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QUANTUM FRACTIONAL ORNSTEIN-UHLENBECK SEMIGROUPS AND ASSOCIATED POTENTIALS

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ABSTRACT. Using an infinite dimensional nuclear space, we introduce the quantum fractional number operator (QFNO) and the associated quantum fractional Ornstein-Uhlenbeck (O-U) semigroups. Then, we solve the Cauchy problems associated with the QFNO and we show that its solutions can be expressed in terms of the aforementioned semigroups. Besides, we prove that the quantum fractional O-U semigroups satisfy the Feller property. Finally, using an adequate definition of the quantum fractional potentials, we give the solutions of the quantum fractional Poisson equations.

1. Introduction

If we go back to the story of the study of fractional powers of operators, we may mention the Abel's work on the tautochrone, the Riemann-Liouville integral and its generalisations by M. Riesz though its general history is recently developed. Hille and Philips tried to prove that one can treated the fractional powers in the framework of an operational calculus which they originated when $-\mathscr{N}$ is the negative of an infinitesimal generator of a bounded semi-group of operators. In 1959, Balakrishmann [1] accomplished this program thoroughly then, in 1960 [2], made a revolution by giving a new definition and extended his theory to a wider class of operators. Meanwhile Krasnoselskii-Sobolevskii [17] and Kato [16] introduced two different definitions which resulted in further results. The theories of Yosida [31], kato [15] and Watanabe [30] also some other classical results on the Riemann-Liouville integral (Hardy-Littleword [12], Love-Young [19]) are reconstructed from a unified point of view. Later, Piech [26] initiated the infinite dimensional analogue of

$$N_d = \sum_{k=1}^{k=d} \left(\frac{\partial}{\partial x_k}\right)^* \left(\frac{\partial}{\partial x_k}\right)$$

on infinite dimensional abstract Wiener space called the number operator N which is given as follows:

$$N = \int_{\mathbb{R}^2} \tau(s,t) a_s^* a_t ds dt,$$

where a_s^* and a_t denote the creation and annihilation operators, respectively, and τ is the usual trace operator. Moreover, he gave the solution of the Cauchy problem associated to the number operator using the semigroup approach. After that, in [18], Kuo formulated the number operator and the O-U semigroup as a continuous linear operator acting on the space of test white noise functionals. Some results about the quantum number operators and their associated quantum O-U semigroups

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was extended in [28]. Moreover, the author showed that such semigroups are Markovian. Later, in [27], the author studied the fractional number operator and gave a probabilistic relation with it as well as some regularity properties about it. Moreover, many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of fractional order (see [5, 14, 23, 22]). The term potential has its origins from mathematical physics and particularly from gravitational problems and electrostatic where the fundamental forces, closely related to the potential energy such as electrostatic or gravity, were given as the gradients of potentials, i.e., equations which satisfy Laplace equations and more generally the Poisson equations. The latter is still in use in many engineering applications and physics fields like electrostatics and heat-conduction. In [29], the quantum potentials are introduced and used to solve the quantum Poisson equations. On the other hand, the Feller property plays an important role for PDE'S to ensure the uniqueness of the invariant measurement for a semigroup $(P_t)_{t\geq 0}$, see [6, 7, 21, 20] and the references therein.

Motivated by the above description, for $\alpha \in (0,1)$:

- We define the QFNO (denoted by $\widetilde{\mathcal{N}}^{\alpha}$, $\widetilde{\mathcal{N}_1}^{\alpha}$ and $\widetilde{\mathcal{N}_2}^{\alpha}$) on the space of entire functions with a certain exponential growth condition with two variables.
- We prove that the associated Cauchy problems can be expressed in terms of the quantum fractional O-U semigroups.
- We show that the quantum fractional O-U semigroups satisfy the Feller property.
- We show that the solutions of the quantum fractional Poisson equations are expressed in terms of the quantum fractional potentials.

