Multiplicity of homoclinic solutions for fractional discrete Laplacian equation

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Abstract: In this paper, the existence and multiplicity of homoclinic solutions are considered for the following equation:

$$(-\Delta_1)^s u(n) + V(n)u(n) = f(n, u(n)) \quad \text{for} \quad n \in \mathbb{Z},$$

where $(-\Delta_1)^s$ denotes the fractional discrete Laplacian, the sequence $V(n)$ is the potential, and $f(n,u)$ is a sequence of functions. Under some conditions on V and f, the existence of ground state sign-changing homoclinic solution u_1 and ground state homoclinic solution u_0 are obtained. Moreover, it is proved that the energy of u_1 is more than twice of the energy of u_0 and so that $u_0 \neq u_1$. We also study the multiplicity of solutions in case of concave-convex nonlinearity. To the best of our knowledge, this is the first attempt in the literature on the multiplicity of homoclinic solutions for fractional discrete Schrödinger type equation. In addition, we also can improve the known results of the corresponding continuous nonlinear Schrödinger equation.

Keywords: Fractional discrete Laplacian, Sign-changing solution, Nonnegative solution, Variational method.

1 Introduction and main results

In this paper, we are interested in the existence, energy property of the ground state homoclinic solutions of the following discrete fractional Schrödinger type equation:

$$(-\Delta_1)^s u(n) + V(n)u(n) = f(n, u(n)) \quad \text{for} \quad n \in \mathbb{Z}, \tag{1.1}$$

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where $(-\Delta_1)^s$ denotes the fractional discrete Laplacian as a special case of $(-\Delta_h)^s$ with $h = 1$, which is defined by

$$(-\Delta_h)^s u(hn) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_h} u(hn) - u(hn)) \frac{dt}{t^{1+s}}$$

and

$$(-\Delta_h) u(nh) = -\frac{1}{h^2}(u((n+1)h) - 2u(nh) + u((n-1)h)).$$

In above formulation, $v(t, nh) = e^{t\Delta_h} u(nh)$ is the flow of the following semidiscrete heat equation

$$\begin{cases} \partial_t v(t, hn) = \Delta_h v(t, hn), & \text{in} \quad \mathbb{Z}_h \times (0, \infty), \\ v(0, hn) = u(hn), & \text{on} \quad \mathbb{Z}_h, \end{cases}$$

where $\mathbb{Z}_h = \{hn : n \in \mathbb{Z}\}$.

A solution u of the equation (1.1) is homoclinic if $\lim_{n \to \pm\infty} u(n) = 0$.

In our paper, the discrete fractional Laplacian operator $(-\Delta_1)^s$ can be defined in a simpler way as follows, for all $n \in \mathbb{Z}$ and $0 < s < 1$,

$$(-\Delta_1)^s u(n) = \sum_{m \in \mathbb{Z}, m \neq n} (u(n) - u(m)) K_s(n - m),$$

where the discrete kernel $K_s(m)$ is given by

$$K_s(m) = \frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} |\Gamma(-s)|} \cdot \frac{\Gamma(|m| - s)}{\Gamma(|m| + 1 + s)},$$

for any $m \in \mathbb{Z} \setminus \{0\}$ and $K_s(0) = 0$. We recall that $K_s(m)$ possesses a delay property, that is, for $0 < s < 1$, there exist two constants $0 < d_s \leq D_s$ such that for any $m \in \mathbb{Z} \setminus \{0\}$, one has

$$\frac{d_s}{|m|^{1+2s}} \leq K_s(m) \leq \frac{D_s}{|m|^{1+2s}}.$$

The delay property can be referred to Theorem 1.1 [8].

The Schrödinger type equation has extensive research background, for examples, it can be used to describe an electron in a planetary system or an electromagnetic field. For more mathematical and physical background on Schrödinger type problems, we refer the readers to [2, 3, 4, 24] and the references therein. After the pioneering work of Lions [14, 15], many interesting results have been obtained for the Schrödinger type equation. Similar problems have been considered over the past few decades, see for examples, [1, 16, 23] and so on. Discrete nonlinear Schrödinger equations are very important nonlinear lattice models in condensed matter physics and biology. A central problem for discrete nonlinear Schrödinger equations is the existence of gap solitons (see [5, 10, 11, 17, 19, 22] and its references). The gap solitons in the discrete nonlinear Schrödinger equations is an isolated standing wave with time frequency in a continuous spectrum gap, which decays to zero at infinity. The main methods to establish the existence of gap solitons include centre manifold reduction and variational methods (see [22]). The gap solitons when optical pulses propagate in a saturated nonlinear medium can be simulated by using the discrete nonlinear Schrödinger equations with a unbounded potential (see for examples, [7, 26, 28]). The $-\Delta + V$ form of discrete Schrödinger operator is widely used in the description of random walks, wave propagation in crystals, nonlinear integrable lattice theory and other fields, for examples, we refer the readers to [25, 27] and its literature. On the other hand, there is a fundamental problem of approximating the continuous fractional Laplace problem in terms of discrete fractional Laplace problem. In recent years, more and more attention has been paid to the fractional Laplace function and its related problems. Fractional Laplace operators appear in the fields as diverse as anomalous diffusion, finance and optimization. We refer the readers to the references [20, 21], which studied a fractional version of the discrete time related nonlinear Schrödinger equation, the low order nonlinear modes and their stability of the system as discrete Laplacian fractional exponential function are numerically calculated.

Usually, one may attempt to solve this kind problem through either variational approaches or topological methods. The existence of nontrivial solutions of nonlinear Schrödinger equation with a compulsive potential $V(n)$ has been studied in the literature. The unbounded potential $V(n)$ guarantees a compact embedding from some subspace of l^2 to l^p for $p \geq 2$, which allows us to deal with the lack of compactness of PS sequences. Zhang and Pankov in [31] have studied the existence of nontrivial solutions for nonlinear Schrödinger equation with unbounded potential $V(n)$, the authors adopted the Nehari manifold minimization method. On the other hand, for discrete fractional case, there have been few results. In [13], the authors considered the multiplicity of homoclinic solutions for a discrete fractional difference equation:

$$(-\Delta_h)^s_p u(hn)+V(n)|u(hn)|^{p-2}u(hn) = \lambda a(n)|u(hn)|^{q-2}u(n)+b(n)|u(hn)|^{r-2}u(hn), \tag{1.2}$$

where $\lambda > 0$ is a parameter, $s \in (0, 1)$, $1 < q < p$, $a(n) \in l^{\frac{p}{p-q}}$, $b(n) \in l^{\infty}$, $(-\Delta_h)^s_p$ is the discrete fractional p-Laplacian operator with $h > 0$ and $V(n)$ is the positive potential, $f(u(hn)) = \lambda a(n)|u(hn)|^{q-2}u(hn) + b(n)|u(hn)|^{r-2}u(hn)$ is a nonlinear function combining convex nonlinearity with concave nonlinearity. Under some appropriate conditions, they showed that the problem possesses two nontrivial and nonnegative homoclinic solutions by using constraint variational method.

Motivated by the works of [8, 16], we first study the homoclinic solutions of the following equation:

$$(-\Delta_1)^s u(n) + V(n)u(n) = f(n, u(n)), \quad n \in \mathbb{Z}. \tag{1.3}$$

For V, we have the following hypotheses:

(V) $V(n) \geq V_0$ for all $n \in \mathbb{Z}$ with constant $V_0 > 0$ and $\lim_{|n| \to \infty} V(n) = +\infty$.

(V') $V(n) \geq 1$ for all $n \in \mathbb{Z}$ and $\sum_{n \in \mathbb{Z}} V^{-1}(n) < +\infty$.

As for $f(n,t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, we assume the following hypotheses:

(f_1) $f(n,t)$ is continuous differentiable with respect to t for every $n \in \mathbb{Z}$; $f(n,t)t > 0$, $t \neq 0$;

(f_2) For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $f(n,t)t \leq \varepsilon t^2 + C_\varepsilon |t|^p$, *for all* $n \in \mathbb{Z}$ and $2 < p < +\infty$;

(f_3) $\lim_{t \to \infty} \frac{F(n,t)}{t^2} = +\infty$, where $F(n,t) = \int_0^t f(n,s)ds$;

(f_4) $\frac{f(n,t)}{|t|}$ is a strictly increasing function of $t \in \mathbb{R} \backslash \{0\}$ for every $n \in \mathbb{Z}$.

