SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. We consider a family of all analytic and univalent functions (i.e., one-to-one) in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. In this paper, we obtain the sharp bounds of the second Hankel determinant of Logarithmic coefficients for some subclasses of analytic functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the class all functions $f \in \mathcal{A}$ that are univalent (i.e., one-to-one) in $\mathbb{D}$. For a general theory of univalent functions, we refer the classical books [7, 9].

For $q, n \in \mathbb{N}$, the *Hankel determinant* $H_{q,n}(f)$ of a function $f \in \mathcal{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

In particular, for $q = 2$ and $n = 1$, $H_{2,1}(f) = a_1 a_3 - a_2^2$ is usually called the second Hankel determinant. It is interesting to note that the second Hankel determinant is related to the Fekete-Szegö functional for $\mu = 1$ as $|H_{2,1}(f)| = |a_1 a_3 - \mu a_2^2|$. For the class $\mathcal{S}$, the bound of $H_{2,1}(f) = a_3 - a_2^2$ was estimated by Bieberbach in 1916. General results for Hankel determinants of any degree studied by Pommerenke [22, 23], Hayman [10] and many others in recent years. It is worth mentioning that Pommerenke [22] gave some applications of Hankel determinants in the study of singularities and the power series with integral coefficients of analytic functions. The problem of computing the bounds of Hankel determinants in a given family of analytic functions attracted the attention of many mathematicians (see [3, 29] and reference therein).

The *Logarithmic coefficients* γ_n of $f \in \mathcal{S}$ are defined by the following series expansion:

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 30C45, 30C50, 30C55.

Key words and phrases. Carathéodory function, Hankel determinant, Logarithmic coefficients, Spiral-like functions, Univalent functions.

The logarithmic coefficients have great importance as they play a crucial role in Milin conjecture [18] (see also [7, p. 155]). Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

where the equality holds if, and only if, f is a rotation of the Koebe function. De Branges [5] proved Milin conjecture which confirmed the famous Bieberbach conjecture. On the other hand, one of reasons for more attention has been given to the Logarithmic coefficients is that the sharp bound for the class $\mathcal{S}$ is known only for γ_1 and γ_2, namely

$$|\gamma_1| \leq 1 \text{ and } |\gamma_2| \leq \frac{1}{2}(1+2e^{-2}) = 0.635\ldots. \tag{1.3}$$

It is still an open problem to find the sharp bounds of γ_n, $n \geq 3$, for the class $\mathcal{S}$. Note that for the Koebe function $k(z) = z/(1-z)^2$, $z \in \mathbb{D}$, it is easy to see that $\gamma_n = 1/n$ for each $n \geq 1$. Therefore it is expected that $|\gamma_n| \leq 1/n$, since the Koebe function plays a role of extremal function in many problems of geometric function theory. But it was shown that, this is not true even for $n = 2$, as we can seen in equation (1.3). The problem of finding the sharp bound of $|\gamma_n|$ for the class $\mathcal{S}$ and for its various subclasses are studied recently by several authors in different contexts, for instance see [1, 2, 8, 14, 26, 31].

If f is given by (1.1), then by differentiating (1.2) and equating coefficients, we obtain

$$\gamma_1 = \frac{1}{2}\,a_2, \gamma_2 = \frac{1}{2}\big(a_3 - \frac{1}{2}\,a_2^2\big), \text{ and } \gamma_3 = \frac{1}{2}\big(a_4 - a_2a_3 + \frac{1}{3}\,a_2^3\big).$$

Due to the great importance of logarithmic coefficients in the recent years, it is appropriate and interesting to compute the Hankel determinant whose entries are logarithmic coefficients. In particular, the second Hankel determinant of $F_f/2$ is defined as

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{1.4}$$

As usual, instead of the whole class $\mathcal{S}$ one can take into account their subclasses for which the problem of finding sharp estimates of Hankel determinant of logarithmic coefficients can be studied. The problem of computing the sharp bounds of $H_{2,1}(F_f/2)$ was considered in [11] for starlike and convex functions.

It is now appropriate to remark that $H_{2,1}(F_f/2)$ is invariant under rotation since for $f_\theta(z) := e^{-i\theta}f(e^{i\theta}z)$, $\theta \in \mathbb{R}$ when $f \in \mathcal{S}$ we have

$$H_{2,1}(F_{f_\theta}/2) = \frac{e^{4i\theta}}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = e^{4i\theta}H_{2,1}(F_f/2).$$

The main purpose of this paper is to obtain the sharp upper bounds of the second Hankel determinant of the logarithmic coefficients, *i.e.*, $|H_{2,1}(F_f/2)|$, for various subclasses of the class $\mathcal{A}$.

2. Preliminary results

In this section, we present key lemmas which will be used to prove the main results of this paper. Let $\mathcal{P}$ denote the class of all analytic functions p having positive real part in

$\mathbb{D}$, with the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 \cdots . \tag{2.1}$$

A member of $\mathcal{P}$ is called a *Carathéodory function.* It is known that $|c_n| \leq 2$, $n \geq 1$ for a function $p \in \mathcal{P}$ (see [7]).

Parametric representations of the coefficients are often useful. Libera and Złotkiewicz [16, 17] derived the following parameterizations of possible values of c_2 and c_3.

Lemma 2.1. *[16, 17] If $p \in \mathcal{P}$ is of the form (2.1) with $c_1 \geq 0$, then*

$$c_1 = 2p_1, \tag{2.2}$$

$$c_2 = 2p_1^2 + 2(1-p_1^2)p_2, \tag{2.3}$$

and

$$c_3 = 2p_1^3 + 4(1-p_1^2)p_1p_2 - 2(1-p_1^2)p_1p_2^2 + 2(1-p_1^2)(1-|p_2|^2)p_3 \tag{2.4}$$

for some $p_1 \in [0,1]$ and $p_2, p_3 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

For $p_1 \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (2.2), namely

$$p(z) = \frac{1+p_1 z}{1-p_1 z}, \quad z \in \mathbb{D}.$$

For $p_1 \in \mathbb{D}$ and $p_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (2.2) and (2.3), namely

$$p(z) = \frac{1 + (p_1 + \overline{p_1}p_2)z + p_2 z^2}{1 - (p_1 - \overline{p_1}p_2)z - p_2 z^2}. \tag{2.5}$$

For $p_1,\ p_2 \in \mathbb{D}$ and $p_3 \in \mathbb{T}$, there is unique function $p \in \mathcal{P}$ with c_1, c_2, and c_3 as in (2.2)-(2.4), namely,

$$p(z) = \frac{1 + (\overline{p_2}p_3 + \overline{p_1}p_2 + p_1)z + (\overline{p_1}p_3 + p_1\overline{p_2}p_3 + p_2)z^2 + p_3 z^3}{1 + (\overline{p_2}p_3 + \overline{p_1}p_2 - p_1)z + (\overline{p_1}p_3 - p_1\overline{p_2}p_3 - p_2)z^2 - p_3 z^3} \quad z \in \mathbb{D}.$$

Next we recall the following well-known result due to Choi *et al.* [6]. Lemma 2.2 plays an important role in the proof of our main results.

