The second nonlinear mixed Jordan triple $*$-derivations on $*$-algebras

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Abstract

Let $\mathcal{A}$ be a unital $*$-algebra. In this paper, under some mild conditions on $\mathcal{A}$, it is shown that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is the second nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation. In particular, we apply the above result to prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras.

Keywords: mixed Jordan triple $*$-derivations; $*$-derivations; von Neumann algebras.

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1 Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, define the skew Lie product of A and B by $[A, B]_* = AB - BA^*$ and the Jordan $*$-product of A and B by $A \bullet B = AB + BA^*$. The skew Lie product and the Jordan $*$-product are fairly meaningful and important in some research topics (see [1, 3, 4, 6, 7, 10–15, 23–25, 28–30, 32]). They were extensively studied because they naturally arise in the problem of representing quadratic functionals with sesquilinear functionals (see [20–22]) and in the problem of characterizing ideals (see [2, 18]).

Recall that an additive map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be an additive derivation if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{A}$. Furthermore, Φ is said to be an

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additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. A map (without the additivity assumption) $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear skew Lie derivation or a nonlinear Jordan $*$-derivation if

$$\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$$

or

$$\Phi(A \bullet B) = \Phi(A) \bullet B + A \bullet \Phi(B)$$

for all $A, B \in \mathcal{A}$. Over the years several works have been published considering Jordan $*$-derivations, skew Lie derivations and triple derivations, such as Jordan triple $*$-derivations and skew Lie triple derivations (see [5, 8, 9, 16, 17, 26, 27]).

Recently, many authors have studied derivations corresponding to some mixed products (see [13, 19, 31, 33, 34]). Y. Zhou, Z. Yang and J. Zhang [33] proved any map Φ from a unital $*$-algebra $\mathcal{A}$ containing a non-trivial projection to itself satisfying

$$\Phi([[A, B]_*, C]) = [[\Phi(A), B]_*, C] + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)]$$

for all $A, B, C \in \mathcal{A}$, is an additive $*$-derivation, where $[A, B] = AB - BA$ is the usual Lie product of A and B. Y. Zhou and J. Zhang [34] proved that any map Φ on factor von Neumann algebra $\mathcal{A}$ satisfying

$$\Phi([[A, B], C]_*) = [[\Phi(A), B], C]_* + [[A, \Phi(B)], C]_* + [[A, B], \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$, is also an additive $*$-derivation. X. Zhao and X. Fang [31] gave similar result on finite von Neumann algebra with no central summands of type I_1. Y. Pang, D. Zhang and D. Ma [19] proved that if Φ is a second nonlinear mixed Jordan triple derivable mapping on a factor von Neumann algebra $\mathcal{A}$, that is,

$$\Phi(A \circ B \bullet C) = \Phi(A) \circ B \bullet C + A \circ \Phi(B) \bullet C + A \circ B \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$, then Φ is an additive $*$-derivation, where $A \circ B = AB + BA$ is the usual Jordan product of A and B.

Very recently, C. Li and D. Zhang [13] considered the nonlinear mixed Jordan triple $*$-derivations. Let $\mathcal{A}$ be a $*$-algebra. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear mixed Jordan triple $*$-derivation if

$$\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet B, C]_* + [A \bullet \Phi(B), C]_* + [A \bullet B, \Phi(C)]_*$$

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for all $A, B, C \in \mathcal{A}$. Under some mild conditions on a $*$-algebra $\mathcal{A}$, C. Li and D. Zhang [13] proved that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation. Similarly, we can give the definition of the second nonlinear mixed Jordan triple $*$-derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be the second nonlinear mixed Jordan triple $*$-derivation if

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$. In this paper, we will give the structure of the second nonlinear mixed Jordan triple $*$-derivations on $*$-algebras. Under some mild conditions on a $*$-algebra $\mathcal{A}$, we prove that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a second nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation. In particular, we apply the above result to prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras.

2 Main result and corollaries

The following is our main result in this paper.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I. Assume that $\mathcal{A}$ contains a nontrivial projection P and satisfies*

$$(\spadesuit) \quad X\mathcal{A}P = 0 \quad \textit{implies} \quad X = 0$$

and

$$(\clubsuit) \quad X\mathcal{A}(I - P) = 0 \quad \textit{implies} \quad X = 0.$$

Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

Recall that an algebra $\mathcal{A}$ is prime if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy $(\spadesuit)$ and $(\clubsuit)$. Applying Theorem 2.1 to prime $*$-algebras, we have the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and P be a nontrivial projection in $\mathcal{A}$. Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. $\mathcal{M}$ is a factor von Neumann algebra if its center only contains the scalar operators. It is well known that a factor von Neumann algebra is prime. Now we have the following corollary.