For equation (1.3), we have give the following three results.

Theorem 1.1. *Suppose that conditions (V), and $(f_1) - (f_4)$ are satisfied. Then the equation (1.1) has a ground state sign-changing homoclinic solution u_1.*

Remark 1.1. *As far as we know, Theorem 1.1 seems to be a new result of fractional discrete Schrödinger equation in sign-changing solutions. We would also like to point out that although the proof idea of Theorem 1.1 is inspired by the reference [9], the proof of Theorem 1.1 is not trivial at all due to the appearance of non-local discrete fractional Laplacian.*

Theorem 1.2. *Suppose the assumption of Theorem 1.1 holds, then we have*

$$I(u_1) > 2I(u_0),$$

where u_1 is the ground state sign-changing homoclinic solution obtained in Theorem 1.1, u_0 is the ground state homoclinic solution and the energy functional $I(*)$ is defined in Section 2 (2.1).

Theorem 1.2 indicates that the sign-changing solution is never the ground state solution of the equation (1.1).

Theorem 1.3. *Suppose that conditions* (V'), *and* (f_1)- (f_2) *are satisfied. Furthermore, we suppose that* $f(n,t)$ *satisfies the following (AR) condition: there exists* $\mu > 2$ *and* $R > 0$ *such that*

$$f(n,t)t \geq \mu F(n,t) > 0, \quad \forall\, n \in \mathbb{Z}, \quad |t| \geq R.$$

Then the equation (1.1) *has a mountain pass type homoclinic solution* u_2.

It is not clear whether $u_2 \neq u_i$ for $i = 0, 1$. For the equation with the presentation as in (1.2), we have the following result for the case of $a(n) \equiv a$, $b(n) \equiv b$ and $p = 2$ with $a > 0$, $b > 0$ which is not included in the consideration of [13]. For simplicity we take $a = 1$ and $b = 1$ without loss of generality.

Theorem 1.4. *Suppose that* $V(n)$ *satisfies the condition* (V), *then there exists two nonnegative homoclinic solutions for the following equation*

$$(-\Delta_1)^s u(n) + V(n)u(n) = \lambda |u(n)|^{q-2}u(n) + |u(n)|^{p-2}u(n) \quad \text{for} \quad n \in \mathbb{Z}, \tag{1.4}$$

where $\lambda \in (0, \Lambda_0)$, Λ_0 *is a positive constant,* $1 < q < 2 < p$, p *is defined in the assumption* (f_2).

Remark 1.2. *Comparing with [13], the term* $\lambda|u(n)|^{q-2}u(n)$ *is not a special case of* $\lambda a(n)|u(n)|^{q-2}u(n)$, *since in [13] it is required that* $a(n) \in l^{\frac{p}{p-q}}$. *On the other hand, in view of* $b(n) \equiv b$, *we can take* $b(n) = 1$, *in the sense of a certain re-scaling.*

To the best of our knowledge, there is little theoretical research on the discrete fractional Laplacian equations except [13, 30], which obtained the homoclinic or positive solutions, our paper seems to be the first work which obtained the sign-changing solutions for discrete fractional Laplacian equations by the sign-changing Nehari set. We also mention that our results are new even for $s = 1$.

The rest of this paper proceeds as follows. Section 2 is devoted to the variational setting and we will prove four lemmas, which will be used in the proofs of our main results. In Section 3, we prove Theorem 1.1-1.4.

2 The variational framework and preliminary results

Firstly, we introduce the Banach space $l^p(\mathbb{Z})$ as follows:

$$l^p(\mathbb{Z}) := \{u(n) : \mathbb{Z} \to \mathbb{R}, \sum_{n \in \mathbb{Z}} |u(n)|^p < +\infty\},$$

with the norm $\|u\|_p = (\sum_{n\in\mathbb{Z}} |u(n)|^p)^{1/p}$. $l^\infty(\mathbb{Z})$ is the set $\{u(n) : \mathbb{Z} \to \mathbb{R}\}$ such that $\|u\|_\infty := \sup_{n\in\mathbb{Z}} |u(n)| < +\infty$. Then ,we have $l^p(\mathbb{Z}) \subset l^q(\mathbb{Z})$ when $1 \le p \le q \le +\infty$ (see [12]).

Let

$$E := \{u \in l^2(\mathbb{Z}) | \|u\| < +\infty\},$$

where $\|\cdot\|$ is defined by $\|u\| = (u,u)^{\frac{1}{2}}$ with

$$(u,v) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (u(n)-u(m))(v(n)-v(m))K_s(n-m) + \sum_{n\in\mathbb{Z}} V(n)u(n)v(n).$$

Then, E is a Hilbert space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. In [30], it was showed that under the condition (V),

$$\|u\|_V = \left(\sum_{n\in\mathbb{Z}} V(n)|u(n)|^2\right)^{\frac{1}{2}}$$

is an equivalent norm of E. According to Lemma 2.1 in [30], we know if $u \in l^2(\mathbb{Z})$, we have

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(n)-u(m)|^2 K_s(n-m) \le C_2\|u\|_2 < +\infty.$$

Then the embedding $E \hookrightarrow l^q(\mathbb{Z})(2 \le q \le +\infty)$ is continuous. That is, there exists a constant C_q such that $\|u\|_q \le C_q\|u\|$ for any $u \in E$, $q \in [2,+\infty]$. Furthermore, according to Lemma 2.3 in [13], we know $E \hookrightarrow l^p(\mathbb{Z})$ is compact for $p \in [2,+\infty)$ under the assumption (V).

The solution of Eq.(1.1) is the critical points of the functional given by

$$I(u) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(n)-u(m)|^2 K_s(n-m) + \frac{1}{2}\sum_{n\in\mathbb{Z}} V(n)|u(n)|^2 - \sum_{n\in\mathbb{Z}} F(n,u(n)), \tag{2.1}$$

where $F(n,u) = \int_0^u f(n,t)dt$. Moreover, under our condition (f_1), I belongs to C^1, so the Fréchet derivative of I is

$$\begin{aligned}\langle I'(u),v\rangle = &\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (u(n)-u(m))(v(n)-v(m))K_s(n-m) \\ &+ \sum_{n\in\mathbb{Z}} V(n)u(n)v(n) - \sum_{n\in\mathbb{Z}} f(n,u(n))v(n),\end{aligned} \tag{2.2}$$

for any $u,v \in E$ (see Lemma 2.5 and Lemma 2.6 in [30]). Furthermore, if $u \in E$ is a solution of (1.1) and $u^\pm \ne 0$, then u is a sign-changing solution of (1.1), where

$$u^+(n) = \max\{u(n),0\}, \quad u^-(n) = \min\{u(n),0\}.$$

Taking $u(n)=u^+(n)+u^-(n)$, the sign-changing solution of Eq.(1.1), into (2.1) and (2.2), we have

$$I(u)=I(u^+(n))+I(u^-(n))+\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m)-u^+(n)u^-(m))K_s(n-m), \tag{2.3}$$

$$\begin{aligned}\langle I'(u),u^+(n)\rangle&=\langle I'(u^+(n)),u^+(n)\rangle\\&+\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m)-u^+(n)u^-(m))K_s(n-m),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\langle I'(u),u^-(n)\rangle&=\langle I'(u^-(n)),u^-(n)\rangle\\&+\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m)-u^+(n)u^-(m))K_s(n-m).\end{aligned} \tag{2.5}$$

Now we define

$$\mathcal{M}=\{u(n)\in E,u^{\pm}(n)\neq 0 \text{ and } \langle I'(u),u^+(n)\rangle=\langle I'(u),u^-(n)\rangle=0\}.$$

We expect to prove that the functional I has a minimum point on $\mathcal{M}$, and then prove that this minimum point is a sign-changing solution of (1.1).

In the beginning, we show that the set $\mathcal{M}\neq\emptyset$ and it has the Nehari manifold structure.