Lemma 2.2. *[6] Let A, B, C be real numbers and*

$$Y(A,B,C) := \max_{z \in \overline{\mathbb{D}}} (|A + Bz + Cz^2| + 1 - |z|^2).$$

(i) If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & \text{for } |B| \geq 2(1-|C|), \\[2ex] 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & \text{for } |B| < 2(1-|C|). \end{cases}$$

(ii) If $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}, \\ R(A,B,C), & \textit{otherwise}, \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| + |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|A| + |C|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \textit{otherwise}. \end{cases}$$

3. Main results

For a better clarity in our presentation, we divide this section into several subsections consisting of different families of functions from the class $\mathcal{A}$ and prove our main results associated with those classes of functions.

3.1. The class $\mathcal{S}_\beta(\alpha)$.

To state our first result we need to introduce the following definitions: A function $f \in \mathcal{A}$ is called *starlike* if $f(\mathbb{D})$ is a starlike domain with respect to origin. The class of univalent starlike functions is denoted by $\mathcal{S}^*$. There is one natural generalization of starlike functions is β-spirallike functions of order α which leads to a useful criterion for univalency. The family $\mathcal{S}_\beta(\alpha)$ of *β-spirallike functions of order α* is defined by

$$\mathcal{S}_\beta(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(e^{-i\beta} \frac{zf'(z)}{f(z)} \right) > \alpha \cos\beta \right\},$$

where $0 \leq \alpha < 1$ and $-\pi/2 < \beta < \pi/2$. It is known that each function in $\mathcal{S}_\beta(\alpha)$ is univalent in $\mathbb{D}$ (see [15]). Functions in $\mathcal{S}_\beta(0)$ are called *β-spirallike*, but they do not necessarily belong to the starlike family $\mathcal{S}^*$. For example, the function $f(z) = z(1-iz)^{i-1}$ is $\pi/4$-spirallike but $f \notin \mathcal{S}^*$. The class $\mathcal{S}_\beta(0)$ was introduced by Špaček [30] (see also [7]). Moreover, $\mathcal{S}_0(\alpha) =: \mathcal{S}^*(\alpha)$ is the usual class of starlike functions of order α, and $\mathcal{S}^*(0) = \mathcal{S}^*$. Recall that the class $\mathcal{S}_\beta(\alpha)$, for $0 \leq \alpha < 1$, is studied by several authors in different perspective (see, for instance [12, 15]).

Now we will prove the first main result of this paper.

Theorem 3.1. *Let* $-\pi/2 < \beta < \pi/2$ *and* $0 \leq \alpha < 1$. *For every* $f \in \mathcal{S}_\beta(\alpha)$ *of the form* (1.1), *we have*

$$|H_{2,1}(F_f/2)| \leq \frac{(1-\alpha)^2 \cos^2\beta}{4}. \tag{3.1}$$

Equality in (3.1) *holds for the rotation of the function*

$$f_1(z) = \frac{z}{(1-z^2)^{(1-\alpha)\cos\beta e^{i\beta}}}.$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_\beta(\alpha)$. Then by the definition, we may consider $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ of the form

$$p(z) = \frac{1}{1-\alpha}\left\{\frac{1}{\cos\beta}\left(e^{-i\beta}\frac{zf'(z)}{f(z)} + i\sin\beta\right) - \alpha\right\}.$$

The above equality we can rewrite as

$$((1-\alpha)p(z)+\alpha)\cos\beta - i\sin\beta = e^{-i\beta}\frac{zf'(z)}{f(z)}.$$

By using the Taylor series representations of the functions f and p, and after comparing the coefficients of z^n $(n = 1, 2, 3)$ on both the sides, we get

$$\begin{aligned} a_2 &= (1-\alpha)e^{i\beta}\cos\beta c_1, \\ 2a_3 &= (1-\alpha)^2 e^{2i\beta}\cos^2\beta c_1^2 + (1-\alpha)e^{i\beta}\cos\beta c_2, \end{aligned}$$

and

$$6a_4 = (1-\alpha)^3 e^{3i\beta}\cos^3\beta c_1^3 + 3c_1c_2(1-\alpha)^2 e^{2i\beta}\cos^2\beta + 2(1-\alpha)e^{i\beta}\cos\beta c_3.$$

Substitution of a_2, a_3, and a_4 in (1.4) gives

$$H_{2,1}(F_f/2) = \frac{(1-\alpha)^2 e^{2i\beta}\cos^2\beta}{48}\,(4c_1c_3 - 3c_2^2).$$

As $H_{2,1}(F_f/2)$ and $\mathcal{S}_\beta(\alpha)$ are invariant under the rotations, therefore to simplify the calculation we assume that c_1 is real. Therefore, by Lemma 2.1, for some $p_1 \in [0,1]$ and $p_2, p_3 \in \overline{\mathbb{D}}$ we have

$$\begin{aligned} H_{2,1}(F_f/2) = \frac{(1-\alpha)^2\cos^2\beta}{12}\Big(&p_1^4 + 2(1-p_1^2)p_1^2p_2 - (1-p_1^2)(3+p_1^2)p_2^2 \\ &+ 4p_1(1-p_1^2)(1-|p_2|^2)p_3\Big). \end{aligned} \tag{3.2}$$

Now, we may have the following cases on p_1:

Case 1: Let $p_1 = 1$. Then from (3.2) we get

$$|H_{2,1}(F_f/2)| = \frac{(1-\alpha)^2\cos^2\beta}{12}.$$

Case 2: Let $p_1 = 0$. Then from (3.2) we get

$$|H_{2,1}(F_f/2)| = \frac{(1-\alpha)^2\cos^2\beta}{12}\,|3p_2^2| \le \frac{(1-\alpha)^2\cos^2\beta}{4}.$$

