CONTINUOUS DEPENDENCE ON BOUNDARY CONDITIONS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Let $n \in \mathbb{N}$ and $\alpha \in (n-1, n)$. Under uniqueness and continuity conditions, we show that solutions to Caputo fractional differential equations depend continuously upon the boundary conditions. We also provide sufficient conditions on interval length between the boundary points in a specific case for $\alpha \in (1, 2)$.

Keywords: continuous dependence, Caputo fractional differential equation, uniqueness
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1. Introduction

Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$ and $a < t_0 < b$ in $\mathbb{R}$. We consider the Caputo differential equation

$$D^{\alpha}_{*t_0} x(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad a < t_0 < t < b, \tag{1.1}$$

where $D^{\alpha}_{*t_0} x$ is the Caputo fractional derivative of order α of the function x.

For $i = 0, 1, \dots, n-1$, let $c_i \in \mathbb{R}$ and define initial conditions for (1.1) at t_0 by

$$x^{(i)}(t_0) = c_i. \tag{1.2}$$

Let $a < t_0 \le t_1 < t_2 < \dots < t_n < b$ and for $i = 1, 2, \dots, n$, $x_i \in \mathbb{R}$. Define boundary conditions for (1.1) as

$$x(t_i) = x_i. \tag{1.3}$$

Throughout this work, we make use of the following assumptions:

(1) $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;
(2) solutions to initial value problems for (1.1) are unique on (a, b);
(3) given points $a < t_0 \le t_1 < t_2 < \dots < t_n < b$, if y and z are solutions of (1.1) such that for $i = 1, 2, \dots, n$, $y(t_i) = z(t_i)$, then $y(t) = z(t)$ on (a, b).

We will explore sufficient conditions for hypothesis (3) in section 3 with $n = 2$.

In this article, we provide a boundary value problem analog of the recent result of Eloe and Masthay [10]. In this work, the authors proved that solutions of initial value problems depend continuously upon initial values when two assumptions are made; namely that f is continuous and that of solutions of Caputo fractional initial value problems are unique. Assumptions (1) and (2) above are those sufficient for the results in [10] and (3) is an additional uniqueness assumption for solutions of Caputo fractional boundary value problems. Also, the authors showed continuous dependence on the initial point from the left when a sequence of uniformly continuous functions converge to f.

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Research into fractional differential equations has seen an explosion of results, [1, 2, 3, 4, 9, 19, 24, 26], to name a small fraction. In fact, there seem to be a limitless number of different ways to define a fractional derivative. However, two definitions have become the source of focus amongst a broad range of researchers in the field; namely the Riemann-Liouville and Caputo fractional derivatives. For expository material on fractional differential equations, we refer the reader to [5, 17, 18, 21].

The novel results presented in this article have broad implications in that much research done for boundary value problems in the ordinary differential equation context over the past 50 years may be translated over to Caputo fractional differential equation. These include uniqueness implies existence results [7, 8, 12, 14, 15, 20] and continuous dependence upon boundary data results [6, 11, 13, 16, 23]. We posit that this result is critical and opens up a wide range of study of Caputo fractional boundary values problems akin to their ordinary differential equation counterparts some of which are listed above.

The remainder of the paper is organized as follows. In Section 2, we introduce fractional integrals and derivatives with specific focus upon Caputo fractional derivatives. Section 3 provides sufficient conditions to invoke hypothesis (3) with a Lipschitz continuous function and $\alpha \in (1,2)$. To conclude, section 4 is where one finds Eloe and Masthay's results and also contains the main results of this work; a boundary value problem analog to continuous dependence of initial value problems.