From the above mentioned papers, in Section 2, we introduce some preliminary results on nuclear algebras of entire functions with two variables. In Section 3, we study some regularity properties about QFNO $\widetilde{\mathcal{N}}^{\alpha}$, left QFNO $\widetilde{\mathcal{N}_1}^{\alpha}$ and right QFNO $\widetilde{\mathcal{N}_2}^{\alpha}$. In Section 4, we construct continuous semigroups with infinitesimal generators $-\widetilde{\mathcal{N}}^{\alpha}$, $-\widetilde{\mathcal{N}_1}^{\alpha}$ and $-\widetilde{\mathcal{N}_2}^{\alpha}$. Then, we study the Cauchy problem associated with the QFNO. Moreover, we prove that the quantum fractional O-U semigroups satisfy the Feller property. In Section 5, we introduce the quantum fractional potentials associated to quantum fractional O-U semigroups and we give the solutions of the quantum fractional Poisson equations.

2. Preliminaries

In this section we shall give some preliminary results which will be used in the sequel. The involvement of these results is stated with reference to [3, 4, 8, 9, 10, 13, 24, 25]. For $i=1,2$, let H_i be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_i$ and norm $|\cdot|_{i,0}$ and let a positive self-adjoint operator denoted $s_i \geq 1$ on H_i such that

$$s_i e_n = \lambda_{i,n} e_{i,n}, \quad n=0,1,2,...,$$

where $\{e_{i,n}\}_{n=1}^{\infty}$ is a complete orthogonal basis for H_i and $\{\lambda_{i,n}\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers satisfying

$$\sum_{n=1}^{\infty} \lambda_{i,n}^{-2} = \left\|s_i^{-1}\right\|_{HS}^{2} < \infty.$$

For $r \in \mathbb{N}$, we define

$$|\xi|_{i,r}^2 := \sum_{n=1}^{\infty} \langle \xi, e_{i,n} \rangle_i^2 \lambda_{i,n}^{2r} = |s_i^r \xi|_{i,0}^2, \quad \xi \in H_i.$$

In fact, let $\{(X_i)_r\}_{r=1}^{\infty}$ a decreasing chain of Hilbert spaces ($(X_i)_q \subseteq (X_i)_r$ for $r \le q$) with norm $|\cdot|_{i,p}$ and with natural continuous inclusions

$$i_{q,r} : (X_i)_q \hookrightarrow (X_i)_r, \; r \le q.$$

Besides, $(X_i)_{-r}$ the $|\cdot|_{i,-r}$-completion of H_i ($p \ge 0$) satisfies

$$(X_i)_{-r} \subseteq (X_i)_{-q}, \; 0 \le r \le q.$$

Defining the countably nuclear space:

$$X_i := proj \lim_{r \to \infty} (X_i)_r.$$

Then, we obtain the real standard Gel'fand triplet:

$$X_i \subset H_i \equiv H_i' \subset X_i'. \tag{2.1}$$

For more details, we refer the reader to [11]. Let $(E_i)_{r \in \mathbb{N}}$ be the complexification of $(X_i)_r$, $i \in \{1,2\}$. We denote by $(E_i)_{-r}$ and E_i' the strong dual space of $(E_i)_r$ and E_i, respectively. Then, we get

$$E_i = proj \lim_{r \to \infty} (E_i)_r \; and \; E_i' = ind \lim_{r \to \infty} (E_i)_{-r}.$$

We denote by $\langle ., . \rangle_i$ the $\mathbb{C}-$bilinear form on $E_i' \times E_i$, $i \in \{1,2\}$. Let $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a Young function (see [10]). The polar function θ^* of θ given by

$$\theta^*(x) = \sup_{s \ge 0} (sx - \theta(s)), \;\; x \ge 0,$$

is also a Young function. Now, we define $(\theta)_n$ as follows:

$$(\theta)_n \;=\; \inf_{x>0} \frac{e^{\theta(x)}}{x^n}, \;\; \text{for every } n \in \mathbb{N}. \tag{2.2}$$

In the sequel, we fix (θ_1, θ_2) a pair of Young function and we suppose that θ_1 and θ_2 satisfy the following condition

$$\lim_{r \to \infty} \frac{\theta_i(r)}{r^2} \;<\; \infty, i \in \{1,2\}. \tag{2.3}$$