Corollary 2.3. Let $\mathcal{M}$ be a factor von Neumann algebra with $\dim(\mathcal{M}) \geq 2$. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{M}$ if and only if Φ is an additive $*$-derivation.

It is shown in [3] and [8] that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{M}$ satifies (♠) and (♣). Now we have the following corollary.

Corollary 2.4. Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{M}$ if and only if Φ is an additive $*$-derivation.

3 The proof of main result

The proof of Theorem 2.1. Let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}_{jk} = P_j \mathcal{A} P_k$, $j, k = 1, 2$. Then $\mathcal{A} = \sum_{j,k=1}^{2} \mathcal{A}_{jk}$. In all that follows, when we write A_{jk}^*, it means that $A_{jk}^* = (A_{jk})^*$. Clearly, we only need prove the necessity. Now we will complete the proof of Theorem 2.1 by proving several claims.

Claim 1. $\Phi(0) = 0$.

Indeed, we have

$$\Phi(0) = \Phi([0, 0]_* \bullet 0) = [\Phi(0), 0]_* \bullet 0 + [0, \Phi(0)]_* \bullet 0 + [0, 0]_* \bullet \Phi(0) = 0.$$

Claim 2. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let

$$T = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22}).$$

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It follows from Claim 1 that

$$[\Phi(P_2), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1 + [P_2, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_* \bullet P_1$$
$$+ [P_2, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_1)$$
$$= \Phi([P_2, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1)$$
$$= \Phi([P_2, C_{21}]_* \bullet P_1)$$
$$= \Phi([P_2, A_{11}]_* \bullet P_1) + \Phi([P_2, B_{12}]_* \bullet P_1) + \Phi([P_2, C_{21}]_* \bullet P_1) + \Phi([P_2, D_{22}]_* \bullet P_1)$$
$$= [\Phi(P_2), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1 + [P_2, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_*$$
$$\bullet P_1 + [P_2, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_1).$$

From this, we get $T_{21} + T_{21}^* = [P_2, T]_* \bullet P_1 = 0$. So $T_{21} = 0$. Similarly, we can prove $T_{12} = 0$.

For every $S_{21} \in \mathcal{A}_{21}$, we have

$$[\Phi(S_{21}), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1 + [S_{21}, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_* \bullet P_1$$
$$+ [S_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_1)$$
$$= \Phi([S_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1)$$
$$= \Phi([S_{21}, A_{11}]_* \bullet P_1)$$
$$= \Phi([S_{21}, A_{11}]_* \bullet P_1) + \Phi([S_{21}, B_{12}]_* \bullet P_1) + \Phi([S_{21}, C_{21}]_* \bullet P_1) + \Phi([S_{21}, D_{22}]_* \bullet P_1)$$
$$= [\Phi(S_{21}), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_1 + [S_{21}, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_*$$
$$\bullet P_1 + [S_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_1),$$

which implies $S_{21}T_{11} + T_{11}^* S_{21}^* = [S_{21}, T]_* \bullet P_1 = 0$. Hence $T_{11}^* S_{21}^* = 0$ for every $S_{21} \in \mathcal{A}_{21}$. It follows from the condition (♣) that $T_{11} = 0$. Similarly, we can prove $T_{22} = 0$, proving the claim.

Claim 3. For every $A_{jk}, B_{jk} \in \mathcal{A}_{jk}, 1 \leq j \neq k \leq 2$, we have

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

Since

$$[-\frac{i}{2}I, i(P_j + A_{jk})]_* \bullet (P_k + B_{jk}) = (A_{jk} + B_{jk}) + A_{jk}^* + B_{jk}A_{jk}^*,$$

