Real structure in non-commutative L_p-spaces

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ABSTRACT. Our work is devoted to the construction of real valued non-commutative L_p spaces associated with real W^*-algebras of different types. We construct such real non-commutative L_p-spaces, $1 \leq p < \infty$, and we prove the theorem of isomorphism for real non-commutative L_p-spaces. Finally, we build an approximation of L_p-spaces associated with real W^*-algebras of type III via L_p-spaces associated with finite real W^*-algebras, analogically to the work of U.Haagerup, M.Junge and Q.Xu (see T.A.M.S. 362 (2010), 2125-2165).

Introduction

The theory of real W^*-algebras is a comparatively new branch of the theory of operator algebras. Its development began in the 1970s with the works of E. Stormer, S. Ayupov and other mathematicians (see [**ARU**] for reference). This theory is closely connected with the theory of complex W^*- algebras and their *-automorphisms. Nevertheless, the structure theory of real W^*- algebras is different from the theory of complex W^*- algebras. Structural theory of real W^*- algebras, the theory of traces and classification by types are now basically completed (see, for example, [**ARU**] and [**L**]).

Non-commutative L_p-spaces are the well-known objects of theory of operator algebras and non-commutative integration. In our article we define the real valued non-commutative L_p-spaces associated

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with semi-finite and infinite real W^*-algebras, and study their isomorphisms. Also, we prove a reduction theorem to approximate real non-commutative L_p-space associate with W^*-algebra of type III (in Haagerup's type construction) by an increasing sequence of real non-commutative L_p-spaces which are associated with finite W^*-algebras.

In Section 1 we define the real valued non-commutative L_p-spaces associated with semi-finite and purely infinite real W^*-algebras and provide some examples. Also, we study the connection between real valued and complex valued non-commutative L_p-spaces (Theorem 1.5)

In Section 2 we prove the theorem of isomorphisms for real valued non-commutative L_p-spaces (Theorem 2.1).

In Section 3 we prove a reduction theorem (Theorem 3.2).

Following the development of this paper, the authors intend to devote a separate article to study of duality of real valued non-commutative L_p-spaces, classification of real valued non-commutative L_p-spaces associated with semi-finite hyper-finite real W^*-algebras, and maximal and individual ergodic theorems on it.

Preliminaries

Let M be a W^*-algebra on the complex Hilbert space H. A **-automorphism (*-anti-automorphism)* θ of M is defined to be a linear map $\theta : M \to M$, such that:

(i) $\theta(x^*) = \theta(x)^*$, and

(ii) $\theta(xy) = \theta(x)\theta(y)$ (respectively, $\theta(xy) = \theta(y)\theta(x)$) for all $x, y \in M$.

The group of all *-automorphisms of M will be denoted by $Aut(M)$ (see [**T1**], [**T2**]).

A *real W^*-algebra* is defined to be a weakly closed real *-subalgebra $R \subset B(H)$ that contains the identity operator and is such that $R \cap \mathbf{i}R = \{0\}$. The least W^*- algebra, containing R (so called *enveloping* W^*-algebra $U(R)$ for R, has the form $U(R) = R + \mathbf{i}R$. The *-subalgebra R defines on $U(R)$ an involutory *-anti-automorphism α_R (of period 2):

$$\alpha_R(x + \mathbf{i}y) = x^* + \mathbf{i}y^*, x, y \in R.$$

In accordance with this, R can be described as follows:

$$R = \{x \in U(R) : \alpha_R(x) = x^*\}.$$

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Let R_1 and R_2 be real W^*-algebras. such that $U(R_1) = U(R_2) = U$. Then R_1 and R_2 are real *-isomorphic if and only if α_{R_1} and α_{R_2} are conjugated in U, that is, there is a *-automorphism $\theta \in Aut(U)$ such that $\alpha_{R_1} \cdot \theta = \theta \cdot \alpha_{R_2}$.

We shall say that a real W^*- algebra is of type $I_{fin}, I_\infty, II_1, II_\infty$ or $III_\lambda, 0 \leq \lambda \leq 1$ if its enveloping W^*- algebra $U(R)$ is of type $I_{fin}, I_\infty, II_1, II_\infty$ or $III_\lambda, 0 \leq \lambda \leq 1$ respectively.