Lemma 2.1. *Assume that assumptions (V) and (f_1)-(f_4) hold, if $u\in E$ with $u^{\pm}\neq 0$, then there is a unique pair (θ_u,t_u) of positive numbers such that $\theta_u u^+(n)+t_u u^-(n)\in\mathcal{M}$.*

Proof. Fix $u=u^++u^-\in E$ with $u^{\pm}\neq 0$. From (f_2), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$f(n,u(n))u(n)\leq \varepsilon|u(n)|^2+C_\varepsilon|u(n)|^p, \quad \textit{for all } n\in\mathbb{Z}. \tag{2.6}$$

Then, for any positive parameters θ,t, by using the embedding $E\hookrightarrow l^q(\mathbb{Z})(2\leq q\leq+\infty)$, i.e. $\|u^+(n)\|_q^q\leq C_q\|u^+(n)\|^q$, we have that

$$\begin{aligned}\langle I'(\theta u^++tu^-),\theta u^+\rangle&\geq \theta^2\|u^+(n)\|^2-\varepsilon\theta^2\sum_{n\in\mathbb{Z}}|u^+(n)|^2-C_\varepsilon\theta^q\sum_{n\in\mathbb{Z}}|u^+(n)|^q\\&+\theta t\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m)-u^+(n)u^-(m))K_s(n-m)\\&\geq\theta^2\|u^+(n)\|^2-\varepsilon C_2\theta^2\|u^+(n)\|^2-C_\varepsilon C_q\theta^q\|u^+(n)\|^q\\&=(1-\varepsilon C_2)\theta^2\|u^+(n)\|^2-C_\varepsilon C_q\theta^q\|u^+(n)\|^q.\end{aligned}$$

Choosing $\varepsilon > 0$ such that $(1-\varepsilon C_2) > 0$. Since $q > 2$, we have that $\langle I'(\theta u^+ + tu^-), \theta u^+\rangle > 0$, for θ small enough and all $t > 0$.

Similarly, we obtain that $\langle I'(\theta u^+ + tu^-), tu^-\rangle > 0$, for t small enough and all $\theta \geq 0$.

Therefore, there is a $\delta_1 > 0$ such that

$$\langle I'(\delta_1 u^+ + tu^-), \delta_1 u^+\rangle > 0, \langle I'(\theta u^+ + \delta_1 u^-), \delta_1 u^-\rangle > 0, \text{ for all } \theta,\ t > 0. \quad (2.7)$$

On the other hand, the assumptions (f_1) - (f_4) imply that

$$F(n,t) \geq 0, n \in \mathbb{Z}. \quad (2.8)$$

By (f_4), we have

$$f(n,t)t - 2F(n,t) > 0, \quad \textit{for all } n \in \mathbb{Z},\ t \in \mathbb{R}\backslash\{0\}, \quad (2.9)$$

and $f(n,t)t - 2F(n,t)$ is an strictly increasing function with respect to t.

Therefore, choose $\theta = \delta_2 > \delta_1$, if $t \in [\delta_1, \delta_2]$ and δ_2 is large enough, by (2.8) and assumptions (f_3), the term $\sum\limits_{n\in\mathbb{Z}} f(n, \delta_2 u^+(n))\delta_2 u^+(n)$ is major when δ_2 is large, so we have

$$\langle I'(\delta_2 u^+ + tu^-), \delta_2 u^+\rangle = (\delta_2)^2\|u^+(n)\|^2 - \sum_{n\in\mathbb{Z}} f(n, \delta_2 u^+(n))\delta_2 u^+(n)$$
$$+ \delta_2 t \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m) < 0.$$

On the other hand, by direct computations, we have

$$\langle I'(\theta u^+ + tu^-), tu^-\rangle = t^2\|u^-(n)\|^2 - \sum_{n\in\mathbb{Z}} f(n, tu^-(n))tu^-(n)$$
$$+ \theta t \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m) < 0.$$

Hence, we deduce that

$$\langle I'(\delta_2 u^+ + tu^-), \delta_2 u^+\rangle < 0, \langle I'(\theta u^+ + \delta_2 u^-), \delta_2 u^-\rangle < 0, \text{ for all } \theta,\ t \in [\delta_1, \delta_2]. \quad (2.10)$$

From (2.7), (2.10), the assumptions of Miranda's Theorem in [18] are satisfied. Thus there is $(\theta_u, t_u) \in (0,\infty)\times(0,\infty)$ such that $\theta_u u^+ + t_u u^- \in \mathcal{M}$.

Now we turn to prove the pair (θ_u, t_u) is unique. If $u \in \mathcal{M}$, in view of the definition of $\mathcal{M}$, we have

$$\|u^{\pm}(n)\|^2 + \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)$$
$$= \sum_{n\in\mathbb{Z}} f(n, u^{\pm}(n))u^{\pm}(n). \tag{2.11}$$

We will show that the pair $(\theta_u, t_u) = (1, 1)$ is the unique pair such that $\theta_u u^+ + t_u u^- \in \mathcal{M}$. Let (θ_0, t_0) be a pair of numbers such that $\theta_0 u^+ + t_0 u^- \in \mathcal{M}$ with $0 < \theta_0 \le t_0$. We have

$$\theta_0^2\|u^+(n)\|^2 + \theta_0 t_0\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)$$
$$= \sum_{n\in\mathbb{Z}} f(n, \theta_0 u^+(n))\theta_0 u^+(n), \tag{2.12}$$

and

$$t_0^2\|u^-(n)\|^2 + \theta_0 t_0\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)$$
$$= \sum_{n\in\mathbb{Z}} f(n, t_0 u^-(n))t_0 u^-(n). \tag{2.13}$$

From (2.13), we deduce that

$$\|u^-(n)\|^2 + \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)$$
$$\ge \sum_{n\in\mathbb{Z}} \frac{f(n, t_0 u^-(n))}{t_0}u^-(n). \tag{2.14}$$

Combining (2.11) with (2.14), we obtain that

$$0 \ge \sum_{n\in\mathbb{Z}}[\frac{f(n, t_0 u^-(n))}{t_0 u^-(n)} - \frac{f(n, u^-(n))}{u^-(n)}](u^-(n))^2.$$

By using the assumption (f_4), we get $t_0 \le 1$. Analogously, from (2.11), (2.12) and $0 < \theta_0 \le t_0$, we have that

$$0 \le \sum_{n\in\mathbb{Z}}[\frac{f(n, \theta_0 u^+(n))}{\theta_0 u^+(n)} - \frac{f(n, u^+(n))}{u^+(n)}](u^+(n))^2.$$

By using the assumption (f_4), we get $\theta_0 \ge 1$. Consequently, $\theta_0 = t_0 = 1$.

In the case $u \notin \mathcal{M}$, we suppose that there are (θ_1, t_1), (θ_2, t_2) such that

$$u_1(n) = \theta_1 u^+(n) + t_1 u^-(n) \in \mathcal{M}, \quad u_2(n) = \theta_2 u^+(n) + t_2 u^-(n) \in \mathcal{M}.$$

Thus,

$$u_2(n) = (\frac{\theta_2}{\theta_1})\theta_1 u^+(n) + (\frac{t_2}{t_1})t_1 u^-(n) = (\frac{\theta_2}{\theta_1})u_1^+(n) + (\frac{t_2}{t_1})u_1^-(n) \in \mathcal{M}.$$

According to $u_1 \in \mathcal{M}$ and the previous case, we have

$$\frac{\theta_2}{\theta_1} = \frac{t_2}{t_1} = 1.$$

Hence, (θ_u, t_u) is the unique pair such that $\theta_u u^+ + t_u u^- \in \mathcal{M}$. □

With the same conditions as in Lemma 2.1, we have the following three results.