Case 3: Let $p_1 \in (0,1)$. Applying the triangle inequality in (3.2) and by using the fact that $|p_3| \leq 1$, we obtain

$$|H_{2,1}(F_f/2)| \leq \frac{(1-\alpha)^2\cos^2\beta}{12}\Big(\big|p_1^4 + 2(1-p_1^2)p_1^2p_2 - (1-p_1^2)(3+p_1^2)p_2^2\big| + 4p_1(1-p_1^2)(1-|p_2|^2)\Big) \tag{3.3}$$

$$= \frac{(1-\alpha)^2\cos^2\beta p_1(1-p_1^2)}{3}\Big(\big|A + Bp_2 + Cp_2^2\big| + 1 - |p_2|^2\Big), \tag{3.4}$$

where

$$A := \frac{p_1^3}{4(1-p_1^2)},\ B := \frac{p_1}{2},\ \text{and } C := -\frac{(3+p_1^2)}{4p_1}.$$

Observe that $AC < 0$, so we can apply case (ii) of Lemma 2.2. Now we check all the conditions of case (ii).

3(a) Note that for $p_1 \in (0,1)$ we have

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 = -\frac{3p_1^2}{p_1^2+3} \leq 0.$$

Also, the inequality $|B| < 2(1-|C|)$ is equivalent to $2p_1^2 - 4p_1 + 3 < 0$ which is not true for $p_1 \in (0,1)$.

3(b) Next, it is easy to check that

$$\min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} = -4AC\left(\frac{1}{C^2}-1\right) \leq B^2,$$

here the last inequality directly follows from 3(a).

3(c) For $0 < p_1 < 1$, it is easy to verify that $|C|(|B| + 4|A|) - |AB| \leq 0$ is not satisfied as $3 + 4p_1^2 \geq 0$.

3(d) We note that the inequality

$$|AB| - |C|(|B| - 4|A|) = \frac{4p_1^4 + 8p_1^2 - 3}{8(1-p_1^2)} \leq 0$$

holds for $0 < p_1 \leq s_1 := \sqrt{\sqrt{7}/2 - 1} \approx 0.568221$. It follows from Lemma 2.2 and the inequality (3.3) that

$$\begin{aligned}|H_{2,1}(F_f/2)| &\leq \frac{(1-\alpha)^2\cos^2\beta p_1(1-p_1^2)}{3}(-|A| + |B| + |C|)\\ &= \frac{(1-\alpha)^2\cos^2\beta(3-4p_1^4)}{12}\\ &\leq \frac{(1-\alpha)^2\cos^2\beta}{4}\end{aligned}$$

for $0 < p_1 \leq s_1$.

3(e) For $s_1 < p_1 < 1$, we use the last case of Lemma 2.2 together with (3.3) to obtain

$$|H_{2,1}(F_f/2)| \leq \frac{(1-\alpha)^2\cos^2\beta p_1(1-p_1^2)}{3}\left(|C|+|A|\right)\sqrt{1-\frac{B^2}{4AC}} = t(p_1),$$

where

$$t(x) := \frac{(1-\alpha)^2\cos^2\beta(3-2x^2)}{6\sqrt{3+x^2}}.$$

Observe that

$$t'(x) = -\frac{(1-\alpha)^2\cos^2\beta(15x+2x^3)}{6(3+x^2)^{3/2}} < 0, \quad s_1 < x < 1,$$

Thus, the function t is decreasing on $s_1 < x < 1$ which yields

$$t(x) \leq t(s_1) = (1-\alpha)^2\cos^2\beta\frac{5-\sqrt{7}}{3\sqrt{8+2\sqrt{7}}} \leq \frac{(1-\alpha)^2\cos^2\beta}{4},$$

for $s_1 < x < 1$. Summarizing parts from Case 1-3, it follows the desired inequality (3.1).

We now proceed to prove the equality part. Consider the function

$$f_1(z) = \frac{z}{(1-z^2)^{(1-\alpha)\cos\beta e^{i\beta}}}, \quad z \in \mathbb{D}.$$

A simple calculation shows that f_1 belongs to $\mathcal{S}_\beta(\alpha)$. The coefficients of f_1 are $a_2 = 0$ and $a_3 = (1-\alpha)\cos\beta e^{i\beta}$. Then from (1.4) we see that the inequality (3.1) is sharp for f_1. This completes the proof. □

For the special case $\beta = 0$, we get the following sharp result for the class of starlike functions of order α:

Corollary 3.2. *Let $f \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$. Then we have*

$$|H_{2,1}(F_f/2)| \leq \frac{(1-\alpha)^2}{4}.$$

The inequality is sharp.

For $\alpha = 0$ and $\beta = 0$, we obtain the estimate for the class $\mathcal{S}^*$ of starlike function.

Corollary 3.3. *Let $f \in \mathcal{S}^*$. Then we have*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{4}.$$

The equality holds for the rotation of the Koebe function.

3.2. **The class $\mathcal{G}(\nu)$.**

Recall that a function $f \in \mathcal{A}$ is said to be *locally univalent function* at a point $z \in \mathbb{D}$ if it is univalent in some neighborhood of z; equivalently $f'(z) \neq 0$. Let $\mathcal{LU}$ denote the subclass of $\mathcal{A}$ consisting of all locally univalent functions; namely, $\mathcal{LU} = \{f \in \mathcal{A} : f'(z) \neq 0, z \in \mathbb{D}\}$. A family $\mathcal{G}(\nu)$, $\nu > 0$, of functions $f \in \mathcal{LU}$ is defined by

$$\mathcal{G}(\nu) = \left\{ f \in \mathcal{LU} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\nu}{2} \right\}.$$

The class $\mathcal{G} := \mathcal{G}(1)$ was first introduced by Ozaki [20] and proved the inclusion relation $\mathcal{G} \subset \mathcal{S}$. The Taylor coefficient problem for the class $\mathcal{G}(\nu)$, $0 < \nu \leq 1$, is discussed in [19]. Recently, the radius of convexity for functions in the class $\mathcal{G}(\nu)$, $\nu > 0$, is obtained in [13]. The class $\mathcal{G}(\nu)$, with special choices of the parameter ν, has also been considered by many researchers in the literature for different purposes; see for instance [13, 25, 27].