The center of W^*- algebra U is a subalgebra $Z(U) \subset U$ such that $Z(U) = \{x \in U : xy = yx, \ \ \forall y \in U\}$. Analogically, the center of real W^*- algebra R is the subalgebra $Z(R) = \{x \in R : xy = yx, \ \forall y \in R\} = Z(U) \cap R$. The real W^*- algebra is called the real factor, if its center is trivial: $Z(R) = \{\mathbb{C} \cdot \mathbf{1}\}$ (see [**ARU**], [**L**]).

Let φ be a weight on real or complex W^*-algebra M_+, that is, a linear map $\varphi : M_+ \to [0, +\infty)$ that satisfies the condition $0 \cdot \infty = 0$. A weight is said to be *normal* if it is ultra-weakly continuous. It is said to be *faithful* if the equality $\varphi(x) = 0$ implies that $x = 0$. It is said to be *semi-finite* if the set $\mathfrak{m}_\varphi = \{x - y : x > 0, y > 0, \varphi(x) < +\infty, \varphi(y) < +\infty\}$ is dense in M. A weight φ satisfying the condition $\varphi(xy) = \varphi(yx) \ \forall x, y \in (\mathfrak{m}_\varphi)_+$ is called a *trace*.

The basic information about the theory of W^*-algebras can be found in [**T1**], [**T2**], about real W^*-algebras - in [**ARU**] and [**L**].

1. Non-commutative L_p-spaces associated with real W^*-algebras

Consider a W^*-algebra U on a Hilbert space H, with a faithful normal finite or semi-finite trace $\bar{\tau}$. Unbounded closed densely defined operator x with a domain $D(x)$ is *affiliated* with U (denoted by $x\eta U$) if $u'xu'^* = x$ for all unitary $u' \in U'$. The operations for x, y affiliated with U are following:

(i) If x, y are linear operators affiliated with U, then $x + y$ with $D(x + y) = D(x) \cap D(y)$ and xy with $D(xy) = \{\xi \in D(y) : y\xi \in D(x)$ are affiliated with U.

(ii) If x is densely defined and $x\eta U$, then $x^*\eta U$.

(iii) If x is closable and $x\eta U$, then for the closure $\bar{x}$ of x we have: $\bar{x}\eta U$.

(iv) Assume that x is densely defined and closed; then for the polar decomposition $x = w|x|$ and the spectral decomposition $|x| = \int_0^\infty \lambda e_\lambda$

we have: $x\eta U$ if and only if $w, e_\lambda \in U$ for all $\lambda \geq 0$. A densely defined closed operator x affiliated with U is called *τ-measurable* if, for any $\delta > 0$, there exists a projecion $e \in U$ such that $eH \subset D(x)$ and $\tau(e^\perp) \leq \delta$.

Let $\tilde{U}$ be the space of all τ-measurable operators. For each $x \in \tilde{U}$ consider a p-norm on $\tilde{U}$:

$$||x||_p = [\bar{\tau}(|x|^p))^{1/p} \in [0, \infty), x \in S.$$

One can show that $||x||_p$ is a norm on S if $1 \leq p < \infty$ and a quasi-norm, if $0 < p < 1$.

The trace $\bar{\tau}$ can be extended to a linear functional on $\tilde{U}$, which will be still denoted by $\bar{\tau}$. A non-commutative L_p-space, associated with $(U, \bar{\tau})$ and denoted by $L_p(U, \bar{\tau})$, is defined as

$$L_p(U) = L_p(U, \bar{\tau}) = \{a \in \tilde{U} : ||a||_p = \bar{\tau}(|a|^p)^{1/p} < \infty\}, /0 < p \leq \infty$$

(references are in [**S**], [**Y**]).

Let $U = U(R) = R + \mathbf{i}R$, where R is a real W^*-algebra generating U. We say that an unbounded closed densely defined operator a is *affiliated* to R, if for the polar decomposition $a = u|a|$ we have: $u \in R$ and all spectral projections of $|a|$ are contained in R.