Lemma 2.2. *Fix $u \in E$ with $u^\pm \neq 0$ such that $\langle I'(u), u^\pm \rangle \leq 0$. Then, the unique pair (θ_u, t_u) in Lemma 2.1 satisfies $0 < \theta_u, t_u \leq 1$.*

Proof. Suppose $\theta_u \geq t_u > 0$, since $\theta_u u^+(n) + t_u u^-(n) \in \mathcal{M}$, then we have

$$\begin{aligned}
&\theta_u^2 \|u^+(n)\|^2 + \theta_u^2 (\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)) \\
&\geq \theta_u^2 \|u^+(n)\|^2 + \theta_u t_u (\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)) \\
&= \sum_{n\in\mathbb{Z}} f(n, \theta_u u^+(n))\theta_u u^+(n). \qquad (2.15)
\end{aligned}$$

On the other hand, the assumption $\langle I'(u), u^+ \rangle \leq 0$ gives that

$$\begin{aligned}
&\|u^+(n)\|^2 + (\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)) \\
&\leq \sum_{n\in\mathbb{Z}} f(n, u^+(n))u^+(n). \qquad (2.16)
\end{aligned}$$

Combining (2.15) with (2.16), we then get

$$0 \geq \sum_{n\in\mathbb{Z}} [\frac{f(n, \theta_u u^+(n))}{\theta_u u^+(n)} - \frac{f(n, u^+(n))}{u^+(n)}](u^+(n))^2 dx,$$

which implies $\theta_u \leq 1$, Therefore, we have $0 < \theta_u, t_u \leq 1$. □

Lemma 2.3. *For fixed $u \in E$ with $u^\pm \neq 0$, (θ_u, t_u) is the unique maximum point of the function $\varphi : (\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ defined as $\varphi(\theta, t) = I(\theta u^+(n) + t u^-(n))$.*

Proof. As proved in Lemma 2.1, (θ_u, t_u) is the unique critical point of φ in $(\mathbb{R}_+ \times \mathbb{R}_+)$, by the assumption (f_3), we obtain $\varphi(\theta, t) \to -\infty$ uniformly as $|(\theta, t)| \to \infty$. Next, we prove that the maximum point of $\varphi(\theta, t)$ cannot be achieved on the boundary of $(\mathbb{R}_+ \times \mathbb{R}_+)$. Indirectly, without loss of generality, we suppose that $(0, \bar{t})$ is a maximum point of $\varphi(\theta, t)$. But, we have

$$\begin{aligned}\varphi(\theta, \bar{t}) &= I(\theta u^+(n) + \bar{t}u^-(n)) = \theta^2 I(u^+(n)) + \bar{t}^2 I(u^-(n)) \\ &+\theta\bar{t}(\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)),\end{aligned}$$

since $\bar{t}(\sum\limits_{n\in\mathbb{Z}}\sum\limits_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m)) > 0$, it is an increasing function with respect to θ with $\theta > 0$ small enough, so we get a contradiction. So far, we have proved this Lemma. □

Next, we turn to prove that the following minimization

$$c_1 := \inf\{I(u) : u(n) \in \mathcal{M}\} \tag{2.17}$$

can be achieved by an element $u_1 \in \mathcal{M}$. This is an important point in proving the existence of sign-changing solutions of equation (1.1).

Lemma 2.4. *$c_1 > 0$ can be achieved by an element $u_1 \in \mathcal{M}$.*

Proof. For every $u \in \mathcal{M}$, we have $\langle I'(u), u\rangle = 0$. Then, for any $\epsilon > 0$, by (f_2), one gets

$$\begin{aligned}\|u\|^2 &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(u(n) - u(m))^2 K_s(n-m) + \sum_{n\in\mathbb{Z}} V(n)|u(n)|^2 \\ &= \sum_{n\in\mathbb{Z}} f(n, u)u \le \epsilon\sum_{n\in\mathbb{Z}}|u|^2 + C_\epsilon\sum_{n\in\mathbb{Z}}|u|^p \\ &\le \epsilon C_2\|u\|^2 + C_\epsilon C_p\|u\|^p.\end{aligned}$$

We may choose $\epsilon = \frac{1}{2C_2}$ such that

$$\|u\| \ge (\frac{1}{2C_\epsilon C_p})^{1/(p-2)} > 0. \tag{2.18}$$

In view of (f_1) and (f_2), there exists $T_0 > 0$, such that for all $n \in \mathbb{Z}$ and $|t| \le T_0$, we have $F(n, t) \le (1/4C_2^2)t^2$. Since $u = 1 \cdot u^+ + 1 \cdot u^- \in \mathcal{M}$, by Lemma 2.3, we have

$$\begin{aligned}I(u(n)) \ge I(\frac{T_0}{\|u\|}u(n)) &= \frac{T_0^2}{2} - \sum_{n\in\mathbb{Z}} F(n, \frac{T_0}{\|u\|}u(n)) \\ &\ge \frac{T_0^2}{2} - \frac{T_0^2}{4C_2^2\|u\|^2}\sum_{n\in\mathbb{Z}}|u(n)|^2 \ge \frac{T_0^2}{4},\end{aligned} \tag{2.19}$$

for all $u \in \mathcal{M}$ and so $c_1 \geq \frac{T_0^2}{4} > 0$ is well defined.

Let $\{\tilde{u}_k\} \subset \mathcal{M}$ such that $I(\tilde{u}_k) \to c_1$. Then in view of the definition of $\{\tilde{u}_k\}$, we are going to prove $\{\tilde{u}_k\}$ is bounded in E. Indirectly, we suppose that $\|\tilde{u}_k\| \to \infty$. We define $v_k = \tilde{u}_k/\|\tilde{u}_k\| \in E$, then we have $\|v_k\| = 1$. By Sobolev's embedding, we may suppose that $v_k \to \bar{v}$ in $l^p(\mathbb{Z})$ for $2 \leq s < +\infty$ in subsequence sense. If $\bar{v} = 0$, for fixed $T > [2(1+c_1)]^{1/2}$, by (f_2) and (f_3), there exists a constant $\widetilde{C}$ such that

$$\limsup_{k\to\infty} \sum_{n\in\mathbb{Z}} F(n, Tv_k) \leq \lim_{k\to\infty} T^2\|v_k\|_2^2 + \widetilde{C} \lim_{k\to\infty} \|v_k\|_p^p = 0. \tag{2.20}$$

Let $l_k = T/\|\tilde{u}_k\|$. In view of above inequality, we have

$$c_1 + o(1) = I(\tilde{u}_k) \geq I(l_k\tilde{u}_k) = \frac{l_k^2}{2}\|\tilde{u}_k\|^2 - \sum_{n\in\mathbb{Z}} F(n, Tv_k) = \frac{T^2}{2} + o(1) > c_1 + 1 + o(1),$$

which is a contradiction. Therefore we deduce that $v \neq 0$.

Now we have $\lim_{k\to\infty} |\tilde{u}_k(n)| = \infty$ for $n \in \{n \in \mathbb{Z} : \bar{v}(n) \neq 0\}$. In fact, otherwise, we have $\lim_{k\to\infty} |v_k| = 0$, which is a contradiction. By (f_3) and the definition of v_k, we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{c_1 + o(1)}{\|\tilde{u}_k\|^2} = \lim_{k\to\infty} \frac{I(\tilde{u}_k)}{\|\tilde{u}_k\|^2} \\
&= \lim_{k\to\infty} [\frac{1}{2}\|v_k\|^2 - \sum_{n\in\mathbb{Z}, \tilde{u}_k(n)\neq 0} \frac{F(n, \tilde{u}_k)}{\tilde{u}_k^2} v_k^2] \\
&\leq \frac{1}{2} - \sum_{n\in\mathbb{Z}, \bar{v}(n)\neq 0} \liminf_{k\to\infty} \frac{F(n, \tilde{u}_k)}{\tilde{u}_k^2} v_k^2 = -\infty.
\end{aligned}$$

This contradiction shows that $\{\tilde{u}_k\}$ is bounded in E. Hence there exists $u_1 \in E$ such that $\tilde{u}_k^{\pm} \rightharpoonup u_1^{\pm}$ in E. By assumption $\{\tilde{u}_k\} \subset \mathcal{M}$, we have $\langle I'(\tilde{u}_k), \tilde{u}_k^{\pm}\rangle = 0$, that is

$$\|\tilde{u}_k^{\pm}\|^2 + \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-\tilde{u}_k^-(n)\tilde{u}_k^+(m) - \tilde{u}_k^+(n)\tilde{u}_k^-(m))K_s(n-m) = \sum_{n\in\mathbb{Z}} f(n, \tilde{u}_k^{\pm})\tilde{u}_k^{\pm}. \tag{2.21}$$