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we get from Claim 2 that

$$\begin{aligned}
&\Phi(A_{jk}+B_{jk})+\Phi(A_{jk}^*)+\Phi(B_{jk}A_{jk}^*)\\
&=\Phi([-\frac{i}{2}I, i(P_j+A_{jk})]_*\bullet(P_k+B_{jk}))\\
&=[\Phi(-\frac{i}{2}I), i(P_j+A_{jk})]_*\bullet(P_k+B_{jk})+[-\frac{i}{2}I,\Phi(i(P_j+A_{jk}))]_*\bullet(P_k+B_{jk})\\
&+[-\frac{i}{2}I, i(P_j+A_{jk})]_*\bullet\Phi(P_k+B_{jk})\\
&=[\Phi(-\frac{i}{2}I), i(P_j+A_{jk})]_*\bullet(P_k+B_{jk})+[-\frac{i}{2}I,\Phi(iP_j)+\Phi(iA_{jk})]_*\bullet(P_k+B_{jk})\\
&+[-\frac{i}{2}I, i(P_j+A_{jk})]_*\bullet(\Phi(P_k)+\Phi(B_{jk}))\\
&=\Phi([-\frac{i}{2}I, iP_j]_*\bullet P_k)+\Phi([-\frac{i}{2}I, iP_j]_*\bullet B_{jk})+\Phi([-\frac{i}{2}I, iA_{jk}]_*\bullet P_k)\\
&+\Phi([-\frac{i}{2}I, iA_{jk}]_*\bullet B_{jk})\\
&=\Phi(B_{jk})+\Phi(A_{jk}+A_{jk}^*)+\Phi(B_{jk}A_{jk}^*)\\
&=\Phi(B_{jk})+\Phi(A_{jk})+\Phi(A_{jk}^*)+\Phi(B_{jk}A_{jk}^*).
\end{aligned}$$

Hence $\Phi(A_{jk}+B_{jk})=\Phi(A_{jk})+\Phi(B_{jk})$.
Claim 4. For every $A_{jj}, B_{jj}\in\mathcal{A}_{jj}, 1\le j\le 2$, we have

$$\Phi(A_{jj}+B_{jj})=\Phi(A_{jj})+\Phi(B_{jj}).$$

Let $T=\Phi(A_{11}+B_{11})-\Phi(A_{11})-\Phi(B_{11})$. Then

$$\begin{aligned}
&[\Phi(P_2), A_{11}+B_{11}]_*\bullet P_1+[P_2,\Phi(A_{11}+B_{11})]_*\bullet P_1+[P_2, A_{11}+B_{11}]_*\bullet\Phi(P_1)\\
&=\Phi([P_2, A_{11}+B_{11}]_*\bullet P_1)\\
&=\Phi([P_2, A_{11}]_*\bullet P_1)+\Phi([P_2, B_{11}]_*\bullet P_1)\\
&=[\Phi(P_2), A_{11}+B_{11}]_*\bullet P_1+[P_2,\Phi(A_{11})+\Phi(B_{11})]_*\bullet P_1\\
&+[P_2, A_{11}+B_{11}]_*\bullet\Phi(P_1).
\end{aligned}$$

From this, we get $T_{21}+T_{21}^*=[P_2,T]_*\bullet P_1=0$. So $T_{21}=0$. Similarly, we can prove $T_{12}=0$.

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For every $S_{12} \in \mathcal{A}_{12}$, we obtain that

$$\begin{aligned}
&[\Phi(S_{12}), A_{11} + B_{11}]_* \bullet P_2 + [S_{12}, \Phi(A_{11} + B_{11})]_* \bullet P_2 + [S_{12}, A_{11} + B_{11}]_* \bullet \Phi(P_2) \\
&= \Phi([S_{12}, A_{11} + B_{11}]_* \bullet P_2) \\
&= \Phi([S_{12}, A_{11}]_* \bullet P_2) + \Phi([S_{12}, B_{11}]_* \bullet P_2) \\
&= [\Phi(S_{12}), A_{11} + B_{11}]_* \bullet P_2 + [S_{12}, \Phi(A_{11}) + \Phi(B_{11})]_* \bullet P_2 \\
&+ [S_{12}, A_{11} + B_{11}]_* \bullet \Phi(P_2),
\end{aligned}$$

which implies that $S_{12}T_{22} + T_{22}^* S_{12}^* = [S_{12}, T]_* \bullet P_2 = 0$. So $T_{22}^* S_{12}^* = 0$ for all $S_{12} \in \mathcal{A}_{12}$. It follows from the condition (♠) that $T_{22} = 0$.