Next, we obtained the following sharp bound of $|H_{2,1}(F_f/2)|$ for $f \in \mathcal{G}(\nu)$.

Theorem 3.4. *Let $0 < \nu \leq 1$. If $f \in \mathcal{G}(\nu)$ given by (1.1), then*

$$|H_{2,1}(F_f/2)| \leq \frac{\nu^2(\nu^2 + 12\nu - 44)}{192(\nu^2 + 8\nu - 32)}.$$

The inequality is sharp.

Proof. Since $f \in \mathcal{G}(\nu)$, then there exist a Carathéodory function p of the form

$$p(z) = \frac{1}{\nu}\left(\nu - \frac{2zf''(z)}{f'(z)}\right).$$

It is equivalent to write

$$\nu(p(z) - 1)f'(z) = -2zf''(z). \tag{3.5}$$

By using the Taylor series representations for functions f and p and equating the coefficients of z, z^2, and z^3 in (3.5), we obtain

$$a_2 = \frac{\nu c_1}{4}, a_3 = \frac{\nu(\nu c_1^2 - 2c_2)}{24}, \text{ and } a_4 = \frac{\nu(6\nu c_1 c_2 - 8c_3 - \nu^2 c_1^2)}{192}.$$

By substituting the above expression for a_2, a_3, and a_4 in (1.4) and then further simplification gives

$$\begin{aligned} H_{2,1}(F_f/2) &= \frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= \frac{\nu^2}{36864}\left(96c_1c_3 - 64c_2^2 - 8\nu c_1^2 c_2 - \nu^2 c_1^4\right). \end{aligned}$$

Noting that $\mathcal{G}(\nu)$ and $H_{2,1}(F_f/2)$ are rotationally invariant. So we can assume that c_1 is real. Thus, in view of Lemma 2.1 and writing c_1, c_2, and c_3 in terms of p_1, p_2, and p_3 we obtain

$$H_{2,1}(F_f/2) = \frac{\nu^2}{2304}\Big((-\nu^2-4\nu+8)p_1^4 + 4(4-\nu)(1-p_1^2)p_1^2p_2 - 8(2+p_1^2)(1-p_1^2)p_2^2 + 24(1-p_1^2)(1-|p_2|^2)p_1p_3\Big) \tag{3.6}$$

with $p_1 \in [0,1]$ and $p_2, p_3 \in \overline{\mathbb{D}}$.

We next divide the proof into three cases:

Case 1: If $p_1 = 1$. Then from (3.6), we obtain

$$|H_{2,1}(F_f/2)| = \frac{\nu^2(-\nu^2-4\nu+8)}{2304}.$$

Case 2: If $p_1 = 0$. Then from (3.6), we obtain

$$|H_{2,1}(F_f/2)| = \frac{16\nu^2|p_2|^2}{2304} \le \frac{\nu^2}{144}.$$

Case 3: Now let $p_1 \in (0,1)$. Then use $|p_3| \le 1$ in (3.6) to obtain

$$\begin{aligned}
&|H_{2,1}(F_f/2)| \\
&\le \frac{24\nu^2 p_1(1-p_1^2)}{2304}\left(\left|\frac{p_1^3(-\nu^2-4\nu+8)}{24(1-p_1^2)} + \frac{(4-\nu)p_1p_2}{6} - \frac{(2+p_1^2)p_2^2}{3p_1}\right| + 1 - |p_2|^2\right) \\
&= \frac{24\nu^2 p_1(1-p_1^2)}{2304}\big(|A + Bp_2 + Cp_2^2| + 1 - |p_2|^2\big),
\end{aligned}$$

where

$$A := \frac{p_1^3(-\nu^2-4\nu+8)}{24(1-p_1^2)},\ B := \frac{(4-\nu)p_1}{6},\ \text{and } C := -\frac{(2+p_1^2)}{3p_1}.$$

Since $AC < 0$, then we apply Lemma 2.2 only for the case (ii). Now we check all the conditions of case (ii).

3(a) Note that

$$-4AC\left(\frac{1}{C^2}-1\right) - B^2 = \frac{p_1^2\big(-\nu^2p_1^2 + (2\nu^2+16\nu-32)\big)}{12(p_1^2+2)} \le 0,$$

for $0 < p_1 < 1$ and $0 < \nu \le 1$. Moreover, it is easy to see that the inequality $|B| < 2(1-|C|)$ does not hold for $p_1 \in (0,1)$.

3(b) Using the above observation, we have the following inequality

$$\min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} = -4AC\left(\frac{1}{C^2}-1\right) \le B^2.$$

3(c) We now show that $|C|(|B|+4|A|)-|AB| > 0$ holds for all $\nu \in (0,1]$ and $p_1 \in (0,1)$. A simple calculation shows that

$$|C|(|B|+4|A|) - |AB| = \frac{-p_1^4(8+\nu)\nu^2 + 8p_1^2(-\nu^2-4\nu+8) + 16(4-\nu)}{144(1-p_1^2)} := g(\nu).$$

It is easily check that g is a decreasing function with respect to ν in $(0,1]$. This implies that

$$g(\nu) \geq g(1) = \frac{16+8p_1^2-3p_1^4}{48(1-p_1^2)} \geq 0.$$

3(d) Next, the inequality

$$\begin{aligned}|AB|-|C|(|B|-4|A|) &= \frac{p_1^4(\nu^3-8\nu^2-64\nu+128)+8p_1^2(-2\nu^2-9\nu+20)-16(4-\nu)}{144(1-p_1^2)}\\ &\leq 0\end{aligned}$$

is equivalent to

$$G(x^2) \leq 0, \quad \nu \in (0,1] \text{ and } x \in (0,1)$$

where

$$G(x) := x^2(\nu^3-8\nu^2-64\nu+128)+8x(-2\nu^2-9\nu+20)-16(4-\nu) \text{ and } x = p_1^2.$$

which is a quadratic polynomial. Note that the discriminant of G is given by $\Delta = 192(304-248\nu+11\nu^2+16\nu^3+\nu^4) > 0$ for $\nu \in (0,1]$. The equation $G(x) = 0$ has following two solutions:

$$x_1 := \frac{-4(-2\nu^2-9\nu+20)-4\sqrt{3(304-248\nu+11\nu^2+16\nu^3+\nu^4)}}{\nu^3-8\nu^2-64\nu+128}$$

and

$$x_2 := \frac{-4(-2\nu^2-9\nu+20)+4\sqrt{3(304-248\nu+11\nu^2+16\nu^3+\nu^4)}}{\nu^3-8\nu^2-64\nu+128}.$$