From (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(n, u(n))u(n) \leq \varepsilon u(n)^2 + C_\varepsilon |u(n)|^p, \quad \textit{for all } n \in \mathbb{Z}.$$

Since $\tilde{u}_k \in \mathcal{M}$, by (2.18), there exists $\mu > 0$ such that

$$\mu \leq \|\tilde{u}_k^{\pm}\|^2 < \sum_{n\in\mathbb{Z}} f(n, \tilde{u}_k^{\pm})\tilde{u}_k^{\pm} \leq \varepsilon \sum_{n\in\mathbb{Z}} |\tilde{u}_k^{\pm}|^2 + C_\varepsilon \sum_{n\in\mathbb{Z}} |\tilde{u}_k^{\pm}|^p.$$

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Since $\{\tilde{u}_k\}$ is bounded in E, there exists a constant $C > 0$ such that

$$\mu \leq \varepsilon C + C_\varepsilon \sum_{n\in\mathbb{Z}} |\tilde{u}_k^{\pm}(n)|^p.$$

Choosing $\varepsilon = \frac{\mu}{2C}$,we get

$$\sum_{n\in\mathbb{Z}} |\tilde{u}_k^{\pm}(n)|^p \geq \frac{\mu}{2C_\varepsilon} > 0.$$

By the compactness of the embedding $E \hookrightarrow l^q(\mathbb{Z})$ for $2 \leq q < +\infty$, we get

$$\sum_{n\in\mathbb{Z}} |u_1^{\pm}(n)|^p \geq \frac{\mu}{2C_\varepsilon} > 0, \tag{2.22}$$

thus we have $u_1^{\pm} \neq 0$. Combining (f_2) with the compactness of $E \hookrightarrow l^q(\mathbb{Z})$ for $2 \leq q < +\infty$, we have

$$\lim_{k\to+\infty} \sum_{n\in\mathbb{Z}} f(n, \tilde{u}_k^{\pm})\tilde{u}_k^{\pm} = \sum_{n\in\mathbb{Z}} f(n, u_1^{\pm})u_1^{\pm}, \lim_{k\to+\infty} \sum_{n\in\mathbb{Z}} F(n, \tilde{u}_k^{\pm}) = \sum_{n\in\mathbb{Z}} F(n, u_1^{\pm}). \tag{2.23}$$

On the other hand, combining the weak semicontinuity of norm $\|u\|$ with

$$(-\tilde{u}_k^-(n)\tilde{u}_k^+(m) - \tilde{u}_k^+(n)\tilde{u}_k^-(m)) \geq 0,$$

by using compactness of the Sobolev embedding, we can obtain $\langle I'(u_1), u_1^{\pm}\rangle \leq 0$. From Lemma 2.2, there exists $(\theta_u, t_u) \in (0, 1] \times (0, 1]$ such that $\bar{u} := \theta_u u_1^+(n) + t_u u_1^-(n) \in \mathcal{M}$. By using (2.9), the fact $\langle I'(\bar{u}), \bar{u}\rangle = 0$, $\langle I'(\tilde{u}_k), \tilde{u}_k\rangle = 0$ and Lemma 2.3, we deduce that

$$\begin{aligned}
c_1 &\leq I(\bar{u}) - \frac{1}{2}\langle I'(\bar{u}), \bar{u}\rangle = \frac{1}{2}\sum_{n\in\mathbb{Z}} (f(n, \bar{u})\bar{u} - 2F(n, \bar{u})) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} (f(n, \theta_u u_1^+(n))\theta_u u_1^+(n) - 2F(n, \theta_u u_1^+(n))) \\
&+ \frac{1}{2}\sum_{n\in\mathbb{Z}} (f(n, t_u u_1^-(n))t_u u_1^-(n) - 2F(n, t_u u_1^-(n))) \\
&\leq \frac{1}{2}\sum_{n\in\mathbb{Z}} (f(n, u_1)u_1 - 2F(n, u_1)) = I(u_1) - \frac{1}{2}\langle I'(u_1), u_1\rangle \\
&\leq \liminf_{k\to\infty} [I(\tilde{u}_k) - \frac{1}{2}\langle I'(\tilde{u}_k), \tilde{u}_k\rangle] = c_1.
\end{aligned}$$

Consequently, $\bar{u} = u_1$ and $I(u_1) = c_1$. □

3 The proof of main results

3.1 Proof of Theorem 1.1.

To prove Theorem 1.1, we only need to prove the following result.

Lemma 3.1. *$u_1 \in \mathcal{M}$ in Lemma 2.4 is a critical point of the functional I in E.*

Proof. For simplicity, we omit the subscript to write $u_1 = u = u^+ + u^- \in \mathcal{M}$ and so $c_1 = I(u)$. We have $\langle I'(u), u^+(n)\rangle = \langle I'(u), u^-(n)\rangle = 0$. In view of Lemma 2.3, for $(\theta, t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, $(1,1)$ is the unique maximum point, we have

$$I(\theta u^+(n) + tu^-(n)) < I(u^+(n) + u^-(n)) = c_1. \tag{3.1}$$

If $I'(u) \neq 0$, there exists $\delta > 0$ and $\gamma > 0$ such that

$$\|I'(v)\| \geq \gamma, \text{ for all } \|v - u\| \leq 3\delta.$$

Since $u \in \mathcal{M}$ bounded in E, similar to the proof of (2.22), if $u \in \mathcal{M}$, we have $\|u^\pm\| > L$, for a constant $L > 0$, we can assume that $6\delta < L$. Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(\theta, t) = \theta u^+(n) + tu^-(n)$. It follows from Lemma 2.3 again that

$$\bar{c}_1 := \max_{\partial D} I \circ g < c_1. \tag{3.2}$$

We let $\varepsilon := \min\{(c_1 - \bar{c}_1)/2, \gamma\delta/8\}$ and $S_\delta := B(u, \delta)$, By using Lemma 2.3 in [29], there exists a deformation η having the following properties:

(a) $\eta(1, u) = u$ if $u \notin I^{-1}([c_1 - 2\varepsilon, c_1 + 2\varepsilon] \cap S_{2\delta})$;

(b) $\eta(1, I^{c_1+\varepsilon} \cap S_{2\delta}) \subset I^{c_1-\varepsilon}$;

(c) $I(\eta(1, u)) \leq I(u)$ for all $u \in E$.

By the above items (a) and (c), we have

$$\max_{(\theta,t)\in\bar{D}} I(\eta(1, g(\theta, t))) < c_1. \tag{3.3}$$

We now prove that $\eta(1, g(D)) \cap \mathcal{M} \neq \varnothing$, which is a contradiction to the definition of c_1. Let $\psi(\theta, t) := \eta(1, g(\theta, t))$ and

$$\Psi(\theta, t) := (\frac{1}{\theta}\langle I'(\psi(\theta, t)), (\psi(\theta, t))^+\rangle, \frac{1}{t}\langle I'(\psi(\theta, t)), (\psi(\theta, t))^-\rangle).$$

The claim holds if there exists $(\theta_0, t_0) \in D$ such that $\Psi(\theta_0, t_0) = (0, 0)$. By the definition of $\|\cdot\|$ and direct computation, we have

$$\begin{aligned}\|g(\theta, t) - u\|^2 &= \|(\theta - 1)u^+ + (t - 1)u^-\|^2 \\ &\geq |\theta - 1|^2\|u^+\|^2 \\ &> |\theta - 1|^2(6\delta)^2,\end{aligned}$$

and $|\theta-1|^2(6\delta)^2 > 4\delta^2 \Leftrightarrow \theta < 2/3$ or $\theta > 4/3$, using (a) and the range of θ, for $\theta = \frac{1}{2}$ and for every $t \in [\frac{1}{2}, \frac{3}{2}]$ we have $g(\frac{1}{2}, t) \notin S_{2\delta}$, so from (a), we have $\psi(\frac{1}{2}, t) = g(\frac{1}{2}, t)$. Thus

$$\Psi(\frac{1}{2}, t) = (2\langle I'(\frac{1}{2}u^+ + tu^-), \frac{1}{2}u^+\rangle, \frac{1}{t}\langle I'(\frac{1}{2}u^+ + tu^-), tu^-\rangle).$$