If a Young function θ satisfies (2.3), there exist two constants numbers γ and β such that

$$(\theta)_n \le \gamma (\frac{2e\beta}{n})^{\frac{n}{2}}.$$

Let $(\mathscr{C}, \|.\|)$ be a complex Banach space and $\mathscr{H}(\mathscr{C})$ be the space of all continuous entire functions on $\mathscr{C}$. For a Young function θ we define the following spaces

$$\mathscr{F}_\theta(E_i') := proj \lim_{r \to \infty; h \downarrow 0} \left\{ g \in \mathscr{H}((E_i)_{-r}, \mathscr{C}); \; \|g\|_{\theta,h} := \sup_{z \in \mathscr{C}} |g(z)| e^{-\theta(h\|z\|)} < \infty \right\}.$$

The space of entire function on $(E_1)_{-r_1}\oplus(E_2)_{-r_2}$ of (θ_1,θ_2)-exponential growth of minimal type is defined by the following expression

$$\begin{aligned}&\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')\\ &= proj\lim_{r_1,r_2\to\infty,h_1,h_2\downarrow 0}\left\{g\in\mathscr{H}((E_1)_{-r_1}\oplus(E_2)_{-r_2});\ \|g\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1,h_2)}<\infty\right\}.\end{aligned}$$

where $\mathscr{H}((E_1)_{-r_1}\oplus(E_2)_{-r_2})$ is the space of entire functions on $(E_1)_{-r_1}\oplus(E_2)_{-r_2}$ and

$$\begin{aligned}&\|g\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1,h_2)}\\ &=\sup_{(a,b)\in(E_1)_{-r_1}\oplus(E_2)_{-r_2}}\left\{|g(a,b)|e^{-\theta_1(h_1|a|_{-r_1})-\theta_2(h_2|b|_{-r_2})}\right\}.\end{aligned}$$

In particular, we have the following identification

$$\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus\{0\})=\mathscr{F}_{\theta_1}(E_1').$$

By using the common symbol $\langle\cdot,\cdot\rangle$ for the canonical $\mathbb{C}$-bilinear form on $(E_1^{\otimes l}\otimes E_2^{\otimes k})'\times E_1^{\otimes l}\otimes E_2^{\otimes k}$, we know that every $\psi\in\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')$ admits the following Taylor expansion:

$$\psi(a,b)=\sum_{l,k=0}^{\infty}\langle a^{\otimes l}\otimes b^{\otimes k},\psi_{l,k}\rangle,\ (a,b)\in E_1'\times E_2', \tag{2.4}$$

where $\psi_{l,k}\in E_1^{\widehat{\otimes} l}\otimes E_2^{\widehat{\otimes} k}$ and $\widehat{\otimes}$ is the symmetric tensor product. Let $F_{\theta_1,\theta_2}(E_1\oplus E_2)$ be the space of all Taylor coefficients $\psi_{l,k}$ obtained from (2.4). It is known that

$$F_{\theta_1,\theta_2}(E_1\oplus E_2)=proj\lim_{r_1,r_2\to\infty;h_1,h_2\downarrow 0}F_{\theta_1,\theta_2,h_1,h_2}((E_1)_{r_1}\oplus(E_2)_{r_2}),$$

where

$$\begin{aligned}&F_{\theta_1,\theta_2,h_1,h_2}((E_1)_{r_1}\oplus(E_2)_{r_2})=\\ &\left\{\overrightarrow{\psi}=(\psi_{l,k})_{l,k\geq 0};\ \psi_{l,k}\in(E_1)_{r_1}^{\widehat{\otimes} l}\otimes(E_2)_{r_2}^{\widehat{\otimes} k},\|\overrightarrow{\psi}\|^2_{(\theta_1,\theta_2),(r_1,r_2),(h_1,h_2)}<\infty\right\}\end{aligned}$$

and

$$\|\overrightarrow{\psi}\|^2_{(\theta_1,\theta_2),(r_1,r_2),(h_1,h_2)}=\sum_{l,k=0}^{\infty}(\theta_1)_l^{-2}h_1^{-l}(\theta_2)_k h_2^{-k}|\psi_{l,k}|^2_{r_1,r_2}.$$