Check that $x_1 < 0$ and $x_2 > 0$ as $892-735\nu+35\nu^2+48\nu^3+3\nu^4 > 0$. Also it is easy to verify that $x_2 < 1$ since $-28672+29696\nu-2816\nu^2-2848\nu^3-8\nu^4+32\nu^5-\nu^6 < 1$. Therefore, the function $G(x)$ has the unique zero $x_2 \in (0,1)$. Hence $G \leq 0$ for $0 < x \leq x_2$ and the condition $|AB| \leq |C|(|B|-4|A|)$ in Lemma 2.2 is satisfied for $0 < p_1 \leq \sqrt{x_2}$. Therefore, Lemma 2.2 yields

$$\begin{aligned}|H_{2,1}(F_f/2)| &\leq \frac{24\nu^2 p_1(1-p_1^2)}{2304}(-|A|+|B|+|C|)\\ &= \frac{\nu^2}{2304}\left((\nu^2+8\nu-32)p_1^4+(8-4\nu)p_1^2+16\right) =: \phi(p_1)\\ &\leq \phi(s_2) = \frac{\nu^2(\nu^2+12\nu-44)}{192(\nu^2+8\nu-32)} \qquad (3.7)\end{aligned}$$

where

$$s_2 := \sqrt{\frac{2(\nu-2)}{\nu^2+8\nu-32}} \qquad (3.8)$$

is the critical point of ϕ and gives the maximum value. A more involved computation shows that $s_2 < \sqrt{x_2}$.

3(e) Next consider the case $\sqrt{x_2} \le p_1 < 1$ and use the last case of the Lemma 2.2

$$|H_{2,1}(F_f/2)| \le \frac{24\nu^2 p_1(1-p_1^2)}{2304}(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}}$$

$$= \frac{\nu^2}{2304}\left(16-8p_1^2-p_1^4\nu(\nu+4)\right)\sqrt{\frac{3(p_1^2\nu^2+\nu^2+8\nu-16)}{2(2+p_1^2)(\nu^2+4\nu-8)}} =: \psi(p_1). \tag{3.9}$$

Since

$$\psi'(x) = \frac{-\nu^2 x}{2304(2+x^2)^2(\nu^2+4\nu-8)}\sqrt{\frac{3(2+x^2)(\nu^2+4\nu-8)}{2(x^2\nu^2+\nu^2+8\nu-16)}} \times \Big(16(-48+24\nu+\nu^2)$$
$$+8x^2(-16-56\nu+23\nu^2+12\nu^3+\nu^4)+x^4\nu(-192+64\nu+76\nu^2+13\nu^3)+$$
$$+4x^6\nu^3(4+\nu)\Big)$$
$$<0,$$

so ψ is decreasing in the interval $[\sqrt{x_2},1)$. Therefore $\psi(p_1) \le \psi(\sqrt{x_2})$ and equation (3.9) leads to

$$|H_{2,1}(F_f/2)| \le \psi(\sqrt{x_2}) = \frac{-16\sqrt{3}\nu^2(\nu^2+4\nu-8)(a+b\sqrt{c})}{2304d^2}\sqrt{\frac{e+4\nu^2\sqrt{c}}{f+2(\nu^2+4\nu-8)\sqrt{c}}},$$

where

$$\begin{aligned}
a &:= 3\nu^4+54\nu^3+18\nu^2-984\nu+1344\\
b &:= 2\nu^2+10\nu-16\\
c &:= 3(304-248\nu+11\nu^2+16\nu^3+\nu^4)\\
d &:= 128-64\nu-8\nu^2+\nu^3\\
e &:= -2048+2048\nu-336\nu^2-108\nu^3+8\nu^4+\nu^5\\
f &:= (\nu^3-4\nu^2-46\nu+88)(\nu^2+4\nu-8).
\end{aligned}$$

A lengthy calculation shows that

$$\psi(\sqrt{x_2}) = \phi(\sqrt{x_2}).$$

Now with the help of last equality along with equations (3.7) and (3.9), we deduce that

$$|H_{2,1}(F_f/2)| \le \psi(p_1) \le \psi(\sqrt{x_2}) \le \phi(s_2) = \frac{\nu^2(\nu^2+12\nu-44)}{192(\nu^2+8\nu-32)}.$$

Thus combining all the above cases 1-3, we find the desired inequality.

To prove the equality part, consider the function

$$p_2(z) = \frac{1-z^2}{1-2s_2z+z^2}$$

is in the class $\mathcal{P}$ follows from Lemma 2.1. Here s_2 is defined by (3.8). For given $p_2 \in \mathcal{P}$, we recall from (3.5) that the function $f_2 \in \mathcal{G}(\nu)$ with

$$a_2 = -\frac{\nu s_2}{2}, \; a_3 = \frac{\nu(1+(\nu-2)s_2^2)}{6}, \text{ and } a_4 = -\frac{\nu(\nu-2)s_2(3+(\nu-4)s_2^2)}{24}.$$

Hence

$$|H_{2,1}(F_f/2)| = \frac{\nu^2(\nu^2+12\nu-44)}{192(\nu^2+8\nu-32)}.$$

This completes the proof of Theorem 3.4. □

3.3. The class $\mathcal{F}_0(\lambda)$.

Let $f \in \mathcal{A}$ be a locally univalent. Then, according to Kaplan's theorem, it follows that f is *close-to-convex* if, and only if,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta},$$

for each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$. If a locally univalent analytic function f defined in $\mathbb{D}$ satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad \text{for } z \in \mathbb{D},$$

then by the Kaplan characterization it follows easily that f is close-to-convex in $\mathbb{D}$, and hence f is univalent in $\mathbb{D}$. This generates the following subclass of the class of close-to-convex (univalent) functions:

$$\mathcal{C}(-1/2) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \text{ for } z \in \mathbb{D} \right\}.$$

Functions in $\mathcal{C}(-1/2)$ are not necessarily starlike but is convex in some direction. Other related results for $f \in \mathcal{C}(-1/2)$ were also obtained in [4, 24]. Robertson [28] considered the following generalization of $\mathcal{C}(-1/2)$ for $-1/2 < \lambda \le 1/2$. The class $\mathcal{F}(\lambda)$, defined for $-1/2 < \lambda \le 1$ by

$$\mathcal{F}(\lambda) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \lambda \text{ for } z \in \mathbb{D} \right\}.$$

We note that $\mathcal{F}(1) = \mathcal{C}(-1/2)$. Moreover, $\mathcal{F}(1/2) =: \mathcal{C}$ is the usual class of convex functions. Functions in $\mathcal{F}(\lambda)$ are close-to-convex for $1/2 \le \lambda \le 1$ but contain non-starlike functions for all $1/2 < \lambda \le 1$ (see [21]). The class $\mathcal{F}(\lambda)$ has also been considered for the restriction $1/2 \le \lambda \le 1$, denote by $\mathcal{F}_0(\lambda)$, and further extensively studied in this regards, we refer to [3].