For every $S_{21} \in \mathcal{A}_{21}$, it follows from Claim 2 and Claim 3 that

$$\begin{aligned}
&[\Phi(S_{21}), A_{11} + B_{11}]_* \bullet P_1 + [S_{21}, \Phi(A_{11} + B_{11})]_* \bullet P_1 + [S_{21}, A_{11} + B_{11}]_* \bullet \Phi(P_1) \\
&= \Phi([S_{21}, A_{11} + B_{11}]_* \bullet P_1) \\
&= \Phi(S_{21}(A_{11} + B_{11})) + \Phi((A_{11} + B_{11})^* S_{21}^*) \\
&= \Phi(S_{21}A_{11}) + \Phi(S_{21}B_{11}) + \Phi(A_{11}^* S_{21}^*) + \Phi(B_{11}^* S_{21}^*) \\
&= \Phi(S_{21}A_{11} + A_{11}^* S_{21}^*) + \Phi(S_{21}B_{11} + B_{11}^* S_{21}^*) \\
&= \Phi([S_{21}, A_{11}]_* \bullet P_1) + \Phi([S_{21}, B_{11}]_* \bullet P_1) \\
&= [\Phi(S_{21}), A_{11} + B_{11}]_* \bullet P_1 + [S_{21}, \Phi(A_{11}) + \Phi(B_{11})]_* \bullet P_1 \\
&+ [S_{21}, A_{11} + B_{11}]_* \bullet \Phi(P_1).
\end{aligned}$$

Hence $S_{21}T_{11} + T_{11}^* S_{21}^* = [S_{21}, T]_* \bullet P_1 = 0$, and then $T_{11}^* S_{21}^* = 0$ for all $S_{21} \in \mathcal{A}_{21}$. It follows from the condition (♣) that $T_{11} = 0$. Now we have proved that $\Phi(A_{11} + B_{11}) = \Phi(A_{11}) + \Phi(B_{11})$. Similarly, we can prove $\Phi(A_{22} + B_{22}) = \Phi(A_{22}) + \Phi(B_{22})$.

Claim 5. Φ is additive.

By Claims 2, 3 and 4, it is easy to show the additivity of Φ.

Claim 6. (1) $\Phi(iI)^* = \Phi(iI) \in \mathcal{Z}(\mathcal{A})$;

(2) $\Phi(I)^* = \Phi(I) \in \mathcal{Z}(\mathcal{A})$.

On using Claim 5, we get

$$\begin{aligned}
-4\Phi(iI) &= \Phi([iI, iI]_* \bullet (iI)) \\
&= [\Phi(iI), iI]_* \bullet (iI) + [iI, \Phi(iI)]_* \bullet (iI) + [iI, iI]_* \bullet \Phi(iI) \\
&= 4\Phi(iI)^* - 8\Phi(iI).
\end{aligned}$$

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So $\Phi(iI)^* = \Phi(iI)$. From here, for all $A \in \mathcal{A}$, we have

$$\begin{aligned} 0 &= \Phi([iI, I]_* \bullet A) \\ &= [\Phi(iI), I]_* \bullet A + [iI, \Phi(I)]_* \bullet A + [iI, I]_* \bullet \Phi(A) \\ &= (2i\Phi(I)) \bullet A \\ &= 2i(\Phi(I)A - A\Phi(I)^*). \end{aligned}$$

Let $A = I$. Then $\Phi(I)^* = \Phi(I)$. So $\Phi(I)A = A\Phi(I)$ for all $A \in \mathcal{A}$, which implies that $\Phi(I) \in \mathcal{Z}(\mathcal{A})$. Similarly, by replacing iI with A in above equation, we can prove (1).

Claim 7. For $1 \leq j \neq k \leq 2$, we have

(1) $P_j\Phi(P_j)P_k = -P_j\Phi(P_k)P_k$ and $P_j\Phi(P_k)P_j = 0$;

(2) $P_j\Phi(iP_j)P_k = -P_j\Phi(iP_k)P_k$ and $P_j\Phi(iP_k)P_j = 0$.

In view of Claim 6, we have

$$\begin{aligned} 0 &= \Phi([iI, P_j]_* \bullet P_k) \\ &= [iI, \Phi(P_j)]_* \bullet P_k + [iI, P_j]_* \bullet \Phi(P_k) \\ &= 2i(\Phi(P_j)P_k - P_k\Phi(P_j)^* + P_j\Phi(P_k) - \Phi(P_k)P_j). \end{aligned}$$

Multiplying by P_j from the left and by P_k from the right, we obtain that $P_j\Phi(P_j)P_k = -P_j\Phi(P_k)P_k$. On the other hand, we also have

$$\begin{aligned} 0 &= \Phi([iP_j, iI]_* \bullet P_k) \\ &= [\Phi(iP_j), iI]_* \bullet P_k + [iP_j, iI]_* \bullet \Phi(P_k) \\ &= i(\Phi(iP_j)P_k - \Phi(iP_j)^*P_k - P_k\Phi(iP_j)^* + P_k\Phi(iP_j)) - 2P_j\Phi(P_k) - 2\Phi(P_k)P_j. \end{aligned}$$

Multiplying by P_j from the both sides, we obtain that $P_j\Phi(P_k)P_j = 0$.