Assume that $U(R)$ is finite, and τ is a normal finite faithful trace on R. Consider then a restriction $\tau_0 = \tau|_{R_+}$ of τ on R_+. It is a normal finite faithful linear functional on A_+, and we can extend τ_0 from R_+ to a linear functional $\bar{\tau}$ on $U(R)_+$ by the following way: $\bar{\tau}(a + \mathbf{i}b) = \tau_0(a)$, where $a, b \in R, a^* = a, b^* = -b$. Then $\bar{\tau}$ is a normal finite faithful α_R-invariant trace on $U(R)_+$ (see [**A1**]). We use this α-invariant trace for constructing of $L_p(R, \tau)$.

Assume now that $U(R)$ is semi-finite, and τ is a normal semi-finite faithful trace on R_+. We can extend this trace to a linear functional $\bar{\tau}$ on $U(R)_+$ by the following: $\bar{\tau}(a + ib) = \tau(a)$, where $a, b \in R, a^* = a, b^* = -b$. Then $\bar{\tau}$ is a normal semi-finite faithful α_R-invariant trace on $U(R)_+$ (see [**A1**]). We use this α-invariant trace for constructing of $L_p(R, \tau)$.

Definition 1.1. A real non-commutative L_p-space, $1 < p < \infty$, associated with (R, τ) and denoted by $L_p(R, \tau)$, is defined as

$$L_p(R, \tau) = \{a \in \tilde{R} : ||a||_p = \tau(|a|^p)^{1/p} < \infty\}.$$

Evidently, $L_p(R, \tau) \subset L_p(U, \bar{\tau})$, $L_p(R, \tau)$ is a $||.||_p$-closed real Banach linear subspace of $L_p(U, \bar{\tau})$, and $L_p(R, \tau) \cap \mathbf{i}L_p(R, \tau) = \{0\}$. Since

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$\mathbf{i}a \notin L_p(U,\tau)$ for any positive a affiliated to R, then the space $L_p(R,\tau)$ is not equal to $L_p(U,\tau)$.

Example 1.2. *Commutative real W^*-algebras.* Consider a commutative real W^*-algebra $R = L^\infty(A,\mu)$ of real valued measurable functions on a measure space (A,μ). Then, using the integration by μ as normal semi-finite faithful trace τ, we obtain a commutative real L_p-space $L_p(R,\tau)$.

Example 1.3. *Non-commutative L_p-spaces associated with a real hyper-finite factor.* It is well known that a (unique) hyper-finite factor R_∞ of type II_∞ contains only one real hyper-finite factor Q of type II_∞, generating R_∞ (see Theorem 4.1, [**G**]). Then a (unique) normal semi-finite faithful trace τ on factor $(R_\infty)_+$ is α_Q-invariant. Consider a normal semi-finite faithful trace $\tau_0 = \tau|_Q$ on Q_+. Then we have built a complex valued non-commutative L_p-space $L_p(Q,\tau)$ and a real valued non-commutative L_p-space $L_p(R_\infty,\tau)$, such that $L_p(Q,\tau) \subset L_p(R_\infty,\tau)$.

Remark 1.4. If we consider a self-adjoint part of $L_p(R,\tau)$, it will be so called a *non-associative L_p-space* associated with JW-algebra $A = (R_{sa}, \circ)$ of all self-adjoint elements of R supplied with a Jordan multiplication $x \circ y = (1/2)(xy + yx), x, y \in R_{sa}$ (see [**A**]).

We define now a non-commutative real L_p-space associated with real W^*-algebra of type III, $1 < p < \infty$, and constructed not by trace only but by any normal semi-finite faithful weight. For this we use the Haagerup's method of constructing of L_p-spaces associated with W^*-algebras of type III ([**H**], [**Te**]).