Moreover, equipped with the projective limit topology, $F_{\theta_1,\theta_2}(E_1\oplus E_2)$ is a nuclear Fréchet space and is isomorphic to $\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')$. Now, let $\psi\sim(\psi_{l,k})_{l,k\geq 0}$ in $\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')$. According to [13], for each $r_1,r_2\geq 0,\ h_1,h_2>0$, there exist $q_1>r_1$ *and* $q_2>r_2$ such that

$$\begin{aligned}|\psi_{l,k}|_{r_1,r_2}\ &\leq\ \|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1,h_2)}\\ &\quad\times e^{l+k}(\theta_1)_l(\theta_2)_k h_1^l h_2^k\|i_{q_1,r_1}\|^l_{HS}\|i_{q_2,r_2}\|^k_{HS}.\end{aligned}\tag{2.5}$$

For locally convex spaces $\mathfrak{X}$ and $\mathfrak{N}$, the space of all continuous linear operator from $\mathfrak{X}$ into $\mathfrak{N}$ is denoted by $\mathscr{L}(\mathfrak{X},\mathfrak{N})$. By the nuclearity of the space $\mathscr{F}_{\theta_i}(E_i')$ ($i\in\{1,2\}$), from the kernel theorem we get the following isomorphism

$$\mathscr{L}(\mathscr{F}^*_{\theta_1}(E_1'),\mathscr{F}_{\theta_2}(E_2'))\simeq(\mathscr{F}_{\theta_1}(E_1')\otimes\mathscr{F}_{\theta_2}(E_2'))\simeq\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2') \tag{2.6}$$

where $\mathscr{F}^*_{\theta_1}(E'_1)$ is the topological dual of $\mathscr{F}_{\theta_1}(E'_1)$. Therefore, for any white noise operator $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1), \mathscr{F}_{\theta_2}(E'_2))$, there exist a unique $\Phi_\Xi \in \mathscr{F}_{\theta_1}(E'_1) \otimes \mathscr{F}_{\theta_2}(E'_2)$ such that

$$\langle\langle g, \Xi f\rangle\rangle = \langle\langle f \otimes g, \Phi_\Xi\rangle\rangle, \quad \forall f \in \mathscr{F}^*_{\theta_1}(E'_1), \quad \forall g \in \mathscr{F}^*_{\theta_2}(E'_2). \tag{2.7}$$

Let $\mathscr{K}$ be the isomorphism between $\mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1), \mathscr{F}_{\theta_2}(E'_2))$ and $\mathscr{F}_{\theta_1,\theta_2}(E'_1 \oplus E'_2)$. Then, we can identify Φ_Ξ with $\mathscr{K}(\Xi)$.

3. Quantum fractional number operators

Throughout this paper, we take $0 < \alpha < 1$. Then, we define the following operators

$$\mathscr{N}^\alpha \psi(a,b) := \sum_{l,k=0;(l,k)\neq(0,0)}^{\infty} (l^\alpha + k^\alpha)\langle a^{\otimes l} \otimes b^{\otimes k}, \psi_{l,k}\rangle, \tag{3.1}$$

$$\mathscr{N}_1^\alpha \psi(a,b) := \sum_{l=1,k=0}^{\infty} l^\alpha \langle a^{\otimes l} \otimes b^{\otimes k}, \psi_{l,k}\rangle, \tag{3.2}$$

$$\mathscr{N}_2^\alpha \psi(a,b) := \sum_{l=0,k=1}^{\infty} k^\alpha \langle a^{\otimes l} \otimes b^{\otimes k}, \psi_{l,k}\rangle, \tag{3.3}$$

where $\psi \in \mathscr{F}_{\theta_1,\theta_2}(E'_1 \oplus E'_2)$ given by

$$\psi(a,b) = \sum_{l,k=0}^{\infty} \langle a^{\otimes l} \otimes b^{\otimes k}, \psi_{l,k}\rangle, \quad (a,b) \in E'_1 \times E'_2.$$

Proposition 1. *The operators $\mathscr{N}^\alpha$, $\mathscr{N}_1^\alpha$ and $\mathscr{N}_2^\alpha$ are linear continuous operators from $\mathscr{F}_{\theta_1,\theta_2}(E'_1 \oplus E'_2)$ into itself.*

Proof. Let $r_1, r_2 \geq 0$ and $\psi(a,b) = \sum_{l,k=0}^{\infty} \langle a^{\otimes l} \otimes b^{\otimes k}, \psi_{l,k}\rangle$ in $\mathscr{F}_{\theta_1,\theta_2}(E'_1 \oplus E'_2)$. In view of the expression of $\mathscr{N}_1^\alpha$, we deduce that

$$|\mathscr{N}_1^\alpha \psi(a,b)| \leq \sum_{l=1,k=0}^{\infty} l^\alpha \; |a|^l_{-r_1} |b|^k_{-r_2} |\psi_{l,k}|_{r_1,r_2}.$$