In the next theorem, we will discuss about the sharp bound for $|H_{2,1}(F_f/2)|$ when the functions f runs over the class $\mathcal{F}_0(\lambda)$.

Theorem 3.5. *Let $f \in \mathcal{F}_0(\lambda)$, for $1/2 \le \lambda \le 1$, given by* (1.1)*. Then*

$$|H_{2,1}(F_f/2)| \le \frac{(2\lambda+1)^2(12\lambda^2-60\lambda-165)}{576(4\lambda^2-12\lambda-39)}.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{F}_0(\lambda)$ be of the form (1.1). Then there exists $p \in \mathcal{P}$ of the form (2.1) such that

$$p(z) = \frac{2}{2\lambda+1}\left(\frac{zf''(z)}{f'(z)} + \frac{2\lambda+1}{2}\right), \quad z \in \mathbb{D}. \tag{3.10}$$

Substituting the series (1.1) and (2.1) into (3.10) and equating the coefficients we obtain

$$a_2 = \frac{2\lambda+1}{4}c_1,$$
$$a_3 = \frac{2\lambda+1}{24}(2c_2 + (2\lambda+1)c_1^2),$$

and

$$a_4 = \frac{(2\lambda+1)}{192}(8c_3 + 6(2\lambda+1)c_1c_2 + c_1^3(2\lambda+1)^2).$$

Since the class $\mathcal{F}_0(\lambda)$ and the functional $H_{2,1}(F_f/2)$ are rotationally invariant, without loss of generality we may assume that $c_1 \in [0, 2]$. Hence by substituting a_2, a_3, and a_4 in (1.4), and Lemma 2.1, we obtain

$$H_{2,1}(F_f/2) = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \tag{3.11}$$
$$= \frac{(2\lambda+1)^2}{36864}\left(96c_1c_3 + 8(2\lambda+1)c_1^2c_2 - (2\lambda+1)^2c_1^4 - 64c_2^2\right)$$
$$= \frac{(2\lambda+1)^2}{2304}\Big((-4\lambda^2+4\lambda+11)p_1^4 + 4(2\lambda+5)(1-p_1^2)p_1^2p_2$$
$$- 8(p_1^2+2)(1-p_1^2)p_2^2 + 24(1-p_1^2)(1-|p_2|^2)p_1p_3\Big). \tag{3.12}$$

The following three possibilities arise:

Case 1: If $p_1 = 1$. Then by (3.11) we have

$$|H_{2,1}(F_f/2)| = \frac{(2\lambda+1)^2}{2304}((-4\lambda^2+4\lambda+11)p_1^4).$$

Case 2: If $p_1 = 0$. Then by (3.11) we have

$$|H_{2,1}(F_f/2)| = \frac{(2\lambda+1)^2}{144}|p_2|^2 \le \frac{(2\lambda+1)^2}{144}.$$

Case 3: Let $p_1 \in (0, 1)$. Since $|p_3| \le 1$, from (3.11) it follows that

$$|H_{2,1}(F_f/2)|$$
$$\le \frac{(2\lambda+1)^2p_1(1-p_1^2)}{96}\left(\left|\frac{(-4\lambda^2+4\lambda+11)p_1^3}{24(1-p_1^2)} + \frac{(2\lambda+5)p_1p_2}{6} - \frac{(2+p_1^2)p_2^2}{3p_1}\right| + 1 - |p_2|^2\right)$$
$$= \frac{(2\lambda+1)^2p_1(1-p_1^2)}{96}(|A + Bp_2 + Cp_2^2| + 1 - |p_2|^2)$$

where

$$A=\frac{(-4\lambda^2+4\lambda+11)p_1^3}{24(1-p_1^2)}, B=\frac{(2\lambda+5)p_1}{6}, \text{ and } C=-\frac{(2+p_1^2)}{3p_1}.$$

We note that $AC<0$. so we can apply case (ii) of Lemma 2.2. Now we check all the conditions of case (ii).

3(a) Note that

$$-4AC\left(\frac{1}{C^2}-1\right)-B^2=-\frac{2(-4\lambda^2+4\lambda+11)(4-p_1^2)+(2+p_1^2)(2\lambda+5^2)}{36(p_1^2+2)}\leq 0.$$

But the inequality $|B|<2(1-|C|)$ is equivalent to $(2\lambda+9)p_1^2-12p_1+8<0$ which is not true for $p_1\in(0,1)$ and $\lambda\in[1/2,1]$.

3(b) For $0<p_1<1$ and $1/2\leq\lambda\leq 1$, an easy computation shows that

$$\min\left\{4(1+|C|)^2,-4AC\left(\frac{1}{C^2}-1\right)\right\}=-4AC\left(\frac{1}{C^2}-1\right)\leq B^2.$$

The last inequality follows from 3(a).

3(c) We first show that $|C|(|B|+4|A|)-|AB|>0$ holds for all $0<p_1<1$ and $1/2\leq\lambda\leq 1$. A simple calculation shows that

$$|C|(|B|+4|A|)-|AB|=\frac{H(p_1)}{144(1-p_1^2)},$$

where

$$H(x):=32\lambda+8(17+6\lambda-8\lambda^2)x^2+8\lambda^3x^4+[80-(20\lambda^2+26\lambda+7)x^4].$$

It is easily seen that $H(x)>0$ for $x\in[0,1]$ and $\lambda\in[1/2,1]$.