Let R be a real W^*-algebra of type III, $U(R)$ - its enveloping W^*-algebra of type III, acting on a Hilbert space H, and φ_0 a normal semi-finite faithful α_R-invariant weight on $U(R)_+$ (about existence of an α_R-invariant φ_0 see [**U**] or [**U1**]). Let $\sigma_t^{\varphi_0}, t \in \mathbb{R}$ be a 1-parametric group of modular *-automorphisms of $U(R)$. Consider a crossed product $N = U(R) \times_{\sigma^{\varphi_0}} \mathbb{R}$ of $U(R)$ on σ^{φ_0}, which is a W^*-algebra generated by $\pi(x), x \in U(R)$ and $\lambda_s, s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = (\sigma_{-t}^{\varphi_0}(x)\xi)(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R},$$

$$(\lambda_s\xi))(t) = \xi(t - s), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}.$$

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The dual actions $\{\theta_s\}$ of $\sigma_t^{\varphi_0}$ extend naturally to *-automorphisms on the extended positive part $\hat{N}_+$ of N (see [**H1**]), and further its extension to continuous *-automorphisms of $\hat{N}$. Consider a mapping

$$\Phi(x) = \pi^{-1}(\int_{-\infty}^{+\infty} \theta_s(x)ds),$$

where $x \in N_+$, and consider also for each normal weight φ on $U(R)$ the weight $\hat{\varphi}$ (an extension of φ to a normal weight on $U(\hat{R})_+$, [**H1**], Prop. 1.10) and a (normal) weight $\tilde{\varphi} = \hat{\varphi} \circ \Phi$. Then the weight $\tilde{\varphi}$ is called the dual weight of φ. There exists a unique normal faithful semi-finite trace τ on N for which we have a Connes' cocycle

$$(\frac{D\tilde{\varphi}_0}{D\tau})_t = \lambda_t,\ t \in \mathbb{R},$$

and $\tau \circ \theta_s = e^{-s}\tau, s \in \mathbb{R}$ (see [**H2**], Lemma 5.2).

Let $U(\tilde{R})$ be a set of all τ-measurable operators affiliated with $U(R)$. For $0 < p < \infty$ the Haagerup L_p-space $L_p(U(R), \varphi_0)$ is defined by

$$L_p(U(R), \varphi) = \{a \in U(\tilde{Q}) : \theta_s(a) = \hat{\sigma^{\varphi}}_s = e^{-s/p}a, \quad s \in \mathbb{R}\}.$$

It was shown in Proposition 3.1 (ii)[**U1**] that on N there exists an involutory *-anti-automorphism $\hat{\alpha_R}$ which is an extension of the involutory *-anti-automorphism α_R on R. Also, it was shown there that $\hat{\alpha_R}$ and $\theta = \hat{\sigma}$ commute on N, and that $\Phi \circ \hat{\alpha_R} = \Phi$ and $\tau \circ \hat{\alpha_R} = \tau$.

Let us denote by Q a real W^*-subalgebra of N defined by an involutory *-anti-automorphism $\hat{\alpha_R}$. Then $N = U(Q)$, and the Radon-Nikodym derivative

$$h_\varphi = \frac{d\tilde{\varphi}}{d\tau}$$

is affiliated with Q (see Theorem 2.1 [**U1**]). The set of all τ-measurable operators affiliated with Q we denote by $\tilde{Q}$.

Definition 1.5. The set

$$L_p(R, \varphi) = \{a \in \tilde{Q} : \theta_s(a) = e^{-s/p}a, s \in \mathbb{R}\}$$

is called the real valued Haagerup's non-commutative L_p-space, or just the real L_p-space $1 < p \leq \infty$, associated with R.

Remark 1.6. Since all $\hat{\alpha}_R$-invariant normal semi-finite faithful weights are connected via Radon-Nikodym derivatives, affiliated to R (Theorem 2.1 [**U**]), then all dual actions θ of modular groups associated with $\bar{\alpha}_R$-invariant normal semi-finite faithful weights on $U(R)_+$ are commuting with $\bar{\alpha_Q}$ and conjugated in Q, and then all real L_p-spaces defined by such dual actions are isometrically *-isomorphic as real valued Banach spaces.