Next, it follows from Claim 6 that

$$\begin{aligned} 0 &= \Phi([iI, iP_j]_* \bullet (iP_k)) \\ &= [iI, \Phi(iP_j)]_* \bullet (iP_k) + [iI, iP_j]_* \bullet \Phi(iP_k) \\ &= -2\Phi(iP_j)P_k + 2P_k\Phi(iP_j)^* - 2P_j\Phi(iP_k) - 2\Phi(iP_k)P_j. \end{aligned}$$

Multiplying by P_j from the left and by P_k from the right, we have $P_j\Phi(iP_j)P_k = -P_j\Phi(iP_k)P_k$. Multiplying by P_j from the both sides, we also have $P_j\Phi(iP_k)P_j =$

0.

Claim 8. $P_j\Phi(P_j)P_j = P_j\Phi(iP_j)P_j = 0$, $j = 1, 2$.

For any $A_{jk} \in \mathcal{A}_{jk}$, $1 \leq j \neq k \leq 2$, using Claims 5 and 6, we have

$$\begin{aligned}2\Phi(iA_{jk}) &= \Phi([iI, P_j]_* \bullet A_{jk}) \\ &= [iI, \Phi(P_j)]_* \bullet A_{jk} + [iI, P_j]_* \bullet \Phi(A_{jk}) \\ &= 2i(\Phi(P_j)A_{jk} - A_{jk}\Phi(P_j)^* + P_j\Phi(A_{jk}) - \Phi(A_{jk})P_j).\end{aligned}$$

Multiplying by P_j from the left and by P_k from the right, by Claim 7 (1), we get

$$P_j\Phi(iA_{jk})P_k = i(P_j\Phi(P_j)A_{jk} + P_j\Phi(A_{jk})P_k). \qquad (3.\ 1)$$

On the other hand, we also have

$$\begin{aligned}-2\Phi(A_{jk}) &= \Phi([iI, P_j]_* \bullet (iA_{jk})) \\ &= [iI, \Phi(P_j)]_* \bullet (iA_{jk}) + [iI, P_j]_* \bullet \Phi(iA_{jk}) \\ &= -2(\Phi(P_j)A_{jk} - A_{jk}\Phi(P_j)^* - iP_j\Phi(iA_{jk}) + i\Phi(iA_{jk})P_j).\end{aligned}$$

Multiplying by P_j from the left and by P_k from the right, by Claim 7 (1), we obtain that $P_j\Phi(A_{jk})P_k = P_j\Phi(P_j)A_{jk} - iP_j\Phi(iA_{jk})P_k$. So

$$P_j\Phi(iA_{jk})P_k = i(P_j\Phi(A_{jk})P_k - P_j\Phi(P_j)A_{jk}). \qquad (3.\ 2)$$

Now, from Eqs. (3. 1) and (3. 2), we have $P_j\Phi(P_j)A_{jk}$=0 for any $A_{jk} \in \mathcal{A}_{jk}$. It follows from (♠) and (♣) that $P_j\Phi(P_j)P_j = 0$.

Moreover, using Claims 5 and 6, we find that

$$\begin{aligned}-2\Phi(A_{jk}) &= \Phi([iI, iP_j]_* \bullet A_{jk}) \\ &= [iI, \Phi(iP_j)]_* \bullet A_{jk} + [iI, iP_j]_* \bullet \Phi(A_{jk}) \\ &= 2i\Phi(iP_j)A_{jk} - 2iA_{jk}\Phi(iP_j)^* - 2P_j\Phi(A_{jk}) - 2\Phi(A_{jk})P_j.\end{aligned}$$

Multiplying by P_j from the left and by P_k from the right, by Claim 7 (2), we obtain that $P_j\Phi(iP_j)A_{jk} = 0$. Thus, it follows from (♠) and (♣) that $P_j\Phi(iP_j)P_j = 0$.