3(d) Next we compute

$$|AB|-|C|(|B|-4|A|)=\frac{J(p_1)}{144(1-p_1^2)},$$

where

$$J(x):=(-8\lambda^3-44\lambda^2+90\lambda+183)x^4+8(-8\lambda^2+10\lambda+27)x^2-16(2\lambda+5).$$

The discriminant of the quadratic equation $J(x)=0$ is given by

$$\Delta=768(137+113\lambda-31\lambda^2-24\lambda^3+4\lambda^4)\geq 0$$

and it has following two solutions

$$y_1:=\frac{-8(-8\lambda^2+10\lambda+27)-\sqrt{\Delta}}{2(-8\lambda^3-44\lambda^2+90\lambda+183)}\leq 0$$

and

$$y_2:=\frac{-8(-8\lambda^2+10\lambda+27)+\sqrt{\Delta}}{2(-8\lambda^3-44\lambda^2+90\lambda+183)}\geq 0.$$

A simple computation shows that $y_2 < 1$. Hence, it follows that

$$\begin{cases} |AB| \le |C|(|B|-4|A|), & \text{for } 0 < p_1 \le \sqrt{y_2}, \\ |AB| \ge |C|(|B|-4|A|), & \text{for } \sqrt{y_2} \le p_1 < 1. \end{cases}$$

Therefore by Lemma 2.2, we obtain

$$|H_{2,1}(F_f/2)| \le \frac{(\lambda+1/2)^2 p_1(1-p_1^2)}{24}(-|A|+|B|+|C|) = h(p_1),$$

where

$$h(x) := \frac{(2\lambda+1)^2}{2304}\Big((4\lambda^2-12\lambda-39)x^4 + 4(2\lambda+3)x^2 + 16\Big).$$

If $1/2 \le \lambda \le 1$, we have $h'(s_3) = 0$, where

$$s_3 := \sqrt{\frac{-2(2\lambda+3)}{4\lambda^2-12\lambda-39}} \tag{3.13}$$

and we note that $s_3 \le \sqrt{y_2}$. Since $h''(s_3) < 0$, we have

$$|H_{2,1}(F_f/2)| \le h(p_1) \le h(s_3) = \frac{(2\lambda+1)^2(12\lambda^2-60\lambda-165)}{576(4\lambda^2-12\lambda-39)}. \tag{3.14}$$

3(e) Next we consider $\sqrt{y_2} \le p_1 < 1$ in order to complete the proof. Then, by Lemma 2.2, we have

$$\begin{aligned} &|H_{2,1}(F_f/2)| \\ &\le \frac{(\lambda+1/2)^2 p_1(1-p_1^2)}{24}(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} \\ &= \frac{(2\lambda+1)^2(16-8p_1^2+(-4\lambda^2+4\lambda+3)p_1^4)}{2304}\sqrt{\frac{3((23+12\lambda-4\lambda^2)-(1+2\lambda)^2p_1^2)}{2(2+p_1^2)(-4\lambda^2+4\lambda+11)}} \\ &=: T(p_1). \end{aligned}$$

By differentiating T, we obtain

$$T'(x) = \frac{(2\lambda+1)^2 x}{2304(2+x^2)^2(11+4\lambda-4\lambda^2)}\sqrt{\frac{3(2+p_1^2)(-4\lambda^2+4\lambda+11)}{2((23+12\lambda-4\lambda^2)-(1+2\lambda)^2x^2)}}K(x,\lambda)$$

where

$$\begin{aligned} K(x,\lambda) := {} & 16(-71-44\lambda+4\lambda^2) + 32x^2(13+35\lambda-7\lambda^2-16\lambda^3+4\lambda^4) \\ & + x^4(193+288\lambda-344\lambda^2-192\lambda^3+208\lambda^4) + 4x^6(-3+2\lambda)(1+2\lambda)^3. \end{aligned}$$

Now differentiate K with respect to x, we get

$$\begin{aligned} \frac{\partial K(x,\lambda)}{\partial x} = {} & 64x(35\lambda-7\lambda^2-16\lambda^3) + 4x^3(193+288\lambda-344\lambda^2-192\lambda^3 \\ & + 208\lambda^4) + 8\big[8x(13+4\lambda^4) + 3x^5(-3+2\lambda)(1+2\lambda)^3\big] > 0 \end{aligned}$$

which implies that $K(x,\lambda)$ is increasing with respect to x and hence

$$K(x,\lambda) \leq K(1,\lambda) = (4\lambda^2-4\lambda-11)(100\lambda^2-76\lambda+49) < 0.$$

The above observation shows that $T'(x) < 0$. Therefore, T is decreasing with respect to $x \in [\sqrt{y_2}, 1)$. Hence

(3.15)

$$T(p_1) \leq T(\sqrt{y_2}) = \frac{32(11+4\lambda-4\lambda^2)(2\lambda+1)^2(a+b\sqrt{c})}{2304\, d^2}\sqrt{\frac{e-24(1+2\lambda)^2\sqrt{c}}{f+16(-4\lambda^2+4\lambda+11)\sqrt{c}}},$$

where

$$\begin{aligned} a &:= 2295+1740\lambda-504\lambda^2-336\lambda^3+48\lambda^4 \\ b &:= -48-24\lambda+16\lambda^2 \\ c &:= 3(137+113\lambda-31\lambda^2-24\lambda^3+4\lambda^4) \\ d &:= -8\lambda^3-44\lambda^2+90\lambda+183 \\ e &:= 3(4317+4738\lambda-104\lambda^2-1040\lambda^3-48\lambda^4+32\lambda^5) \\ f &:= 4(-11-4\lambda+4\lambda^2)(-129-70\lambda+28\lambda^2+8\lambda^3). \end{aligned}$$

A tedious computations show that

$$h(\sqrt{y_2}) = T(\sqrt{y_2}),$$

for each $\lambda \in [1/2, 1]$. Therefore (3.15) together with (3.14) leads to

$$T(\sqrt{y_2}) \leq h(s_3) = \frac{(2\lambda+1)^2(12\lambda^2-60\lambda-165)}{576(4\lambda^2-12\lambda-39)}.$$

Summarizing, from parts 1-3 it follows that

$$|H_{2,1}(F_f/2)| \leq \frac{(2\lambda+1)^2(12\lambda^2-60\lambda-165)}{576(4\lambda^2-12\lambda-39)}.$$

