\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)\right.\\ &\left.+F\left(\tau\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)\left(\pi_1+\pi_2-\varkappa\right)\right)\right].\end{aligned} \tag{2.10}$$

Integrating to inequality (2.10) with respect to τ over $[0,1]$ after multiplying $\tau^{m-1}\left(1-\tau\right)^{n-1}$, we have

$$\begin{aligned}&F\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\beta\left(m,n\right)\\ \leq\ &\frac{1}{2}\left[\int_0^1\tau^{m-1}\left(1-\tau\right)^{n-1}F\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)d\tau\right.\\ &\left.+\int_0^1\tau^{m-1}\left(1-\tau\right)^{n-1}F\left(\tau\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)\left(\pi_1+\pi_2-\varkappa\right)\right)d\tau\right].\end{aligned}$$

Thus, we obtain the first inequality in (2.8) by using the change of variables of integration.

We use the convexity of F to prove the second inequality in (2.8) and we obtain

$$\begin{aligned}&F\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)\\ \leq\ &\tau F\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)F\left(\pi_1+\pi_2-\gamma\right)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&F\left(\tau\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)\left(\pi_1+\pi_2-\varkappa\right)\right)\\ \leq\ &\tau F\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)F\left(\pi_1+\pi_2-\varkappa\right).\end{aligned} \tag{2.12}$$

By applying Jensen-Mercer inequality after addition of (2.11) and (2.12), we have

$$\begin{aligned}&F\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)\\ &+F\left(\tau\left(\pi_1+\pi_2-\gamma\right)+(1-\tau)\left(\pi_1+\pi_2-\varkappa\right)\right)\\ \leq\ &F\left(\pi_1+\pi_2-\gamma\right)+F\left(\pi_1+\pi_2-\varkappa\right)\\ \leq\ &2\left[F\left(\pi_1\right)+F\left(\pi_2\right)\right]-\left[F\left(\varkappa\right)+F\left(\gamma\right)\right].\end{aligned} \tag{2.13}$$

Thus, we obtain second and third inequality of (2.8) by integrating to inequality (2.13) with respect to τ over $[0,1]$ after multiplying $\tau^{m-1}\left(1-\tau\right)^{n-1}$ and using the change of variables of integration. The proof is completed. □

Remark 3. *In Theorem 6, if we set $m=n=1$, then we recapture inequality (1.5).*

Remark 4. *In Theorem 6, if we set $m=1$ and $n=\alpha$ or $m=\alpha$ and $n=1$, then we recapture inequality (2.2) in [19, Theorem 2].*

Remark 5. *In Theorem 6, if we set $m=1$ and $n=\frac{\alpha}{k}$ or $m=\frac{\alpha}{k}$ and $n=1$, then we obtain [2, Theorem 2.1].*

Remark 6. *In Theorem 5, if we set* $\varkappa = \pi_1$ *and* $\gamma = \pi_2$*, then we recapture inequality (1.7).*

3. Trapezoidal Type Inequalities

In this section, we prove some trapezoid formula type inequalities involving Beta functions. Let's start with the following new Lemma.

Lemma 1. *For a differentiable mapping* $F : [\pi_1, \pi_2] \to \mathbb{R}$ *over* (π_1, π_2) *and* $F \in L[\pi_1, \pi_2]$*, then the following equality holds:*

$$\begin{aligned}
(3.1)\qquad & \frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n) \\
& -\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw \\
= & \frac{\gamma-\varkappa}{2}\int_0^1 \beta_\tau(m,n)\begin{bmatrix} F'(\pi_1+\pi_2-(\tau\varkappa+(1-\tau)\gamma)) \\ +F'(\pi_1+\pi_2-(\tau\gamma+(1-\tau)\varkappa)) \end{bmatrix} d\tau \\
= & \frac{\gamma-\varkappa}{2}\int_0^1 [\beta_\tau(m,n)-\beta_{1-\tau}(m,n)]\,F'(\pi_1+\pi_2-(\tau\varkappa+(1-\tau)\gamma))\,d\tau,
\end{aligned}$$

where

$$\beta_\tau(m,n) = \int_0^\tau z^{m-1}(1-z)^{n-1}\,dz$$

and $\varkappa, \gamma \in [\pi_1, \pi_2]$.