By direct computations, we know that

$$\begin{aligned}
&\langle I'(\frac{1}{2}u^+ + tu^-), \frac{1}{2}u^+\rangle = \langle I'(\frac{1}{2}u^+), \frac{1}{2}u^+\rangle \\
&+ \frac{t}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m) \geq \langle I'(\frac{1}{2}u), \frac{1}{2}u^+\rangle > 0,
\end{aligned}$$

from which we obtain

$$\langle I'(\frac{1}{2}u^+ + tu^-), \frac{1}{2}u^+\rangle > 0, \quad \text{for every } \ t \in [\frac{1}{2}, \frac{3}{2}]. \tag{3.4}$$

Similarly, for $\theta = \frac{3}{2}$ and for every $t \in [\frac{1}{2}, \frac{3}{2}]$ we have $\psi(\frac{3}{2}, t) = g(\frac{3}{2}, t)$, so that

$$\begin{aligned}
&\langle I'(\frac{3}{2}u^+ + tu^-), \frac{3}{2}u^+\rangle = \langle I'(\frac{3}{2}u^+), \frac{3}{2}u^+\rangle \\
&+ \frac{3t}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(-u^-(n)u^+(m) - u^+(n)u^-(m))K_s(n-m) \leq \langle I'(\frac{3}{2}u), \frac{3}{2}u^+\rangle < 0,
\end{aligned}$$

so that

$$\langle I'(\frac{3}{2}u^+ + tu^-), \frac{3}{2}u^+\rangle < 0, \quad \text{for every } \ t \in [\frac{1}{2}, \frac{3}{2}]. \tag{3.5}$$

Similarly we have,

$$\begin{aligned}
&\langle I'(\theta u^+ + \frac{1}{2}u^-), \frac{1}{2}u^-\rangle > 0, \quad \text{for every } \ \theta \in [\frac{1}{2}, \frac{3}{2}], \\
&\langle I'(\theta u^+ + \frac{3}{2}u^-), \frac{3}{2}u^-\rangle < 0, \quad \text{for every } \ \theta \in [\frac{1}{2}, \frac{3}{2}].
\end{aligned} \tag{3.6}$$

Since Ψ is continuous on D, according to (3.4)-(3.6), by Miranda's theorem (see [18]), we have $\Psi(\theta_0, t_0) = 0$ for some $(\theta_0, t_0) \in D$, so $\eta(1, g(\theta_0, t_0)) = \psi(\theta_0, t_0) \in \mathcal{M}$, which is contradicted to (3.3). From this, we can conclude that u is a critical point of I in E. □

By Lemma 3.1, we obtain a sign-changing solution of equation (1.1), since $u_1 \in E \subset l^2(\mathbb{Z})$. By the integrability of $u_1 \in l^2(\mathbb{Z})$, we obtain $\lim_{n\to\pm\infty} u_1(n) = 0$, so it is homoclinic.

3.2 Proof of Theorem 1.2.

Let $u_1 \in E$ be a sign-changing solution of (1.1), and denote by

$$\mathcal{N} := \{u \in E\backslash\{0\} : \langle I'(u), u\rangle = 0\}.$$

Similar to Lemma 2.4, we see that the following positive number

$$c_0 := \inf_{u\in\mathcal{N}} I(u) \tag{3.7}$$

is well defined. We will prove $c_1 > 2c_0$.

Proof of Theorem 1.2. As standard processes as in proving Lemma 2.1-Lemma 3.1 above, by using the Nehari fiber mapping skills, one can see that there exists $u_0 \in \mathcal{N}$ such that $I'(u_0) = 0$ in E and $I(u_0) = c_0$. That is, u_0 is a nonnegative ground state solution of equation (1.1).

Reviewing that $u_1 = u^+(n) + u^-(n)$ is a sign-changing solution obtained in Theorem 1.1. Similarly as the proof of Lemma 2.1, there is a unique positive number $\theta_{u^+(n)}$ such that

$$\theta_{u^+(n)}u^+(n) \in \mathcal{N}. \tag{3.8}$$

In view of (2.4), since $\langle I'(u_1), u^+(n)\rangle = 0$, we have

$$\langle I'(u^+(n)), u^+(n)\rangle < 0. \tag{3.9}$$

Using (3.8) and (3.9), similar to the proof of Lemma 2.2, we obtain

$$\theta_{u^+(n)} \in (0, 1).$$

Similarly, we can prove that there is an unique $\theta_{u^-(n)} \in (0, 1)$ such that

$$\theta_{u^-(n)}u^-(n) \in \mathcal{N}.$$

Finally, by direct calculation, since $u \in \mathcal{M}$, by Lemma 2.3, $(1, 1)$ is the unique maximum point of the function $\varphi(\theta, t) = I(\theta u^+(n) + tu^-(n))$, one gets

$$\begin{aligned} 2c_0 &\le I(\theta_{u^+(n)}u^+(n)) + I(\theta_{u^-(n)}u^-(n)) \\ &\le I(\theta_{u^+(n)}u^+(n) + \theta_{u^-(n)}u^-(n)) < I(u^+(n) + u^-(n)) = c_1. \end{aligned}$$

That is $I(u_1) > 2c_0$. The proof of Theorem 1.2 is complete.

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3.3 Proof of Theorem 1.3.

We are ready to prove Theorem 1.3.

Proof. We can easily check that the functional I satisfies the mountain pass geometric structure. In fact, $I(0)=0$. For any $w\in E\setminus\{0\}$, by (f_2) and Sobolev embedding, we have

$$\begin{aligned} I(w) &= \frac{1}{2}\|w\|^2 - \sum_{n\in\mathbb{Z}} F(n,w) \\ &\geq (\frac{1}{2}-\varepsilon C_2)\|w\|^2 - C_\varepsilon C_p\|w\|^p. \end{aligned}$$

Since $p>2$, we can choose ε small enough and small $\rho>0$ such that $I(w)\geq\alpha>0$ with $\|w\|=\rho$. On the other hand, for fixed $\varphi\in E\setminus\{0\}$, by (AR) condition, we know $I(t\varphi)=\frac{t^2}{2}\|\varphi\|^2-\sum\limits_{n\in\mathbb{Z}}F(n,t\varphi)\to-\infty$ as $t\to+\infty$. Hence, we deduce that there exists $e=t_0\varphi$ for t_0 large enough such that $\|e\|>\rho$ and $I(e)<0$.

By using the (AR) condition, we deduce that the $(PS)_{\tilde{c}}$ sequence $\{\tilde{u}_k\}\subset E$ (satisfying $I(\tilde{u}_k)\to\tilde{c}$ and $I'(\tilde{u}_k)\to 0$) is bounded in E and satisfies the $(PS)_{\tilde{c}}$ conditions. In fact, for k large enough, by using the (AR) condition, one has

$$\begin{aligned} C+o(1)\|\tilde{u}_k\| &\geq I(\tilde{u}_k)-\frac{1}{\mu}\langle I'(\tilde{u}_k),\tilde{u}_k\rangle \\ &\geq (\frac{1}{2}-\frac{1}{\mu})\|\tilde{u}_k\|^2 - \sum_{\{n\in\mathbb{Z}:|\tilde{u}_k(n)|\leq R\}}(F(n,\tilde{u}_k(n))-\frac{1}{\mu}f(n,\tilde{u}_k(n))\tilde{u}_k(n)). \end{aligned}$$