Consider now a semi-finite normal faithful trace τ on real W^*-algebra Q, the enveloping W^*-algebra $U(R) = R + \mathbf{i}R$ of R, an involutory *-anti-automorphism α of $U(R)$, generating R, and a trace $\bar{\tau}$ which is a linear extension of the trace τ on R to $U(R)$. We know that $L_p(R,\tau) \subset L_p(U(R),\bar{\tau})$ by definition of $L_p(R,\tau)$.

Theorem 1.7. If an extended trace $\bar{\tau}$ on $U(R)$ is α-invariant, then

$$L_p(U(R),\bar{\tau}) = L_p(R,\tau) + \mathbf{i}L_p(R,\tau).$$

Proof. Consider an unbounded operator $x = a + \mathbf{i}b, a, b \in Q$, affiliated to $U(R)$. Assume that $x \geq 0$. Then $a^* = a \geq 0, b^* = -b$, and x has a spectral decomposition $\int_0^\infty \lambda dP_\lambda$, where $\{P_\lambda\}$ is an spectral family of projectors from $U(R)$. By spectral theorem and by continuity of α in strong topology, $\alpha(x) = \int_0^\infty \lambda d\alpha(P_\lambda)$. If $x \in L_p(U(R),\bar{\tau}$, then

$$||x||_p^p = \bar{\tau}(|x|^p) = \bar{\tau}(x^p) = \int_0^\infty \lambda^p d\bar{\tau}(P_\lambda) < \infty,$$

and then

$$||\alpha(x)||_p^p = \bar{\tau}((\alpha(x))^p) = \int_0^\infty \lambda^p d\bar{\tau}(\alpha(P_\lambda)).$$

Since $\bar{\tau}$ is α-invariant, then

$$\bar{\tau}(\alpha(P_\lambda)) = \bar{\tau}(P_\lambda), \quad \lambda \in \mathbb{R}.$$

Thus, $||\alpha(x)||_p^p < \infty$, too, and then $\alpha(x)$ is also contained in $L_p(U(R),\bar{\tau})$. So,

$$a = 1/2(x + \alpha(x)) = 1/2(a + \mathbf{i}b + a - \mathbf{i}b) \in L_p(U(R),\bar{\tau}),$$

and

$$b = \frac{1}{\mathbf{i}}(x - a) \in L_p(U(R),\bar{\tau}).$$

Consider now a self-adjoint unbounded operator k, affiliated with $U(R)$. We know that $k = k_1 - k_2$,

$$k_1 = \int_0^\infty \lambda dP_\lambda^{(1)},\ k_2 = \int_0^\infty \lambda dP_\lambda^{(2)},$$

$supp(k_1)$ and $supp(k_2)$ are orthogonal. Then we can prove similarly that if $k \in L_p(U(R), \bar{\tau})$, then $\alpha(k) \in L_p(U(R), \bar{\tau})$.

Since any unbounded operator x in $L_p(U(R), \bar{\tau})$ represents a linear combination of self-adjoint operators:

$$x = \frac{1}{2}(x + x^*) + \frac{1}{2\mathbf{i}}(\mathbf{i}x - \mathbf{i}x^*),$$

then $\alpha(x) \in L_p(U(R), \bar{\tau})$ for any $x \in L_p(U(R), \bar{\tau})$, and for $x = a + \mathbf{i}b, a, b \in R$ if $x \in L_p(U(R), \bar{\tau})$ then $a, b \in L_p(U(R), \bar{\tau})$. Since $\bar{\tau}|_R = \tau$, then $a, b \in L_p(R, \tau)$.

So,

$$L_p(U(R), \bar{\tau}) \subset L_p(R, \tau) + \mathbf{i}L_p(R, \tau),$$

and then

$$L_p(U(R), \bar{\tau}) = L_p(R, \tau) + \mathbf{i}L_p(R, \tau).\square$$

2. Isomorphisms of real non-commutative L_p-spaces associated with real semi-finite W^*-algebras

The following theorem is a real-valued version of the theorem of K. Watanabe (see [**W**], Theorem 3.6):

Theorem 2.1. Let $1 < p < \infty$, $p \neq 2$. Let R_1 and R_2 be σ-finite W^*- algebras with normal semi-finite faithful traces τ_1 and τ_2 correspondingly. Let T be a *-preserving linear isometry from $L_p(R_1, \tau_1)$ to $L_p(R_2, \tau_2)$. Then there exists a Jordan *-isomorphism from R_1 to R_2.