Proof. From integration by parts, we have

$$\begin{aligned}
(3.2)\qquad & \int_0^1 \beta_\tau(m,n)\,F'(\pi_1+\pi_2-(\tau\varkappa+(1-\tau)\gamma))\,d\tau \\
= & \beta_\tau(m,n)\frac{F(\pi_1+\pi_2-\varkappa)}{\gamma-\varkappa} \\
& -\frac{1}{\gamma-\varkappa}\int_0^1 \tau^{m-1}(1-\tau)^{n-1}F(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\,d\tau \\
= & \beta_\tau(m,n)\frac{F(\pi_1+\pi_2-\varkappa)}{\gamma-\varkappa} \\
& -\frac{1}{(\gamma-\varkappa)^{m+n}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}(\pi_1+\pi_2-\varkappa-w)^{n-1}(w-(\pi_1+\pi_2-\gamma))^{m-1}F(w)\,dw
\end{aligned}$$

and

$$\begin{aligned}
(3.3)\qquad & \int_0^1 \beta_\tau(m,n)\,F'(\pi_1+\pi_2-(\tau\gamma+(1-\tau)\varkappa))\,d\tau \\
= & \beta_\tau(m,n)\frac{F(\pi_1+\pi_2-\gamma)}{\gamma-\varkappa} \\
& -\frac{1}{(\gamma-\varkappa)^{m+n}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}(\pi_1+\pi_2-\varkappa-w)^{m-1}(w-(\pi_1+\pi_2-\gamma))^{n-1}F(w)\,dw.
\end{aligned}$$

Thus, we obtain the required equality by subtracting (3.2) from (3.3) and later multiplying $\frac{2}{(\gamma-\varkappa)}$. The proof is completed. ☐

Remark 7. *In Lemma 1, if we set $m = n = 1$, then Lemma 1 becomes [19, Corollary 1].*

Remark 8. *In Lemma 1, if we set $m = 1$ and $n = \alpha$ or $m = \alpha$ and $n = 1$, then we obtain [19, Lemma 1].*

Remark 9. *In Lemma 1, if we set $m = 1$ and $n = \frac{\alpha}{k}$ or $m = \frac{\alpha}{k}$ and $n = 1$, then we obtain equality (3.3) in [25, Remark 3.1].*

Remark 10. *In Lemma 1, if we set $\varkappa = \pi_1$ and $\gamma = \pi_2$, then we obtain [24, Lemma 5].*

Theorem 7. *We assume that the conditions of Lemma 1 hold. If $|F'|$ is a convex function, then we have the following trapezoid formula type inequality:*

$$
\begin{aligned}
&\left|\frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n)\right. \\
&\left.-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw\right| \\
\leq\;& (\gamma-\varkappa)\left(|F'(\pi_1)|+|F'(\pi_2)|-\frac{|F'(\varkappa)|+|F'(\gamma)|}{2}\right) \\
&\times\int_0^{\frac{1}{2}}\left[\beta_{1-\tau}(m,n)-\beta_\tau(m,n)\right]d\tau.
\end{aligned}
\tag{3.4}
$$

Proof. Taking modulus in (3.1) and using Jensen-Mercer inequality, we have

$$
\begin{aligned}
&\left|\frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n)\right. \\
&\left.-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw\right| \\
\leq\;& \frac{\gamma-\varkappa}{2}\int_0^1\left|\left[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)\right]\right|\left|F'(\pi_1+\pi_2-(\tau\varkappa+(1-\tau)\gamma))\right|d\tau \\
\leq\;& \frac{\gamma-\varkappa}{2}\left[\int_0^{\frac{1}{2}}\left[\beta_{1-\tau}(m,n)-\beta_\tau(m,n)\right]\left[\begin{array}{c}|F'(\pi_1)|+|F'(\pi_2)| \\ -\tau|F'(\varkappa)|-(1-\tau)|F'(\gamma)|\end{array}\right]d\tau\right. \\
&\left.+\int_{\frac{1}{2}}^1\left[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)\right]\left[|F'(\pi_1)|+|F'(\pi_2)|-\tau|F'(\varkappa)|-(1-\tau)|F'(\gamma)|\right]d\tau\right] \\
=\;& (\gamma-\varkappa)\left(|F'(\pi_1)|+|F'(\pi_2)|-\frac{|F'(\varkappa)|+|F'(\gamma)|}{2}\right)\int_0^{\frac{1}{2}}\left[\beta_{1-\tau}(m,n)-\beta_\tau(m,n)\right]d\tau
\end{aligned}
$$

and the proof is completed. □

Remark 11. *In Theorem 7, if we set $m = n = 1$, then Theorem 7 becomes [25, Corollary3.5].*