For the case $|\tilde{u}_k(n)|\leq R$, by choosing $\varepsilon=1$ in (f_2), there exists a constant $C>0$ such that $|f(n,u)|\leq C$ for $|u|\leq R$ and any $n\in\mathbb{Z}$, then we have $|F(n,u)-\frac{1}{\mu}f(n,u)u|\leq C|u|$. Using Hölder inequality and that for any $u\in E$, $\sum\limits_{n\in\mathbb{Z}}|u|\leq(\sum\limits_{n\in\mathbb{Z}}V^{-1}(n))^{\frac{1}{2}}\|u\|$, there holds

$$C+o(1)\|\tilde{u}_k\|\geq(\frac{1}{2}-\frac{1}{\mu})\|\tilde{u}_k\|^2-\sum_{\{n\in\mathbb{Z}:|\tilde{u}_k(n)|\leq R\}}C|\tilde{u}_k|\geq C_1(\|\tilde{u}_k\|^2-\|\tilde{u}_k\|),$$

we deduce that $\|\tilde{u}_k\|$ is bounded. Then there exists $u_2\in E$ such that $\tilde{u}_k\rightharpoonup u_2$ in E and $\tilde{u}_k\to u_2$ in $l^2(\mathbb{Z})$ in subsequence sense. Therefore, in view of $I'(\tilde{u}_k)\to 0$ and $\tilde{u}_k\rightharpoonup u_2$ in E, we have

$$\lim_{k\to\infty}\langle I'(\tilde{u}_k)-I'(u_2),\tilde{u}_k-u_2\rangle=0. \tag{3.10}$$

Using a similar discussion as proof of (2.23), one can deduce that

$$\lim_{k\to+\infty}\sum_{n\in\mathbb{Z}}(f(n,\tilde{u}_k)-f(n,u_2))(\tilde{u}_k-u_2)=0.$$

This fact together with (3.10) yields that $\|\tilde{u}_k-u_2\|\to 0$ as $k\to+\infty$. That is, $\tilde{u}_k\to u_2$ in E. Hence, we have proved the existence of mountain pass type solution u_2 for the equation (1.1). □

3.4 Proof of Theorem 1.4.

Now we consider the problem (1.4), the method used to prove existence of two nonnegative homoclinic solutions is similar to [6], with a little difference. We define the functional associated with equation (1.4) as follows:

$$\begin{aligned}J(u)&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(n)-u(m)|^2K_s(n-m)\\&+\frac{1}{2}\sum_{n\in\mathbb{Z}}V(n)|u(n)|^2-\sum_{n\in\mathbb{Z}}(\frac{\lambda}{q}|u(n)|^q+\frac{1}{p}|u(n)|^p),\end{aligned}\tag{3.11}$$

then, $J(u)\in C^1$ (see Lemma 2.5 and Lemma 2.6 in [13]) and the Fréchet derivative of J is

$$\begin{aligned}\langle J'(u),v\rangle&=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(u(n)-u(m))(v(n)-v(m))K_s(n-m)\\&+\sum_{n\in\mathbb{Z}}V(n)u(n)v(n)-\sum_{n\in\mathbb{Z}}(\lambda|u(n)|^{q-2}u(n)+|u(n)|^{p-2}u(n))v(n).\end{aligned}$$

We define the fibering map $\Phi_u(t):=J(tu)$, so

$$\begin{aligned}\Phi_u'(t)=J_t'(tu)&=t\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(n)-u(m)|^2K_s(n-m)\\&+t\sum_{n\in\mathbb{Z}}V(n)|u(n)|^2-\sum_{n\in\mathbb{Z}}(\lambda t^{q-1}|u(n)|^q+t^{p-1}|u(n)|^p),\end{aligned}$$

$$\begin{aligned}\Phi_u''(t)=J_t''(tu)&=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(n)-u(m)|^2K_s(n-m)+\sum_{n\in\mathbb{Z}}V(n)|u(n)|^2\\&-\sum_{n\in\mathbb{Z}}(\lambda(q-1)t^{q-2}|u(n)|^q+(p-1)t^{p-2}|u(n)|^p).\end{aligned}$$

Now define three subsets $\mathcal{N}^+$, $\mathcal{N}^-$ and $\mathcal{N}^0$ of $\mathcal{N}_J := \{u \in E\backslash\{0\} : \langle J'(u), u\rangle = 0\}$ respectively as follows:

$$\mathcal{N}^+ = \{u \in \mathcal{N}_J : \Phi_u''(1) > 0\},$$
$$\mathcal{N}^0 = \{u \in \mathcal{N}_J : \Phi_u''(1) = 0\},$$
$$\mathcal{N}^- = \{u \in \mathcal{N}_J : \Phi_u''(1) < 0\}.$$

In order to prove Theorem 1.4, we present two Lemmas, which are important materials used in proving Theorem 1.4 and different from the method in [6].

Lemma 3.2. *There exists $\Lambda_0 > 0$, such that for any $\lambda \in (0, \Lambda_0)$, $\mathcal{N}^0 = \emptyset$.*

Proof. Suppose that $\mathcal{N}^0 \neq \emptyset$ for any $\lambda > 0$. Therefore, for $u \in \mathcal{N}^0$, we have $\langle J'(u), u\rangle = 0$ and $\Phi_u''(1) = 0$. That is,

$$\|u\|^2 = \sum_{n\in\mathbb{Z}}(\lambda|u(n)|^q + |u(n)|^p),$$
$$\|u\|^2 = \sum_{n\in\mathbb{Z}}(\lambda(q-1)|u(n)|^q + (p-1)|u(n)|^p).$$

From the above equalities, we have

$$(2-q)\|u\|^2 = (p-q)\sum_{n\in\mathbb{Z}}|u(n)|^p,$$
$$(p-2)\|u\|^2 = \lambda(p-q)\sum_{n\in\mathbb{Z}}|u(n)|^q.$$

By the Sobolev embedding, we have

$$\|u\|^2 \leq C_p\frac{p-q}{2-q}\|u\|^p, \quad \|u\|^2 \leq \lambda C_q\frac{p-q}{p-2}\|u\|^q.$$

It follows that there exists two positive constants C' and C'' such that $C' \leq \|u\| \leq \lambda C''$. This is a contradiction if λ is sufficiently small. Hence, we deduce that there exists a positive constant Λ_0 such that $\mathcal{N}^0 = \emptyset$ for all $\lambda \in (0, \Lambda_0)$. □

Lemma 3.3. *If u is a local minimizer of J on $\mathcal{N}_J$ and $u \notin \mathcal{N}^0$, then $J'(u) = 0$.*

Proof. Suppose that u is a local minimizer of J on $\mathcal{N}_J$. By Lagrange multipliers, there exists $\mu \in \mathbb{R}$, such that $J'(u) = \mu K'(u)$, where $K(u)$ is defined by

$$K(u) = \|u\|^2 - \sum_{n\in\mathbb{Z}}(\lambda|u(n)|^q + |u(n)|^p).$$

Since $u \in \mathcal{N}_J$, we have $\mu\langle K'(u), u\rangle = 0$. The fact $u \notin \mathcal{N}^0$ implies that $\langle K'(u), u\rangle = \Phi_u''(1) \neq 0$. Therefore, $\mu = 0$. It follows that $J'(u) = 0$. □

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In order to understand the fibering map, we define the function π_u as follows:

$$\pi_u(t) = t^{2-q}\|u\|^2 - t^{p-q}\sum_{n\in\mathbb{Z}}|u(n)|^p.$$

We notice that $tu \in \mathcal{N}_J$ if and only if $\pi_u(t) = \lambda \sum_{n\in\mathbb{Z}}|u(n)|^q$, for $t > 0$. Moreover,

$$\pi'_u(t) = (2-q)t^{1-q}\|u\|^2 - (p-q)t^{p-q-1}\sum_{n\in\mathbb{Z}}|u(n)|^p. \tag{3.12}$$

By direct computations, we deduce that if $tu \in \mathcal{N}_J$, then

$$t^{q-1}\pi'_u(t) = \Phi''_u(t). \tag{3.13}$$

Therefore, $tu \in \mathcal{N}^+$ (or $tu \in \mathcal{N}^-$) if and only if $tu \in \mathcal{N}$ and $\pi'_u(t) > 0$ (or $\pi'_u(t) < 0$).

Fixed $u \in E$ and $u \neq 0$. By (3.12), $\pi_u(t)$ satisfies the following properties:

(a) $\pi_u(t)$ has a unique maximum at $t = t_{\max} = \left(\frac{(2-q)\|u\|^2}{(p-q)(\sum_{n\in\mathbb{Z}}|u(n)|^p)}\right)^{1/(p-2)}$;

(b) $\pi'_u(t) > 0$ on $(0, t_{\max})$ and $\pi'_u(t) < 0$ on $(t_{\max}, +\infty)$;

(c) $\pi_u(0) = 0$ and $\lim_{t\to+\infty} \pi_u(t) = -\infty$.