Proof. We begin the proof from a technical lemma 2.2:

Lemma 2.2. Let $1 < p < \infty$. Consider a σ-finite real W^*- algebra R. Then for any two equivalent projections $e, f \in R$, $e = u^*u$ and $f = uu^*$ for some partial isometry $u \in R$, we can find an element $a \in L_p(R, \tau)$, $\tau \circ \alpha_R = \tau$, such that the right support $r(a)$ of a is equal to e and left support $l(a)$ of a is equal to f.

Proof. Since R is σ-finite, there is a faithful normal state φ on R, such that $\varphi \circ \alpha_R = \varphi$. Consider a finite normal functional $\varphi(u^*.u)$ with a support uu^*. Then

$$\varphi(u^*(\mathbf{1} - uu^*u) = 0.$$

Also, if $q \in R$ is any projection such that $\varphi(u^*(\mathbf{1}-q)u) = 0$ we have: $u^*(\mathbf{1} - q)u = 0$, and then $uu^* \leq q$. Define an element $a \in L_p(R, \tau$ by $a = u^*(uh_0u^*)^{1/p}$, where $h_0 = \frac{d\varphi}{d\tau} > 0$ is a Radon-Nikodym derivative of φ with respect to τ. Since φ and τ both are α_R-invariant, then h_0 is affiliated to R (see [**U**], Theorem 2.1). Then $a = u^*(uh_0u^*)^{1/p}$ is also a polar decomposition of a, and then $r(a) = uu^* = f$ and $l(a) = u^*u = e$. The proof is completed. □

Corollary 2.3. Let $1 < p < \infty$, and let R be a σ-finite real W^*- algebra. Then for any projection e there exists an element $a \in L_p(R, \tau)_+$ such that the support $s(a)$ of a is equal to e.

Let $P(R)$ be a set of all projections in R.

Definition 2.4. A *projection ortho-isomorphism* between R_1 and R_2 is the map $\theta : P(R_1) \to P(R_2)$ which is one to one, onto, and such that $ef = 0$ if and only if $\theta(e)\theta(f) = 0$ for $e, f \in R_1$.

Lemma 2.5. Let $1 < p < \infty$, $p \neq 2$, and let R_1 and R_2 are σ-finite real W^*- algebras. Assume that T is a *-preserving linear isometry from $a \in L_p(R_1, \tau_1)$ onto $a \in L_p(R_2, \tau_2)$. Then there exists an ortho-isomorphism between $U(R_1)$ and $U(R_2)$ such that $\theta(P(R_1)) = P(R_2)$.

Proof. Consider an extension $\bar{T}$ of T by linearity: $\bar{T} : L_p(U(R_1), \bar{\tau_1}) \to L_p(U(R_2), \bar{\tau_1})$. Then it is a *-preserving linear isometry $\bar{T}$ from $L_p(U(R_1), \bar{\tau_1})$ onto $L_p(U(R_2), \bar{\tau_2})$. For $\bar{T}$ in Proposition 3.5 ([**W**]) it was defined a mapping $J : P(U(R_1))$ to $P(U(R_2))$:

$$J(s(a)) = s(T(a)), a \in L_p(U(R_1), \bar{\tau_1})_{sa}.$$

Then if a is affiliated to R_1, then $\bar{T}(a) = T(a)$ by construction of the linear *-isometry $\bar{T}$, and then $s(T(a)) \in R_2$. Thus, $J(s(a)) = s(\bar{T}(a)) = s(T(a)) \in P(R_2)$, if a is affiliated to R_1. So, by Proposition 3.5 ([**W**]), J is an ortho-isomorphism, and $J(P(R_1)) = P(R_2)$. □

By Theorem 3.6 and Lemma 6 and 7, and also by Theorem 1 and Corollary from the article of H. Dye (see [**D**]), there exists a Jordan *-isomorphism ρ between $U(R_1)$ and $U(R_2)$. By construction of ρ (see Lemma 6 of [**D**]), $\rho(R_1) = R_2$, which is completing the proof of Theorem 2.1.□

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Corollary 2.6. If R_1 and R_2 are not *isomorphic, then there exists no *-preserving surjective linear isometry from $L_p(R_1, \tau_1)$ to $L_p(R_2, \tau_2)$, $1 < p < \infty$, $p \neq 2$.

