VANISHING COEFFICIENTS IN POWERS OF THETA FUNCTIONS WITH ODD MODULI

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Abstract. Following the recent investigation of Somashekara and Thulasi on vanishing coefficients in a certain infinite q-product, we further study this phenomenon in a family of powers of theta functions with odd moduli, and prove that some arithmetic progressions on vanishing coefficients could be classified into a unified family, which significantly extends a result obtained by Somashekara and Thulasi.

1. Introduction

Throughout, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The study on vanishing coefficients with arithmetic progressions in quotients of infinite q-products, originated from the work by Richmond and Szekeres [20] in 1978, and was further investigated by Andrews and Bressoud [2], Alladi and Gordon [1], Hirschhorn [11], Chen and Huang [5], Chan and Yesilyurt [4], Xia and Yao [31,32], Lin [14], and Mc Laughlin [15]. In a private communication with Ernest X. W. Xia, Hirschhorn [13] investiagated vanishing coefficients in two infinite q-products, which also have the property that when the product is expanded as a formal power series in q, then the coefficients in one or more arithmetic progressions vanish. Let the sequences $\{a(n)\}_{n\geq 0}$ and $\{b(n)\}_{n\geq 0}$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n = (-q,-q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3,$$
$$\sum_{n=0}^{\infty} b(n)q^n = (-q^2,-q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3.$$

Hirschhorn [13] proved that for any $n \geq 0$,

$$a(5n+2) = a(5n+4) = b(5n+1) = b(5n+4) = 0. \tag{1.1}$$

Date: October 21, 2022.

2010 *Mathematics Subject Classification.* 11F27, 11B65, 30B10.

Key words and phrases. Vanishing coefficients; powers of theta functions; arithmetic progressions; odd moduli.

Hirschhorn's results reignited the enthusiasm for the investigation of vanishing coefficients in infinite q-series expansions. Since then, many scholars successively considered vanishing coefficients with arithmetic progressions in various types of q-series expansions, see, for example, [3, 6, 7, 9, 10, 17–19, 22–30, 33].

Quite recently, Somashekara and Thulasi [21] utilized some q-series techniques to study vanishing coefficients in several infinite q-products, and obtained several identities similar to (1.1). In particular, they [21, Theorem 3.1] proved that if

$$\sum_{n=0}^{\infty} t(n)q^n = (q, q^2; q^3)_\infty^3 (-q^3, -q^3; q^6)_\infty,$$

then for any $n \geq 0$,

$$t(3n+2) = 0. \tag{1.2}$$

One readily finds that the powers of q in infinite product $(-q^3, -q^3; q^6)_\infty$ are always the multiples of 3. Therefore, (1.2) is equivalent to

$$\widetilde{t}(3n+2) = 0, \tag{1.3}$$

where the sequence $\{\widetilde{t}(n)\}_{n\geq 0}$ is defined by

$$\sum_{n=0}^{\infty} \widetilde{t}(n)q^n = (q, q^2; q^3)_\infty^3.$$

Actually, the identity (1.3) follows immediately from the following 3-dissection identity due to Hirschhorn [12, Equations (21.3.1) and (21.3.4)]:

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (6n+1) q^{3n(3n+1)/2} - 3q(q^9;q^9)_\infty^3.$$

Roughly speaking, the infinite product $(\pm q^k, \pm q^{M-k}; q^M)_\infty$ is called a theta function, where M is a positive integer greater than or equal to 2 and $1 \leq k \leq M-1$. In some contexts, the number M is called the *modulus* of the corresponding theta function. For (1.3), a natural question is whether this is an isolated phenomenon, or whether there are more such identities in a family of powers of theta functions, including the infinite product $(q, q^2; q^3)_\infty^3$ as a special case. For the sake of convenience, define the sequence $\{\omega_{j,r,k}(n)\}_{n\geq 0}$ by

$$\sum_{n=0}^{\infty} \omega_{j,r,k}(n) q^n = (q^j, q^{r-j}; q^r)_\infty^k, \tag{1.4}$$

where $1 \leq j < r$, r is an integer greater than 2 and k is a positive integer.

In this paper, we further investigate vanishing coefficients in (1.4) with odd moduli, and prove that some arithmetic progressions on vanishing coefficients could be classified into a unified family. More precisely, we prove the following general identity.