We now show that the above inequality is sharp by constructing extreme function. Consider the function p_3 of the form

$$p_3(z) = \frac{1-z^2}{1-2s_3z+z^2} = 1+2s_3z+(4s_3^2-2)z^2+(8s_3^3-6s_3)z^3+\cdots$$

with s_3 given by (3.13) and it belongs to the class $\mathcal{P}$ follows from Lemma 2.1. The corresponding function f_3 can be obtain from (3.10) and coefficients of f_3 are given by

$$a_2 = \frac{(2\lambda+1)s_3}{2},\ a_3 = \frac{(2\lambda+1)((3+2\lambda)s_3^2-1)}{6},$$

and,

$$a_4 = \frac{(2\lambda+1)(2\lambda+3)((2\lambda+5)s_3^2-3)s_3}{24}.$$

From (1.4), it is clear that inequality is sharp for f_3. This completes the proof of the theorem. □

Choosing $\lambda = 1/2$ and $\lambda = 1$, we deduce the following sharp inequalities:

Corollary 3.6. *If $f \in \mathcal{C}$, then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{33}.$$

If $f \in \mathcal{C}(-1/2)$ given by (1.1), *then*

$$|H_{2,1}(F_f/2)| \leq \frac{213}{3008}.$$

The inequalities are sharp.

Acknowledgement. The first author thank SERB-CRG and the second author thank IIT Bhubaneswar for providing Institute Post Doctoral fellowship. The authors thank the referee for careful reading and helpful comments for improving this paper.

Compliance of Ethical Standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. M. F. Ali and A. Vasudevarao, On logarithmic coefficients of some close-to-convex functions, *Proc. Amer. Math. Soc.* **146** (2018), 1131–1142.
2. M. F. Ali and A. Vasudevarao, On logarithmic coefficients of some close-to-convex functions, *Bull. Aust. Math. Soc.* **97**(2) (2018), 253–264.
3. V. Allu, A. Lecko, and D. K. Thomas, Hankel, Toeplitz and Hermitian-Toeplitz Determinants for Ozaki Close-to-convex Functions, *Mediterr. J. Math.* **19** (2022) Paper No. 22, 17 pp.
4. V. Arora and S. K. Sahoo, Meromorphic functions with small Schwarzian derivative, *Stud. Univ. Babe-Bolyai Math.* **63** (3) (2018), 355–370.
5. L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), 137–152.
6. N. E. Cho, Y. C. Kim, and T. Sugawa, A general approach to the Fekete-Szegö problem, *J. Math. Soc. Japan.* **59** (2007), 707–727.
7. P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
8. D. Girela, Logarithmic coefficients of univalent functions, *Ann. Acad. Sci. Fenn. Math.* **25** (2000), 337–350.
9. A. W. Goodman, Univalent functions, Vols. 1–2, Mariner Publishing Co., Tampa, FL, 1983.
10. W. K. Hayman, On the second Hankel determinant of mean univalent functions, *Proc. London Math. Soc.* **18** (1968), 77–94.
11. B. Kowalczyk and A. Lecko, Second hankel determinant of logarithmic coefficients of convex and starlike functions, *Bull. Aust. Math. Soc.* DOI: 10.1017/S0004972721000836.
12. S. Kumar and S. K. Sahoo, Preserving properties and pre-Schwarzian norms of nonlinear integral transforms, *Acta Math. Hungar.* **162** (2020), 84–97.
13. S. Kumar and S. K. Sahoo, Radius of convexity for integral operators involving Hornich operations, *J. Math. Anal. Appl.* **502** (2) (2021), 125265.

14. U. P. Kumar and A. Vasudevarao, Logarithmic coefficients for certain subclasses of close-to-convex functions, *Monatsh. Math.* **187** (2018), 543–563.
15. R. J. Libera, Univalent α-spiral functions, *Canad. J. Math.* **19** (1967), 449–456.
16. R. J. Libera and E. J. Złotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* **85** (1982), 225–230.
17. R. J. Libera and E. J. Złotkiewicz, Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$, *Proc. Amer. Math. Soc.* **87** (1983), 251–257.
18. I. M. Milin, Univalent Functions and Orthonormal Systems, Izdat. "Nauka", Moscow, 1971 (in Russian); English transl. American Mathematical Society, Providence (1977).
19. M. Obradovic, S. Ponnusamy, and K.-J. Wirths, Coefficient characterizations and sections for some univalent functions, *Sib. Math. J.* **54**(1) (2013), 679–696.
20. S. Ozaki, On the theory of multivalent functions. II, *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A.* **4** (1941), 45–87.
21. J. A. Pfaltzgraff, M. O. Reade, and T. Umezawa, Sufficient conditions for univalence, *Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math.-Phys.* **2**(2) (1976), 211–218.
22. C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.* **41** (1966), 111–122.
23. C. Pommerenke, On the Hankel determinants of Univalent functions, *Mathematika* **14** (1967), 108–112.
24. S. Ponnusamy, S. K. Sahoo, and H. Yanagihara, Radius of convexity of partial sums of functions in the close-to-convex family, *Nonlinear Anal.* **95** (2014), 219–228.
25. S. Ponnusamy and V. Singh, Univalence of certain integral transforms, *Glas. Mat. Ser. III* **31**(2) (51) (1996), 253–261.
26. S. Ponnusamy and T. Sugawa, Sharp inequalities for logarithmic coefficients and their applications, *Bull. Sci. Math.* **166** (2021), 23 pp, Article 102931.
27. S. Ponnusamy and A. Vasudevarao, Region of variability of two subclasses of univalent functions, *J. Math. Anal. Appl.* **332**(2) (2007), 1323–1334.
28. M. S. Robertson, On the theory of univalent functions, *Ann. Math. (2)* **37**(2) (1936), 374–408.
29. Y.J. Sim, A. Lecko, and D.K. Thomas, The second Hankel determinant for strongly convex and Ozaki closetoconvex functions, *Ann. Mat. Pura Appl.* **200**(6) (2021), 2515–2533.
30. L. ŠPAČEK, Contribution à la théorie des fonctions univalentes (in Czech), *Časop Pěst. Mat.-Fys.* **62** (1933), 12–19.
31. P. Zaprawa, Initial logarithmic coefficients for functions starlike with respect to symmetric points, *Bol. Soc. Mat. Mex.* **27** (2021)(3), pp. 62, 13 pp.

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