2. Preliminaries

Let $\alpha > 0$. The Riemann-Liouville fractional integral of a function x of order α, denoted $I^{\alpha}_{t_0}x$, is defined as

$$I^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds, \quad t_0 \le t,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n-1 < \alpha \le n$. The Riemann-Liouville fractional derivative of order α of the function x, denoted $D^{\alpha}_{t_0}x$, is defined as

$$D^{\alpha}_{t_0}x(t) = D^{n}I^{n-\alpha}_{t_0}x(t),$$

provided the right-hand side exists. If a function x is such that

$$D^{\alpha}_{t_0}\left(x(t) - \sum_{i=0}^{n-1} x^{(i)}(t_0)\frac{(t-t_0)^i}{i!}\right)$$

exists, then the Caputo fractional derivative of order α of x is defined by

$$D^{\alpha}_{*t_0}x(t) = D^{\alpha}_{t_0}\left(x(t) - \sum_{i=0}^{n-1} x^{(i)}(t_0)\frac{(t-t_0)^i}{i!}\right).$$

Remark 2.1. A sufficient condition to guarantee the existence of the Caputo fractional derivative is the absolute continuity of the $(n-1)$st derivative of x. See Theorem 3.1 in [5] and discussion thereafter.

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3. Hypothesis (3)

We present sufficient conditions to satisfy hypothesis (3) when $n = 2$, i.e. $\alpha \in (1, 2]$.

For $a < t_0 \le t_1 < t_2 < b$ and continuous $f : (a, b) \times \mathbb{R} \to \mathbb{R}$, we consider the Caputo fractional boundary value problem

$$D^{\alpha}_{*t_0} x(t) = f(t, x(t)), \tag{3.1}$$

$$x(t_1) = x_1 \quad x(t_2) = x_2. \tag{3.2}$$

Lemma 3.1. *Let $g : (a, b) \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = I^{\alpha}_{t_0} g(t) - \frac{t - t_1}{t_2 - t_1} I^{\alpha}_{t_0} g(t_2) - \frac{t_2 - t}{t_2 - t_1} I^{\alpha}_{t_0} g(t_1) + x_1 \left(\frac{t_2 - t}{t_2 - t_1}\right) + x_2 \left(\frac{t - t_1}{t_2 - t_1}\right)$$

if and only if x is a solution of the Caputo fractional boundary value problem

$$D^{\alpha}_{*t_0} x(t) = g(t), \quad a < t_0 < t < b,$$
$$x(t_1) = x_1 \quad x(t_2) = x_2.$$

Proof. ($\Rightarrow$) First note that for $\alpha \in (1, 2]$ that the Caputo fractional derivative of order α of a linear function is 0.

Apply $D^{\alpha}_{*t_0}$ to both sides of the integral equation to get

$$\begin{aligned} D^{\alpha}_{*t_0} x(t) =& D^{\alpha}_{*t_0} \left[I^{\alpha}_{t_0} g(t) - \frac{t - t_1}{t_2 - t_1} I^{\alpha}_{t_0} g(t_2) - \frac{t_2 - t}{t_2 - t_1} I^{\alpha}_{t_0} g(t_1) + x_1 \left(\frac{t_2 - t}{t_2 - t_1}\right) + x_2 \left(\frac{t - t_1}{t_2 - t_1}\right)\right] \\ =& g(t) \end{aligned}$$

Now, it is clear that $x(t_1) = x_1$ and $x(t_2) = x_2$. Therefore, $x(t)$ solves the Caputo fractional boundary value problem.

($\Leftarrow$) Next, apply $I^{\alpha}_{t_0}$ to both sides of $D^{\alpha}_{*t_0} x(t) = g(t)$ to get

$$I^{\alpha}_{t_0} \left(D^{\alpha}_{*t_0} x(t)\right) = I^{\alpha}_{t_0}(g(t)).$$

From Lemma 2.4 [25], we have

$$x(t) + c_0 + c_1(t - t_0) = I^{\alpha}_{t_0} g(t)$$

or

$$x(t) = I^{\alpha}_{t_0} g(t) - c_0 - c_1(t - t_0).$$

Set $t = t_1$ and replace $x(t_1) = x_1$ to see

$$x_1 = I^{\alpha}_{t_0} g(t_1) - c_0 - c_1(t_1 - t_0).$$

Similarly, we have

$$x_2 = I^{\alpha}_{t_0} g(t_2) - c_0 - c_1(t_2 - t_0).$$

Subtract the two equations to get

$$x_2 - x_1 = I^{\alpha}_{t_0} g(t_2) - I^{\alpha}_{t_0} g(t_1) - c_1(t_2 - t_1).$$