Remark 12. *In Theorem 7, if we set $m = 1$ and $n = \alpha$ or $m = \alpha$ and $n = 1$, then we obtain [19, Theorem 4].*

Remark 13. *In Theorem 7, if we set $m = 1$ and $n = \frac{\alpha}{k}$ or $m = \frac{\alpha}{k}$ and $n = 1$, then we recapture inequality (3.11) in [25, Remark 3.7].*

Remark 14. *In Theorem 7, if we set $\varkappa = \pi_1$ and $\gamma = \pi_2$, then we obtain [24, Theorem 6].*

Theorem 8. *We assume that the conditions of Lemma 1 hold. If* $|F'|^q$, $q>1$ *is convex functions, then we have the following trapezoidal type inequality:*

$$\begin{aligned}&\left|\frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n)\right.\\&\left.-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw\right|\\ \leq\ &\frac{\gamma-\varkappa}{2}\left(\int_0^1|[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)]|^p\,d\tau\right)^{\frac{1}{p}}\\&\times\left(|F'(\pi_1)|^q+|F'(\pi_2)|^q-\frac{|F'(\varkappa)|^q+|F'(\gamma)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus in equality (3.1) and applying Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n)\right.\\&\left.-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw\right|\\ \leq\ &\frac{\gamma-\varkappa}{2}\left(\int_0^1|[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)]|^p\,d\tau\right)^{\frac{1}{p}}\\&\times\left(\int_0^1|F'(\pi_1+\pi_2-(\tau\varkappa+(1-\tau)\gamma))|^q\,d\tau\right)^{\frac{1}{q}}.\end{aligned}$$

From Jensen-Mercer inequality, we have

$$\begin{aligned}&\left|\frac{F(\pi_1+\pi_2-\gamma)+F(\pi_1+\pi_2-\varkappa)}{2}\beta(m,n)\right.\\&\left.-\frac{1}{2(\gamma-\varkappa)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\Delta(w)F(w)\,dw\right|\\ \leq\ &\frac{\gamma-\varkappa}{2}\left(\int_0^1|[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)]|^p\,d\tau\right)^{\frac{1}{p}}\\&\times\left(\int_0^1\left[|F'(\pi_1)|^q+|F'(\pi_2)|^q-\tau|F'(\varkappa)|^q-(1-\tau)|F'(\gamma)|^q\right]d\tau\right)^{\frac{1}{q}}\\ =\ &\frac{\gamma-\varkappa}{2}\left(\int_0^1|[\beta_\tau(m,n)-\beta_{1-\tau}(m,n)]|^p\,d\tau\right)^{\frac{1}{p}}\\&\times\left(|F'(\pi_1)|^q+|F'(\pi_2)|^q-\frac{|F'(\varkappa)|^q+|F'(\gamma)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

and the proof is completed. □

Remark 15. *In Theorem 8, if we set* $m=n=1$, *then Theorem 8 becomes [25, Corollary 3.8].*

Remark 16. *In Theorem 8, if we set $m=1$ and $n=\alpha$ or $m=\alpha$ and $n=1$, then we obtain [25, Corollary 3.9].*

Remark 17. *In Theorem 8, if we set $m=1$ and $n=\frac{\alpha}{k}$ or $m=\frac{\alpha}{k}$ and $n=1$, then we recapture inequality (3.11) in [25, Corollary 3.10].*

Remark 18. *In Theorem 8, if we set $\varkappa=\pi_1$ and $\gamma=\pi_2$, then we obtain [24, Theorem 7].*

4. Application to Special Means

In this section, we give applications of newly established inequalities in the context of special means of real numbers. For arbitrary positive real numbers π_1 and π_2 ($\pi_1 \neq \pi_2$), we consider the means as follows:

(1) The arithmetic mean
$$\mathcal{A}=\mathcal{A}\left(\pi_1,\pi_2\right)=\frac{\pi_1+\pi_2}{2}.$$
(2) The harmonic mean
$$\mathcal{H}=\mathcal{H}\left(\pi_1,\pi_2\right)=\frac{2\pi_1\pi_2}{\pi_1+\pi_2}.$$
(3) The geometric mean
$$\mathcal{G}=\mathcal{G}\left(\pi_1,\pi_2\right)=\sqrt{\pi_1\pi_2}.$$