Moreover, consider now a σ-finite factor M of type III_λ, where $0 < \lambda < 1$. In [**Sta**] P. J. Stacey proved that M contains only two non-isomorphic real factors R_1 and R_2, generating M. Let ϕ_1 be a normal semi-finite faithful weight on R_1 and ϕ_2 - a normal semi-finite faithful weight on R_2.

Corollary 2.7. There exists no *-preserving surjective linear isometry from $L_p(R_1, \phi_1)$ to $L_p(R_2, \phi_2)$, $1 < p < \infty$, $p \neq 2$.

Thus, we have two non-isomorphic real L_p-spaces associated with real sub-factors generating the same factor of type III_λ.

3. Reduction to L_p-spaces affiliated with finite real W^*-algebras

In this section we are constructing an approximation of Haagerup's real L_p spaces, built in Section 1, by L_p-spaces associated with finite real W^*-algebras, as a real valued analogy of Theorem 2.1 [**HJX**].

Let G be a discrete subgroup $\bigcup_{n>0} 2^{-n}\mathbb{Z}$ of $\mathbb{R}$. Let M be a σ-finite W^*-algebra with real W^*-subalgebra of M, generated by α_R, such that $M = U(R)$, and with normal faithful α_R-invariant state φ. Consider the crossed product $M = U(R) \times_{\sigma^\varphi} \mathbb{R}$, where the modular group σ_t^φ is an automorphic representation of G on U (see [**HJX**], p. 2131). Let $\hat{\varphi}$ denote a dual weight of φ, which is α_1-invariant (see [U]). Let Q be a real W^*-subalgebra generated by α_1 in N, $N = U(Q) = Q + \mathbf{i}Q$.

Theorem 3.1. There exists an increasing sequence $\{Q_n\}_{n>0}$ of finite real W^*-subalgebras of Q such that:

(i) $\bigcup_{n\geq 1} Q_n$ is w^*-dense in Q;

(ii) For every $n \in N$ there exists a normal faithful conditional expectation Φ_n from Q to Q_n such that

$$\hat{\varphi} \circ \Phi_n = \hat{\varphi}$$

and

$$\sigma_t^\varphi \circ \Phi_n = \Phi_n \circ \sigma_t^\varphi. \ \ t \in R$$

Proof. It is well known that since $M = U(R)$ is a W^*-algebra of type III, then $N = U(Q)$ is a semi-finite W^*-algebra, by Takesaki's duality (see [**T3**]). Then

$$\sigma_s^{\hat{\varphi}}(x) = \lambda(s)x\lambda(s)^*, \quad x \in M, s \in \mathbb{R},$$

where $\{\lambda(s)\}$ are unitary operators from N. Following the Lemma 2.3 from [**HJX**], define the unique element

$$b_n = -\mathbf{i}\log(\lambda(2^{-n})),$$

where $0 < Im(\log z) < 2\pi$, $z \in C\ \{0\}$. Then $0 \leq b_n \leq 2\pi\mathbf{1}$, $e^{\mathbf{i}b_n} = \lambda(2^{-n})$ and $b_n \in Z(M_{\hat{\varphi}}) = \{x \in M : \sigma_s^{\hat{\varphi}}(x) = x\}$. Define now

$$\varphi_n(x) = \hat{\varphi}(e^{2^n b_n}x), \quad x \in M,\ n \geq 1.$$

By Lemma 2.4 from [**HJX**], $\sigma_s^{\varphi_n}$ is 2^{-n} periodic, and for $N_n = N_{\varphi_n}$ there exists a unique normal faithful conditional expectation Φ_n from N onto N_n such that

$$\hat{\varphi} \circ \Phi_n = \hat{\varphi}$$

and

$$\sigma_s^{\hat{\varphi}} \circ \Phi_n = \Phi_n \circ \sigma_s^{\hat{\varphi}}, \quad t \in \mathbb{R}, n \geq 1,$$

together with $N_n \subset N_{n+1}$. Now let us note that $\hat{\varphi}$ is α_Q-invariant (see [**U1**]), and then $\lambda(s)$ could be represented as $\lambda(s) = h^{\mathbf{i}s}$, where h is affiliated to real W^*-algebra $Q = \{x \in N : \alpha_Q(x) = x^*\}$, associated with involutory *-anti-automorphism α_Q.