Thus,

$$c_1 = \frac{I^{\alpha}_{t_0} g(t_2) - I^{\alpha}_{t_0} g(t_1)}{t_2 - t_1} - \frac{x_2 - x_1}{t_2 - t_1}.$$

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Substituting back, we find, in terms of c_1, that

$$c_0 = I^{\alpha}_{t_0} g(t_2) - x_2 - c_1(t_2 - t_0).$$

Substituting c_0 and c_1, simplification, and rearrangement yield the desired integral equation $x(t)$. □

A standard application of the Banach fixed point theorem is used to prove the following.

Theorem 3.1. *Let $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

(A1) $|f(t, u) - f(t, v)| \leq L|u - v|$ *for* $t \in [t_0, t_2]$ *and* $u, v \in \mathbb{R}$ *and*

(A2) $\dfrac{(2 + T)L(t_2 - t_0)^{\alpha}}{\Gamma(\alpha + 1)} < 1$

where L is a Lipschitz constant and $T := \dfrac{t_2 - t_0}{t_2 - t_1} \geq 1$. *Then,* (3.1), (3.2) *has a unique solution on* $[t_0, t_2]$.

Proof. We transform the problem into a fixed point problem to apply the Banach fixed point theorem. Set

$$M := \sup_{t \in [t_0, t_2]} |f(t, 0)|, \quad \Lambda := \frac{(2 + T)(t_2 - t_0)^{\alpha}}{\Gamma(\alpha + 1)},$$

and choose r satisfying

$$r \geq \frac{\Lambda M + T|x_1| + |x_2|}{1 - \Lambda L} > 0.$$

Define the operator $T : C([t_0, t_2], \mathbb{R}) \to C([t_0, t_2], \mathbb{R})$ by

$$\begin{aligned}(Tx)(t) =& I^{\alpha}_{t_0} f(t, x(t)) - \frac{t - t_1}{t_2 - t_1} I^{\alpha}_{t_0} f(t_2, x(t_2)) \\ & - \frac{t_2 - t}{t_2 - t_1} I^{\alpha}_{t_0} f(t_1, x(t_1)) + x_1 \left(\frac{t_2 - t}{t_2 - t_1}\right) + x_2 \left(\frac{t - t_1}{t_2 - t_1}\right).\end{aligned}$$

First, we show that $TB_r \subset B_r$ for $B_r = \{x \in C[t_0, t_2] : ||x|| \leq r\}$. Let $x \in B_r$. Then,

$$\begin{aligned}||Tx|| =& \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f(s, x(s)) ds - \frac{t - t_1}{\Gamma(\alpha)(t_2 - t_1)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} f(s, x(s)) ds \right. \\ & \left. - \frac{t_2 - t}{\Gamma(\alpha)(t_2 - t_1)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha - 1} f(s, x(s)) ds + x_1 \left(\frac{t_2 - t}{t_2 - t_1}\right) + x_2 \left(\frac{t - t_1}{t_2 - t_1}\right) \right| \\ \leq& \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s))| ds + \frac{t_2 - t_1}{(t_2 - t_1)\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s))| ds \\ & + \frac{T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s))| ds + T|x_1| + |x_2| \\ =& \frac{2 + T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s)) - f(s, 0) + f(s, 0)| ds + T|x_1| + |x_2| \\ \leq& \frac{2 + T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s)) - f(s, 0)| ds + \frac{2 + T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, 0)| ds \\ & + T|x_1| + |x_2|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2+T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} L|x(s)-0|ds + \frac{2+T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} M ds + T|x_1| + |x_2| \\
&\leq \frac{2+T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} L||x||ds + \frac{2+T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} M ds + T|x_1| + |x_2| \\
&= \frac{(2+T)(L||x||+M)}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} ds + T|x_1| + |x_2| \\
&= \frac{(2+T)(t_2-t_0)^{\alpha}}{\Gamma(\alpha+1)} (L||x||+M) + T|x_1| + |x_2| \\
&< \Lambda(Lr+M) + T|x_1| + |x_2| \leq r.
\end{aligned}
$$

Therefore, $TB_r \subset B_r$.