On the other hand, if $0 < \lambda \sum_{n\in\mathbb{Z}}|u(n)|^p < \Phi_u(t_{\max})$, then there exists t_1 and t_2 with $0 < t_1 < t_{\max} < t_2$ such that $\pi_u(t_1) = \pi_u(t_2) = \lambda \sum_{n\in\mathbb{Z}}|u(n)|^p$ and $\pi'_u(t_1) > 0$, $\pi'_u(t_2) < 0$. (3.12) implies that $\Phi'_u(t_1) = \Phi'_u(t_2) = 0$. By (3.13), we also know that $\Phi''_u(t_1) > 0$, $\Phi''_u(t_2) < 0$. Therefore, the fibering map $\Phi_u(t)$ has a local minimum at t_1 and a local maximum at t_2 such that $t_1u \in \mathcal{N}^+$ and $t_2u \in \mathcal{N}^-$. By property (b), we know $tu \notin \mathcal{N}_J$ for $t \neq t_i$, $i = 1, 2$. This fact implies that $\Phi'_u(t) \neq 0$ for $t \in (0, t_1)$. Combining with the fact that t_1 is a local minimum of $\Phi_u(t)$ and $\Phi'_u(t) \in C[0, +\infty)$, we deduce that $\Phi'_u(t) < 0$ for $t \in (0, t_1)$. Moreover, noticing $\Phi_u(0) = 0$, we know $\Phi_u(t_1) = J(t_1u) < 0$.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. In view of Lemma 3.2, we can choose a constant $\Lambda_0 > 0$ such that $\mathcal{N}_J = \mathcal{N}^+ \cup \mathcal{N}^-$ for all $\lambda \in (0, \Lambda_0)$ and $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$. Since $u \in \mathcal{N}_J$, we have

$$J(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2 - \lambda(\frac{1}{q} - \frac{1}{p})\sum_{n\in\mathbb{Z}}|u(n)|^q.$$

Since $1 < q < 2$ and $J(u) \in C^1$, by Sobolev embedding, we have $J(u)$ is coercive, bounded from below in $\mathcal{N}_J$. So, we take the minimizing sequence $\{\tilde{u}_k\} \subset \mathcal{N}^+$

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satisfying $\lim_{k\to\infty} J(\tilde{u}_k) = \inf_{u\in\mathcal{N}^+} J(u)$, which is bounded in E, and converges to $\tilde{u}$ weakly. In fact, $\tilde{u} \neq 0$, since $J(\tilde{u}) \leq \liminf_{k\to\infty} J(\tilde{u}_k) < 0$. Moreover, there exists $t_3 > 0$ such that $t_3\tilde{u} \in \mathcal{N}^+$ and $J(t_3\tilde{u}) < 0$. Hence, it follows that $\inf_{u\in\mathcal{N}^+} J(u) < 0$.

Next, we show that $\tilde{u}_k \to \tilde{u}$ in E. If not, then $\|\tilde{u}\| < \liminf_{k\to\infty} \|\tilde{u}_k\|$. Thus, for $\{\tilde{u}_k\} \subset \mathcal{N}^+$, since $\tilde{u}_k \to \tilde{u}$ in $l^r(\mathbb{Z})$ for $2 \leq r < +\infty$ (in subsequence sense), we have

$$\begin{aligned}\lim_{k\to\infty} \Phi'_{\tilde{u}_k}(t_3) &= \lim_{k\to\infty} \big(t_3\|\tilde{u}_k\|^2 - \sum_{n\in\mathbb{Z}}(\lambda t_3^{q-1}|\tilde{u}_k(n)|^q + t_3^{p-1}|\tilde{u}_k(n)|^p)\big) \\ &> t_3\|\tilde{u}\|^2 - \sum_{n\in\mathbb{Z}}(\lambda t_3^{q-1}|\tilde{u}(n)|^q + t_3^{p-1}|\tilde{u}(n)|^p) = \Phi'_{\tilde{u}}(t_3) = 0.\end{aligned}$$

That is, $\Phi'_{\tilde{u}_k}(t_3) > 0$ for k large. Since $\tilde{u}_k = 1 \cdot \tilde{u}_k \in \mathcal{N}^+$, that is, $t_1 = 1$, we have $\Phi'_{\tilde{u}_k}(t) < 0$ for $t \in (0,1)$ and $\Phi'_{\tilde{u}_k}(1) = 0$. Hence, $t_3 > 1$. The fact that $\Phi_{\tilde{u}}(t)$ is decreasing on $(0, t_3)$ implies that

$$J(t_3\tilde{u}) \leq J(\tilde{u}) < \lim_{k\to\infty} J(\tilde{u}_k) = \inf_{u\in\mathcal{N}^+} J(u) < 0,$$

which is a contradiction. Therefore $\tilde{u}_k \to \tilde{u} \neq 0$ strongly in E. Moreover, reviewing that $\mathcal{N}^0 = \emptyset$ for any $\lambda \in (0, \Lambda_0)$ and $\Phi''_{\tilde{u}}(1) = \lim_{k\to\infty} \Phi''_{\tilde{u}_k}(1) \geq 0$, we obtain $\tilde{u} \in \mathcal{N}^+$. This implies

$$\lim_{k\to\infty} J(\tilde{u}_k) = J(\tilde{u}) = \inf_{u\in\mathcal{N}^+} J(u).$$

That is, $\tilde{u}$ is a minimizer of $J(u)$ on $\mathcal{N}^+$. Next, we claim that $|\tilde{u}|$ is also a minimizer of $J(u)$ on $\mathcal{N}^+$. Since $J(|\tilde{u}|) \leq J(\tilde{u}) < 0$ and $\Phi''_{\tilde{u}}(1) = (2-q)\lambda \sum_{n\in\mathbb{Z}} |\tilde{u}(n)|^q + (2-p)\sum_{n\in\mathbb{Z}} |\tilde{u}(n)|^p > 0$, we only need to show that $\Phi'_{|\tilde{u}|}(1) = 0$. In fact, $\Phi'_{|\tilde{u}|}(1) \leq 0$ since $\||\tilde{u}|\| \leq \|\tilde{u}\|$. By contradiction, we suppose that $\Phi'_{|\tilde{u}|}(1) < 0$, then, from the analysis of fibering map $\Phi_{|\tilde{u}|}(t)$, we know that there is $t_4 > 0$ such that $t_4\tilde{|u|} \in \mathcal{N}^+$, i.e., $\Phi''_{|\tilde{u}|}(t_4) > 0$ and $\Phi'_{|\tilde{u}|}(t_4) = 0$. Hence, $t_4 \neq 1$. Similarly, we notice that t_4 is a minimizer of $\Phi_{|\tilde{u}|}(t) = J(t|\tilde{u}|)$, we have

$$J(t_4|\tilde{u}|) < J(|\tilde{u}|) \leq J(\tilde{u}) = \inf_{u\in\mathcal{N}^+} J(u),$$

which is a contradiction. Thus, $\Phi'_{|\tilde{u}|}(1) = 0$ and then, $\Phi''_{|\tilde{u}|}(1) = \Phi''_{\tilde{u}}(1) = (2-q)\lambda \sum_{n\in\mathbb{Z}} |\tilde{u}(n)|^q + (2-p)\sum_{n\in\mathbb{Z}} |\tilde{u}(n)|^p > 0$. Therefore, $|\tilde{u}| \in \mathcal{N}^+ \subset \mathcal{N}_J$ and $J(|\tilde{u}|) = \inf_{u\in\mathcal{N}^+} J(u)$. That is, $|\tilde{u}|$ is a nonnegative minimizer of $J(u)$ on $\mathcal{N}^+$. The existence of nonnegative minimizer for $J(u)$ on $\mathcal{N}^-$ follows similar arguments as above.

Therefore, by above fibering map discussions, we deduce that $J(u)$ has a nontrivial nonnegative local minimizer on $\mathcal{N}^+$ and $\mathcal{N}^-$ respectively. By using Lemma 3.3, $J(u)$ has two nontrivial critical points in $E \subset l^2(\mathbb{Z})$, which are two nonnegative homoclinic solutions of problem (1.4). The proof is complete.

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