Theorem 1.1. *For all n and any $m \geq 1$,*

$$\omega_{k,2m+1,2m+1}\big((2m+1)n + 2mk^2\big) = 0, \tag{1.5}$$

where $1 \leq k \leq m$ and $\gcd(k, 2m+1) = 1$.

Remark 1.2. Several remarks on Theorem 1.1 are necessary. First, the condition "for all n" implies that the arithmetic progression $(2m+1)n+2mk^2$ is well-defined in (1.4) for given m and k. Next, (1.3) is the special case $(k, m) = (1, 1)$ in (1.5). Finally, it is worthwhile to emphasize that unlike most results of vanishing coefficients in the aforementioned literature, where the powers in quotients or products of theta functions under study are usually fixed, (1.5) is valid for infinitely many cases.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need to use the following identity.

Lemma 2.1. *Let μ be a positive integer and let k be an integer. Then for any $m \geq 2$,*

$$f(-q^k, -q^{\mu-k})^m = \sum_{j=0}^{m-1} (-q^k)^j f\big((-1)^m q^{mk+j\mu}, (-1)^m q^{-mk+(m-j)\mu}\big) M_{\mu,j,m}, \tag{2.1}$$

where each $M_{\mu,j,m}$ is certain formal power series in q^μ, independent of k, and $f(a, b)$ is the Ramanujan theta function, which satisfies the following celebrated Jacobi triple product identity:

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, \qquad |ab| < 1.$$

Proof. Mc Laughlin [16, Equation (4.4)] deduced that

$$\begin{aligned} f(zq, q/z)^m = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{m-1}=0}^{m-1} & z^{j_{m-1}} q^{j_1^2 + (j_2-j_1)^2 + \cdots + (j_{m-1}-j_{m-2})^2} \\ & \times f\big(q^{2+2j_1}, q^{2-2j_1}\big) f\big(z^m q^{m+2j_{m-1}}, q^{m-2j_{m-1}}/z^m\big) \\ & \times \prod_{i=2}^{m-1} f\big(q^{i(i+1)+2(i+1)j_{i-1}-2ij_i}, q^{i(i+1)-2(i+1)j_{i-1}+2ij_i}\big). \end{aligned} \tag{2.2}$$

Replacing q by $q^{\mu/2}$ and setting $z = -q^{-\mu/2+k}$ in (2.2) yields

$$\begin{aligned} & f(-q^k, -q^{\mu-k})^m \\ & = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{m-1}=0}^{m-1} (-q^k)^{j_{m-1}} q^{(j_1^2+j_2^2+\cdots+j_{m-2}^2 - j_1 j_2 - j_2 j_3 - \cdots - j_{m-2} j_{m-1})\mu + (j_{m-1}^2 - j_{m-1})\mu/2} \\ & \qquad \times f\big(q^{(1+j_1)\mu}, q^{(1-j_1)\mu}\big) f\big((-1)^m q^{mk+j_{m-1}\mu}, (-1)^m q^{(\mu-k)m - j_{m-1}\mu}\big) \end{aligned}$$

$$\times \prod_{i=2}^{m-1} f\big(q^{(i(i+1)/2+(i+1)j_{i-1}-ij_i)\mu}, q^{(i(i+1)/2-(i+1)j_{i-1}+ij_i)\mu}\big),$$

from which we obtain that for $0 \le j \le m-1$,

$$\begin{aligned} M_{\mu,j,m} = &\sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{m-2}=0}^{m-2} q^{(j_1^2+j_2^2+\cdots+j_{m-2}^2-j_1j_2-j_2j_3-\cdots-j_{m-2}j)\mu+(j^2-j)\mu/2} \\ &\times f\big(q^{(1+j_1)\mu}, q^{(1-j_1)\mu}\big)\prod_{i=2}^{m-2} f\big(q^{(i(i+1)/2+(i+1)j_{i-1}-ij_i)\mu}, q^{(i(i+1)/2-(i+1)j_{i-1}+ij_i)\mu}\big) \\ &\times f\big(q^{(m(m-1)/2+mj_{2-1}-(m-1)j)\mu}, q^{(m(m-1)/2-mj_{m-2}+(m-1)j)\mu}\big). \end{aligned}$$

This completes the proof. □

Now it is time to prove Theorem 1.1.