Now, we show that T is a contraction. For any $x, y \in B_r$ and for each $t \in [t_0, t_2]$, we have

$$
\begin{aligned}
||Tx-Ty|| = &\Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))] ds \\
&- \frac{t-t_1}{(t_2-t_1)\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))] ds \\
&- \frac{t_2-t}{(t_2-t_1)\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))] ds \Bigg| \\
\leq &\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds \\
&+ \frac{t_2-t_1}{(t_2-t_1)\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds \\
&+ \frac{t_2-t_0}{(t_2-t_1)\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds \\
\leq &\frac{2+T}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds \\
\leq &\frac{2+T}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} L|x(s)-y(s)| ds \\
\leq &\frac{(2+T)L||x-y||}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} ds \\
\leq &\frac{(2+T)L(t_2-t_0)^{\alpha}}{\Gamma(\alpha+1)} ||x-y|| \\
= &L\Lambda||x-y|| < ||x-y||.
\end{aligned}
$$

Thus, T is a contraction. Therefore, there is a unique fixed point which is a solution to the Caputo fractional boundary value problem (3.1), (3.2). □

Remark 3.1. Assumption (A2) is not sharp and could be improved.

Using assumption (A2) in Theorem 3.1 to find values t_0, t_1, and t_2 looks daunting at first. So, we present the following to make it easier to see sufficient choices.

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Choose $0 < B < 1$, $\alpha \in (1, 2]$, and $t_0 \in \mathbb{R}$. Assume f is continuous with Lipschitz constant L.

To satisfy (A2), choose

$$t_2 < t_0 + \left[\frac{B\Gamma(\alpha+1)}{2L}\right]^{1/\alpha}$$

and

$$t_1 < t_2 - \frac{B(t_2 - t_0)}{2(1-B)}.$$

Note that you could choose $t_0 \leq t_1 < t_2$ in $\mathbb{R}$ and then, seek a function with a suitable Lipschitz constant L. This is easily seen when solving for L in assumption (A2).

Remark 3.2. What we presented thus far in this section corresponds to $t_0 \leq t_1$. However, it is worth noting that the results simplify if you do impose equality. Having $t_0 = t_1$ is a typical scenario in a Caputo fractional boundary value problem. One may want the fractional integral to begin at the left most boundary condition. We present two corollaries to show this simplification.

Corollary 3.1. *Let $g : (a, b) \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} g(s)ds \\ &- \frac{t-t_1}{\Gamma(\alpha)(t_2-t_1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(s)ds + x_1 \left(1 - \frac{t-t_1}{t_2-t_1}\right) + x_2 \left(\frac{t-t_1}{t_2-t_1}\right) \end{aligned}$$

if and only if x is a solution of the fractional boundary value problem

$$D^{\alpha}_{*t_1} x(t) = g(t), \quad a < t_1 < t < b,$$

$$x(t_1) = x_1 \quad x(t_2) = x_2.$$

Corollary 3.2. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

(A1) $|f(t, u) - f(t, v)| \leq L|u - v|$ *for all* $t \in [t_1, t_2]$ *and* $u, v \in \mathbb{R}$ *and*

(A2) $\dfrac{2L(t_2 - t_1)^{\alpha}}{\Gamma(\alpha + 1)} < 1$

where L is a Lipschitz constant. Then,

$$D^{\alpha}_{*t_1} x(t) = f(t, x(t)),$$

$$x(t_1) = x_1 \quad x(t_2) = x_2.$$

has a unique solution on $[t_1, t_2]$.

Remark 3.3. The proofs of these two corollaries follow the same style as that of the more generalized theorems. Of note, when $t_0 = t_1$, the constant $T = \frac{t_2 - t_0}{t_2 - t_1} = 1$. By Theorem 3.1, one may then expect condition (A2) of the last corollary to have $2 + T = 2 + 1 = 3$. However, it clearly does not. So, condition (A2) of this last corollary is indeed sharper.