Proposition 1. *For $0<\pi_1<\pi_2$ and $p\in\mathbb{N}$, $p\geq 2$, the following inequality holds for all $\varkappa,\gamma\in[\pi_1,\pi_2]$:*
$$\begin{aligned}
&\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa,\gamma\right)\right)^p\beta\left(m,n\right)\\
\leq\;& 2\mathcal{A}\left(\pi_1^p,\pi_2^p\right)\beta\left(m,n\right)-\frac{1}{2\left(\gamma-\varkappa\right)^{m+n-1}}\int_{\varkappa}^{\gamma}\Lambda\left(w\right)w^p dw\\
\leq\;& \left(2\mathcal{A}\left(\pi_1^p,\pi_2^p\right)-\mathcal{A}^p\left(\varkappa,\gamma\right)\right)\beta\left(m,n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 5 to the convex function $F(w)=w^p$, $w>0$. □

Proposition 2. *For $0<\pi_1<\pi_2$, the following inequality holds for all $\varkappa,\gamma\in[\pi_1,\pi_2]$:*
$$\begin{aligned}
&\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa,\gamma\right)\right)^{-1}\beta\left(m,n\right)\\
\leq\;& 2\mathcal{H}^{-1}\left(\pi_1,\pi_2\right)\beta\left(m,n\right)-\frac{1}{2\left(\gamma-\varkappa\right)^{m+n-1}}\int_{\varkappa}^{\gamma}\frac{\Lambda\left(w\right)}{w}dw\\
\leq\;& \left(2\mathcal{H}^{-1}\left(\pi_1,\pi_2\right)-\mathcal{A}^{-1}\left(\pi_1,\pi_2\right)\right)\beta\left(m,n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 5 to the convex function $F(w)=\frac{1}{w}$, $w\neq 0$. □

Proposition 3. *For $0<\pi_1<\pi_2$, the following inequality holds for all $\varkappa,\gamma\in[\pi_1,\pi_2]$:*
$$\begin{aligned}
&\ln\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa,\gamma\right)\right)\beta\left(m,n\right)\\
\leq\;& 2\ln\left(\mathcal{G}\left(\pi_1,\pi_2\right)\right)\beta\left(m,n\right)-\frac{1}{2\left(\gamma-\varkappa\right)^{m+n-1}}\int_{\varkappa}^{\gamma}\Lambda\left(w\right)\ln w dw\\
\leq\;& \left(2\ln\left(\mathcal{G}\left(\pi_1,\pi_2\right)\right)-\ln\left(\mathcal{A}\left(\varkappa,\gamma\right)\right)\right)\beta\left(m,n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 5 to the convex function $F(w) = \ln w$. □

Proposition 4. *For $0 < \pi_1 < \pi_2$ and $p \in \mathbb{N}$, $p \geq 2$, the following inequality holds for all $\varkappa, \gamma \in [\pi_1, \pi_2]$:*

$$\begin{aligned}
& \left(2\mathcal{A}\left(\pi_1, \pi_2\right) - \mathcal{A}\left(\varkappa, \gamma\right)\right)^p \beta\left(m, n\right) \\
\leq\ & \frac{1}{2\left(\gamma - \varkappa\right)^{m+n-1}} \int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \Delta\left(w\right) w^p dw \\
\leq\ & \mathcal{A}\left(\left(\pi_1 + \pi_2 - \varkappa\right)^p, \left(\pi_1 + \pi_2 - \gamma\right)^p\right) \beta\left(m, n\right) \\
\leq\ & \left(2\mathcal{A}\left(\pi_1^p, \pi_2^p\right) - \mathcal{A}\left(\varkappa^p, \gamma^p\right)\right) \beta\left(m, n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 6 to the convex function $F(w) = w^p$, $w > 0$. □

Proposition 5. *For $0 < \pi_1 < \pi_2$, the following inequality holds for all $\varkappa, \gamma \in [\pi_1, \pi_2]$:*

$$\begin{aligned}
& \left(2\mathcal{A}\left(\pi_1, \pi_2\right) - \mathcal{A}\left(\varkappa, \gamma\right)\right)^{-1} \beta\left(m, n\right) \\
\leq\ & \frac{1}{2\left(\gamma - \varkappa\right)^{m+n-1}} \int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \frac{\Delta\left(w\right)}{w} dw \\
\leq\ & \mathcal{H}^{-1}\left(\pi_1 + \pi_2 - \varkappa, \pi_1 + \pi_2 - \gamma\right) \beta\left(m, n\right) \\
\leq\ & \left(2\mathcal{H}^{-1}\left(\pi_1, \pi_2\right) - \mathcal{H}^{-1}\left(\varkappa, \gamma\right)\right) \beta\left(m, n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 6 to the convex function $F(w) = \frac{1}{w}$, $w \neq 0$. □