Claim 9. $\Phi(I) = \Phi(iI) = 0$.

By Claims 5, 7 and 8, we have

$$\begin{aligned}\Phi(I) &= \Phi(P_1 + P_2) = \Phi(P_1) + \Phi(P_2) \\ &= P_1\Phi(P_1)P_2 + P_2\Phi(P_1)P_1 + P_1\Phi(P_2)P_2 + P_2\Phi(P_2)P_1 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Phi(iI) &= \Phi(i(P_1+P_2)) = \Phi(iP_1)+\Phi(iP_2)\\ &= P_1\Phi(iP_1)P_2 + P_2\Phi(iP_1)P_1 + P_1\Phi(iP_2)P_2 + P_2\Phi(iP_2)P_1\\ &= 0.\end{aligned}$$

Claim 10. For any $A \in \mathcal{A}$, $\Phi(iA) = i\Phi(A)$.

Let $N = -N^* \in \mathcal{A}$. Using Claims 5 and 9, we have

$$-4\Phi(N) = \Phi([iI, N]_* \bullet (iI)) = [iI, \Phi(N)]_* \bullet (iI) = -2\Phi(N) + 2\Phi(N)^*,$$

which implies that

$$\Phi(N)^* = -\Phi(N).$$

From this, we get

$$4\Phi(iN) = \Phi([N, iI]_* \bullet I) = [\Phi(N), iI]_* \bullet I = 4i\Phi(N).$$

So

$$\Phi(iN) = i\Phi(N).$$

For any $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1^* = -A_1$ and $A_2^* = -A_2$. It follows that

$$\begin{aligned}\Phi(iA) &= \Phi(i(A_1+iA_2)) = \Phi(iA_1 - A_2) = \Phi(iA_1) - \Phi(A_2)\\ &= i\Phi(A_1) - \Phi(A_2) = i(\Phi(A_1) + i\Phi(A_2))\\ &= i(\Phi(A_1) + \Phi(iA_2)) = i\Phi(A_1 + iA_2)\\ &= i\Phi(A).\end{aligned}$$

Claim 11. For any $A, B \in \mathcal{A}$, we have $\Phi(A \bullet B) = \Phi(A) \bullet B + A \bullet \Phi(B)$.

In view of Claims 5, 9 and 10, we have

$$\begin{aligned}-2\Phi(A \bullet B) &= \Phi([iI, iA]_* \bullet B)\\ &= [iI, i\Phi(A)]_* \bullet B + [iI, iA]_* \bullet \Phi(B)\\ &= -2(\Phi(A) \bullet B + A \bullet \Phi(B)),\end{aligned}$$

which yields that $\Phi(A \bullet B) = \Phi(A) \bullet B + A \bullet \Phi(B)$.

Claim 12. For any $A \in \mathcal{A}$, $\Phi(A^*) = \Phi(A)^*$.

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Let $A \in \mathcal{A}$. It follows from Claims 5, 9 and 11 that

$$\Phi(A) + \Phi(A^*) = \Phi(A + A^*) = \Phi(A \bullet I) = \Phi(A) \bullet I = \Phi(A) + \Phi(A)^*.$$

Hence $\Phi(A^*) = \Phi(A)^*$.

Claim 13. Φ is a derivation.

By Claims 5 and 11, we have

$$\begin{aligned}\Phi(AB) + \Phi(BA^*) &= \Phi(AB + BA^*) = \Phi(A \bullet B) \\ &= \Phi(A) \bullet B + A \bullet \Phi(B) \\ &= \Phi(A)B + B\Phi(A)^* + A\Phi(B) + \Phi(B)A^*. \end{aligned} \tag{3. 3}$$

On the other hand, by Claims 5, 10 and 11, we have

$$\begin{aligned}-\Phi(AB) + \Phi(BA^*) &= \Phi(-AB + BA^*) = \Phi((iA) \bullet (iB)) \\ &= (i\Phi(A)) \bullet (iB) + (iA) \bullet (i\Phi(B)) \\ &= -\Phi(A)B + B\Phi(A)^* - A\Phi(B) + \Phi(B)A^*. \end{aligned} \tag{3. 4}$$

From Eqs. (3. 3) and (3. 4), we obtain that

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

Now, from Claims 5, 12 and 13, we can conclude that Φ is an additive $*$-derivation. This completes the proof of Theorem 2.1.

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