Proof of Theorem 1.1. According to (2.1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty}\omega_{k,2m+1,2m+1}(n)q^n &= (q^k, q^{2m+1-k}; q^{2m+1})_\infty^{2m+1} \\ &= \frac{1}{(q^{2m+1}; q^{2m+1})_\infty^{2m+1}} f(-q^k, -q^{2m+1-k}) \\ &\quad\times \sum_{j=0}^{2m-1}(-q^k)^j f\big(q^{2mk+(2m+1)j}, q^{2m(2m+1)-(2m+1)j-2mk}\big) M_{2m+1,j,2m} \\ &= \frac{1}{(q^{2m+1}; q^{2m+1})_\infty^{2m+1}}\sum_{j=0}^{2m-1}(-1)^j S_{2m,j,k}M_{2m+1,j,2m}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} S_{2m,j,k} &= q^{jk} f(-q^k, -q^{2m+1-k}) f\big(q^{2mk+(2m+1)j}, q^{2m(2m+1)-(2m+1)j-2mk}\big) \\ &= \sum_{x,y=-\infty}^{\infty}(-1)^x q^{(2m+1)x(x-1)/2+kx+m(2m+1)y(y-1)+2mky+(2m+1)jy+jk}. \end{aligned} \tag{2.4}$$

In $S_{2m,j,k}$, assume that $kx + 2mky + jk \equiv 2mk^2 \pmod{2m+1}$, then the condition $\gcd(k, 2m+1) = 1$ implies that $x + 2my + j \equiv -k \pmod{2m+1}$. That is, $x - y \equiv -k - j \pmod{2m+1}$, or, equivalently, $2mx + y \equiv k + j \pmod{2m+1}$. Thus one can write $x - y = (2m+1)r - k - j$ and $2mx + y = (2m+1)s + k + j$, where r and s are some integers. We therefore deduce that $x = r + s$ and $y = -2mr + s + k + j$. Substituting these values into (2.4) gives

$$U_{2m+1}\big(q^{-2mk^2}S_{2m,j,k}\big) = \sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{Q_1(r,s)},$$

where the Atkin U_m-operator is defined by

$$U_m\left(\sum_{n=n_0}^{\infty} a(n)q^n\right) = \sum_{n=\lceil n_0/m\rceil}^{\infty} a(mn)q^n,$$

and

$$\begin{aligned}Q_1(r,s) = j^2 + 2jk - jm + j^2m - km + 2jkm + k^2m - r/2 + kr - 2jmr \\ - 2kmr + 2m^2r - 4jm^2r - 4km^2r + r^2/2 + 4m^3r^2 - s/2 \\ + js + ks - ms + 2jms + 2kms + rs - 4m^2rs + s^2/2 + ms^2. \quad (2.5)\end{aligned}$$

In order to cancel these two "cross terms" in (2.5), that is, $rs - 4m^2rs$, thus we need to make a change of variable $s \mapsto (2m-1)r + s$. After simplification, we find that

$$U_{2m+1}\big(q^{-2mk^2}S_{2m,j,k}\big) = \sum_{r,s=-\infty}^{\infty} (-1)^{2mr+s}q^{Q_2(r,s)},$$

where

$$\begin{aligned}Q_2(r,s) = j^2 + 2jk - jm + j^2m - km + 2jkm + k^2m - jr - 2jmr - 2kmr \\ + mr^2 + 2m^2r^2 - s/2 + js + ks - ms + 2jms + 2kms + s^2/2 + ms^2.\end{aligned}$$

Next, we further make a change of variable $s \mapsto s - j - k$ to deduce that

$$U_{2m+1}\big(q^{-2mk^2}S_{2m,j,k}\big) = \sum_{r,s=-\infty}^{\infty} (-1)^{s-j-k}q^{Q_3(r,s)},$$

where

$$\begin{aligned}Q_3(r,s) = j/2 + j^2/2 + k/2 - k^2/2 + jk - jr - 2jmr \\ - 2kmr + mr^2 + 2m^2r^2 - s/2 - ms + s^2/2 + ms^2, \quad (2.6)\end{aligned}$$

from which we obtain that

$$\begin{aligned}U_{2m+1}\big(q^{-2mk^2}S_{2m,j,k}\big) = (-1)^{j+k}q^{j/2+j^2/2+k/2-k^2/2+jk}f\big(-1,-q^{(2m+1)}\big) \\ \times f\Big(q^{(2m+1)m-(2km+2jm+j)}, q^{(2m+1)m+(2km+2jm+j)}\Big) = 0. \quad (2.7)\end{aligned}$$