4. Main Result

We establish and prove that under assumptions (1)-(3) that solutions of (1.1), (1.3) depend continuously upon the boundary conditions.

Let $[c,d] \subset \mathbb{R}$ and for $x \in C[c,d]$, define

$$\|x\|_{0,[c,d]} = \max_{t\in[c,d]} |x(t)|.$$

If $k \in \mathbb{N}$, for $x \in C^k[c,d]$, define

$$\|x\|_{k,[c,d]} = \max\{\|x\|_{0,[c,d]}, \|x'\|_{0,[c,d]}, \ldots, \|x^{(k)}\|_{0,[c,d]}\}.$$

The following result from [10], Corollary 2.4, establishes the continuous dependence on initial values for initial value problems.

Lemma 4.1. *Assume $E \subset \mathbb{R}\times\mathbb{R}^n$ is open, connected, and convex, and let $f : E \to \mathbb{R}$ be continuous. Assume that solutions of initial value problems for* (1.1) *with initial conditions in E are unique. Given any $(t_0, z_0, z_1, \ldots, z_{n-1}) \in E$, let $x(t; t_0, \mathbf{z})$ denote the solution of* (1.1) *with initial conditions $x^{(i)}(t_0) = z_i$, $i = 0, 1, \ldots, n-1$ and maximal interval of existence $[t_0, \omega_{\mathbf{z}})$. Then, for all $\epsilon > 0$ and each compact $[c,d] \subset [t_0, \omega_{\mathbf{z}})$, there exists a $\delta > 0$ such that if $(t_0, v_0, v_1, \ldots, v_{n-1}) \in E$ and $\max\limits_{i=0,1,\ldots,n-1} |z_i - v_i| < \delta$, then $[c,d] \subset [t_0, \omega_{\mathbf{v}})$, the right maximal interval of existence of the solution $x(t; t_0, \mathbf{v})$, and*

$$\|x(t; t_0, \mathbf{z}) - x(t; t_0, \mathbf{v})\|_{n-1,[c,d]} < \epsilon.$$

We now present the first of two main results. This result establishes the continuous dependence on boundary conditions whenever the Caputo fractional operator begins left of the first boundary condition, i.e. $t_0 < t_1$.

Theorem 4.1 (Case when $t_0 < t_1$)**.** *Assume that hypotheses* (1)-(3) *hold. Let $x(t)$ be a solution of* (1.1) *on $[t_0, b)$, $[c,d] \subset [t_0, b)$ with points $c < t_1 < t_2 < \ldots < t_n < d$, and $\epsilon > 0$. Then, there exists a $\delta(\epsilon, [c,d]) > 0$ such that, if for $i = 1, 2 \ldots, n$, $|t_i - \tau_i| < \delta$ with $c < \tau_1 < \tau_2 < \tau_3 < \ldots, \tau_n < d$ and $|x(t_i) - y_i| < \delta$ with $y_i \in \mathbb{R}$, then there exists a solution $y(t)$ of* (1.1) *satisfying $y(\tau_i) = y_i$. Also,*

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$

Proof. Define the set

$$G = \{(t_1, t_2, \ldots, t_n, C_0, C_1, \ldots, C_{n-1}) | c < t_1 < t_2 < \cdots < t_n < b \text{ and } C_0, C_1, \ldots, C_{n-1} \in \mathbb{R}\}.$$