By construction of b_n and by spectral theorem we can see that $b_n \in Q$, and then φ_n is α_Q-invariant for every $n \geq 1$. From Proposition 3.1 [**U1**] we can see that Φ_n are commuting with α_Q (which means that we can correctly reduce the action of Φ_n to the conditional expectation from Q to Q_n), and also that $N_n = Q_n + \mathbf{i}Q_n$, where $Q_n = N_n \cap Q$. Also, we have $Q_n \subset Q_{n+1}$. Really, $\varphi_{n+1}(x) = \varphi(h_n x)$ (see Lemma 2.4 from [**HJX**]), and $h_n = e^{-(2^{-(n+1}b_{n+1} - 2^{-n}b_n)} \in N_n$, and easy to check that $h_n \in Q_n = N_n \cap Q$. Finally, by Theorem 2.1 from [**HJX**], $\bigcup_{\leq 1} N_n$ is $w*$-dense in N, so, $\bigcup_{n\leq 1} Q_n$ is dense in Q. The proof of Theorem is completed.□

Theorem 3.2. Let R be a σ-finite real W^*-algebra and $1 \leq p < \infty$. Let $L_p(R)$ be a Haagerup's type real non-commutative L_p-space associated with R. Then there exist a real valued Banach space X_p, a sequence Q_n of finite real W^*-algebras, each equipped with a finite normal faithful trace τ_n, and for each $n \geq 1$ an isometric embedding

$J_n : L_p(Q_n, \tau_n) \to Y_p$ such that

(i) the sequence $\{J_n(L_p(Q_n, \tau_n))\}$ is increasing;

(ii)$\bigcup_{n\leq 1} J_n(L_p(Q_n, \tau_n))$ is dense in Y_p;

(iii) $L_p(R)$ is isometric to a subspace of $\tilde{Y_p}$ of Y_p.

Proof. Consider a normal faithful α_R-invariant state on M. Let us build the Banach space $Y_p = L_p(Q, \hat{\varphi})$, and a sequence of Banach spaces $L_p(Q_n, \hat{\varphi})|_{Q_n}) = L_p(Q_n, \tau_n)$, which we constructed in the proof of Theorem 3.1. Its satisfy properties of (i)-(iii) of Corollary 3.2, and it follows from Theorem 3.1 [**HJX**] that $\bigcup_{n\geq 1} L_p(N_n, \hat{\varphi}_n)$ is dense in $L_p(N, \hat{\varphi})$. Then it follows from Theorem 1.5 that $\bigcup_{n\geq 1} L_p(Q_n, \hat{\varphi})|_{Q_n}) = \bigcup_{n\geq 1} L_p(Q_n, \tau_n)$ is dense in $L_p(Q, \tau)$.□

Presented construction of real non-commutative L_p-spaces allows to state few structural problems, resolved in 2000-2004 for (complex) non-commutative L_p-spaces:

Questions: 1. Is the Dichotomy Principle of Kadec-Pelczynski for closed linear subspaces ([**KP**]) correct for all non-isomorphic real non-commutative L_p-spaces, $p > 2$, generating the isomorphic (complex) non-commutative L_p-space (for the complex non-commutative L_p spaces refer [**RX**])?

2. Is the Subsequence Splitting Lemma for bounded sequences in non-commutative L_p-spaces, $0 < p < \infty$ (see [**R**], [**R1**]) correct for the real non-commutative L_p spaces?

3. Build a full classification of real L_p spaces, $1 < p < \infty$, associated with real semi-finite W^*-algebras (refer [**HRS**]).

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