According to (2.3) and (2.7), we arrive at

$$\begin{aligned}&U_{2m+1}\left(q^{-2mk^2}\sum_{n=0}^{\infty}\omega_{k,2m+1,2m+1}(n)q^n\right) \\ &= U_{2m+1}\big(q^{-2mk^2}(q^k, q^{2m+1-k}; q^{2m+1})_\infty^{2m+1}\big) \\ &= \frac{1}{(q;q)_\infty^{2m+1}}\sum_{j=0}^{2m-1}(-1)^jU_{2m+1}\big(q^{-2mk^2}S_{2m,j,k}\big)U_{2m+1}\big(M_{2m+1,j,2m}\big) = 0,\end{aligned}$$

where the penultimate step follows from the fact that the $M_{2m+1,j,2m}$ for $0 \leq j \leq 2m-1$ are formal power series in which all powers of q are multiples of $2m+1$. Therefore, we conclude that

$$\omega_{k,2m+1,2m+1}\big((2m+1)n+2mk^2\big)=0.$$

This finishes the proof of Theorem 1.1. □

3. Concluding remarks

We conclude this paper with three remarks.

Firstly, following a similar strategy of proving Theorem 1.1, one can also obtain that for all n and any $m \geq 1$,

$$\varpi_{k,2m+1,2m+1}\big((2m+1)n+2mk^2\big)=0, \tag{3.1}$$

where $1 \leq k \leq m$, $\gcd(k, 2m+1) = 1$, and the sequence $\{\varpi_{j,r,k}(n)\}_{n\geq 0}$ is defined by

$$\sum_{n=0}^{\infty}\varpi_{j,r,k}(n)q^n=(q^j,q^{r-j};q^r)_\infty(-q^j,-q^{r-j};q^r)_\infty^{k-1}, \tag{3.2}$$

where j and r satisfy the same condition as (1.4). To prove (3.1), we need to utilize the following identity similar to (2.1), which can also be derived by (2.2):

$$f(q^k,q^{\mu-k})^m=\sum_{j=0}^{m-1}q^{kj}f\left(q^{mk+j\mu},q^{-mk+(m-j)\mu}\right)M_{\mu,j,m},$$

where $M_{\mu,j,m}$ is defined as Lemma 2.1. And the remainder of the proof of (3.1) is essentially the same as that of (1.5).

Secondly, with the help of a computer, we can not find any identities similar to (1.5) and (3.1) in the sequences $\{\omega_{k,2m,2m}(n)\}_{n\geq 0}$ and $\{\varpi_{k,2m,2m}(n)\}_{n\geq 0}$. Moreover, a follow-up question worth investigation is whether there are other arithmetic progressions on vanishing coefficients in the sequences $\{\omega_{k,2m+1,2m+1}(n)\}_{n\geq 0}$ and $\{\varpi_{k,2m+1,2m+1}(n)\}_{n\geq 0}$ except (1.5) and (3.1).

Finally, in a recent paper [8], Chern and the author further studied vanishing coefficients in several families of quotients or products of theta functions. We established two strategies that can provide a unified treatment on vanishing coefficients with arithmetic progressions in these families of quotients or products of theta functions. Our method can be used to not only verify experimentally discovered coefficient-vanishing results, but also to generate a number of general phenomena. Therefore, the appearances of (1.5) and (3.1) are just the tip of the iceberg in the general phenomena on vanishing coefficients enjoyed by certain families of products of theta functions.

21 Oct 2022 06:59:21 PDT
220604-Tang Version 2 - Submitted to Rocky Mountain J. Math.

Acknowledgements

The author would like to acknowledge the anonymous referee for his/her careful reading and constructive comments that have improved this paper to a great extent. This work was partially supported by the National Natural Science Foundation of China (No. 12201093), the Science and Technology Research Program of Chongqing Municipal Education Commission (No. KJQN202200509) and the Doctoral start-up research grant (No. 21XLB038) of Chongqing Normal University.

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