The set G is an open subset of $\mathbb{R}^{2n}$. Next, define a mapping $\phi : G \to \mathbb{R}^{2n}$ by

$$\phi(t_1, t_2, \ldots, t_n, C_0, C_1, \ldots, C_{n-1}) = (t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \ldots, u(t_n)),$$

where $u(t)$ is the solution of (1.1) satisfying the initial conditions $u(t_0) = C_0$, $u'(t_0) = C_1$, $\ldots, u^{(n-1)}(t_0) = C_{n-1}$. Assumptions (1)-(2) and Lemma 4.1 imply the continuity of solutions of initial value problems for (1.1) with respect to initial conditions. Thus, ϕ is continuous. Moreover, assumption (3) implies that ϕ is one-to-one. It follows from the Brouwer Invariance of Domain Theorem, page 199 in [22], that $\phi(G)$ is an open subset of $\mathbb{R}^{2n}$ and that ϕ is a homeomorphism from G to $\phi(G)$. Thus, we have the result since ϕ^{-1} is continuous and the fact that $\phi(G)$ is open. $\square$

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As mentioned before, it is often the case the we want the Caputo fractional derivative and left most boundary condition to start at the same value, i.e. $t_0 = t_1$. To that end, we present Corollary 2.3 from [10] to establish continuous dependence upon initial points and initial values. The added hypothesis is namely that there must exist a sequence of continuous functions f_k that converge uniformly to f.

Lemma 4.2. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$ is open, connected, and convex, and let $f_k : E \to \mathbb{R}$ denote a sequence of functions that converge uniformly to a function f on every compact subset of E. Assume $(t_0, c_0, c_1, \ldots, c_{n-1}) \in E$ and for each $k \geq 1$, consider an initial value problem*

$$D^{\alpha}_{*t_0^k} x(t) = f_k(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad a < t_0^k < t < \omega^k,$$

$$x^{(i)}(t_0^k) = c_i^k, \quad i = 0, 1, \ldots, n-1,$$

and let $x^k(t)$ denote the solution on right maximal interval I^k. Further, assume t_0^k is an increasing sequence and $t_0^k \uparrow t_0^-$ and $(t_0^k, c_0^k, \ldots, c_{n-1}^k) \to (t_0, c_0, \ldots, c_{n-1})$ as $k \to \infty$. Assume the solution $x(t)$ of (1.1), (1.2) is unique with maximal interval of existence $[t_0, \omega)$. Let $[c, d] \subset [t_0, \omega)$. Then, there exists K such that if $k \geq K$, then $[c, d] \subset I^k$, and

$$||x^k(t) - x(t)||_{n-1,[c,d]} \to 0 \text{ as } k \to \infty.$$

Now, we present the second main result establishing the continuous dependence of boundary conditions whenever the Caputo fractional operator begins at the first boundary condition, i.e. $t_0 = t_1$.

Theorem 4.2 (Case when $t_0 = t_1$)**.** *Assume that hypotheses (1)-(3) hold. Let $x(t)$ be a solution of (1.1) on $[t_1, b)$, $[c, d] \subset [t_1, b)$ with points $t_1 = c < t_2 < \ldots < t_n < d$, and $\epsilon > 0$. Then, there exists a $\delta(\epsilon, [c, d]) > 0$ such that, if for $i = 2, 3, \ldots, n$, $|t_i - \tau_i| < \delta$ with $c < \tau_2 < \tau_3 < \ldots, \tau_n < d$ and for $i = 1, 2, \ldots, n$, $|x(t_i) - y_i| < \delta$ with $y_i \in \mathbb{R}$, then there exists a solution $y(t)$ of (1.1) satisfying $y(t_1) = y_1$ and for $i = 2, 3, \ldots, n$, $y(\tau_i) = y_i$. Also,*

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$

Additionally, if $f_k : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is a sequence of continuous functions that converge uniformly to f on compact subsets of $[c, d] \times \mathbb{R}^n$ and for $k \geq 1$, t_1^k is an increasing sequence such that $t_1^k \uparrow t_1^-$ as $k \to \infty$, then there exists a K such that if $k \geq K$, then

$$||x_k(t) - x(t)||_{n-1,[c,d]} \to 0 \quad as \quad k \to \infty.$$

The proof of the first part is identical except for the omission of t_1 in the definition of the set G as this is now fixed due to the operator and that $t_0 = t_1$. Next, with the added condition, we may apply Lemma 4.2 to establish the continuous dependence of $t_0 = t_1$ from the left as well.

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