Proposition 6. *For $0 < \pi_1 < \pi_2$, the following inequality holds for all $\varkappa, \gamma \in [\pi_1, \pi_2]$:*

$$\begin{aligned}
& \ln\left(2\mathcal{A}\left(\pi_1, \pi_2\right) - \mathcal{A}\left(\varkappa, \gamma\right)\right) \beta\left(m, n\right) \\
\leq\ & \frac{1}{2\left(\gamma - \varkappa\right)^{m+n-1}} \int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \Delta\left(w\right) \ln w dw \\
\leq\ & \ln\left(\mathcal{G}\left(\pi_1 + \pi_2 - \varkappa, \pi_1 + \pi_2 - \gamma\right)\right) \beta\left(m, n\right) \\
\leq\ & \left(2\ln\left(\mathcal{G}\left(\pi_1, \pi_2\right)\right) - \ln\left(\mathcal{G}\left(\varkappa, \gamma\right)\right)\right) \beta\left(m, n\right).
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 6 to the convex function $F(w) = \ln w$. □

Proposition 7. *For $0 < \pi_1 < \pi_2$ and $p \in \mathbb{N}$, $p \geq 2$, the following inequality holds for all $\varkappa, \gamma \in [\pi_1, \pi_2]$:*

$$\begin{aligned}
& \left|\mathcal{A}\left(\left(\pi_1 + \pi_2 - \varkappa\right)^p, \left(\pi_1 + \pi_2 - \gamma\right)^p\right) \beta\left(m, n\right)\right. \\
& \left. - \frac{1}{2\left(\gamma - \varkappa\right)^{m+n-1}} \int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \Delta\left(w\right) w^p dw \right| \\
\leq\ & p\left(\gamma - \varkappa\right) \left(2\mathcal{A}\left(\pi_1^{p-1}, \pi_2^{p-1}\right) - \mathcal{A}\left(\varkappa^{p-1}, \gamma^{p-1}\right)\right) \int_0^{\frac{1}{2}} \left[\beta_{1-\tau}\left(m, n\right) - \beta_{\tau}\left(m, n\right)\right] d\tau.
\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 7 to the convex function $F(w) = w^p$, $w > 0$. □

Proposition 8. *For $0<\pi_1<\pi_2$, the following inequality holds for all $\varkappa,\gamma\in[\pi_1,\pi_2]$:*

$$\begin{aligned}&\Big|\mathcal{H}^{-1}\left(\pi_1+\pi_2-\varkappa,\pi_1+\pi_2-\gamma\right)\beta\left(m,n\right)\\&-\frac{1}{2\left(\gamma-\varkappa\right)^{m+n-1}}\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa}\frac{\Delta\left(w\right)}{w}dw\Big|\\\leq\;&\left(\varkappa-\gamma\right)\left(2\mathcal{H}^{-1}\left(\pi_1^2,\pi_2^2\right)-\mathcal{H}^{-1}\left(\varkappa^2,\gamma^2\right)\right)\int_0^{\frac{1}{2}}\left[\beta_{1-\tau}\left(m,n\right)-\beta_{\tau}\left(m,n\right)\right]d\tau.\end{aligned}$$

Proof. The needed inequality can be obtained by applying Theorem 7 to the convex function $F\left(w\right)=\frac{1}{w}$, $w\neq0$. □

5. Conclusions

Using the beta function, we demonstrated several new generalizations of Hermite–Hadamard-Mercer type inequalities. Additionally, for differentiable convex functions, we demonstrated some novel trapezoidal type inequalities involving the Beta function. The fundamental benefit of these inequalities is that they can assist us in determining the error bounds for the trapezoidal formula in both fractional calculus and classical calculus. The first connection between Mercer inequalities and beta functions is provided by these inequalities, and this connection opens up new possibilities for the theory of special functions. In our remarks, we also emphasized the significance of applying beta functions to recently discovered inequalities. Finally, we demonstrated some applications of recently discovered inequalities using special real number. Researchers can obtain comparable inequalities for different types of convexity and special functions in their future work, which is a new and interesting problem.

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