Furthermore, using (2.5) and the expression $l^\alpha \leq 2^{\alpha l}$, we deduce

$$|\mathscr{N}_1^\alpha \psi(a,b)| \leq \|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1,h_2)}$$

$$\times \sum_{l,k=0}^{\infty} \{2^\alpha m_1 e \|i_{q_1,r_1}\|_{HS}\}^l |a|^l_{-r_1} (\theta_1)_l \{m_2 e \|i_{q_2,r_2}\|_{HS}\}^k |b|^k_{-r_2} (\theta_2)_k,$$

where $q_1 > r_1$, $q_2 > r_2$ and $h_1, h_2 > 0$. Therefore, for $h'_1, h'_2, h_1, h_2 > 0$, $q_1 > r_1$ and $q_2 > r_2$ such that

$$\max\left\{2^\alpha \frac{h_1}{h'_1} e \|i_{q_1,r_1}\|_{HS}, \frac{h_2}{h'_2} e \|i_{q_2,r_2}\|_{HS}\right\} < 1,$$

we can easily check that

$$\|\mathscr{N}_1^\alpha \psi\|_{(\theta_1,\theta_2);(r_1,r_2);(h'_1,h'_2)} \leq c \times \|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1,h_2)},$$

where

$$c := c(r_1, r_2, q_1, q_2) = \{1 - 2^{\alpha}\frac{h_1}{h_1'}e\|i_{q_1,r_1}\|_{HS}\}^{-1}\{1 - \frac{h_2}{h_2'}e\|i_{q_2,r_2}\|_{HS}\}^{-1}.$$

Hence, we prove the continuity of $\mathcal{N}_1^{\alpha}$. Using a similar calculus, we prove the continuity of $\mathcal{N}_2^{\alpha}$ and $\mathcal{N}^{\alpha}$. □

Remark 1. *We can easily note the following decompositions*

$$\mathcal{N}_1^{\alpha} = N^{\alpha} \otimes I; \ \mathcal{N}_2^{\alpha} = I \otimes N^{\alpha} \text{ and } \mathcal{N}^{\alpha} = N^{\alpha} \otimes I + I \otimes N^{\alpha},$$

where N^{α} is the standard fractional number operator (see [27]) on $\mathcal{F}_{\theta_i}(E_i')$ given by

$$N^{\alpha}\psi(a) = \sum_{l=1}^{\infty}\langle a^{\otimes l}, l^{\alpha}\psi_l\rangle_i, \text{ for } \psi(a) = \sum_{l=0}^{\infty}\langle a^{\otimes l}, \psi_l\rangle_i \in \mathcal{F}_{\theta_i}(E_i').$$

Definition 1. *On the space of white noise operators $\mathcal{L}(\mathcal{F}_{\theta_1}^*(E_1'), \mathcal{F}_{\theta_2}(E_2'))$, we introduce the following operators $\widetilde{\mathcal{N}_1}^{\alpha} = \mathcal{K}^{-1}(\mathcal{N}_1^{\alpha})\mathcal{K}$, $\widetilde{\mathcal{N}_2}^{\alpha} = \mathcal{K}^{-1}(\mathcal{N}_2^{\alpha})\mathcal{K}$ and $\widetilde{\mathcal{N}}^{\alpha} = \mathcal{K}^{-1}\mathcal{N}^{\alpha}\mathcal{K}$.*

The operator $\widetilde{\mathcal{N}_1}^{\alpha}$ is named left QFNO, $\widetilde{\mathcal{N}_2}^{\alpha}$ is named right QFNO and $\widetilde{\mathcal{N}}^{\alpha}$ is named QFNO.

Proposition 2. *For $\Xi \in \mathcal{L}(\mathcal{F}_{\theta_1}^*(E_1'), \mathcal{F}_{\theta_2}(E_2'))$, we have*

$$\widetilde{\mathcal{N}_1}^{\alpha}(\Xi) = \Xi N^{\alpha}, \ \widetilde{\mathcal{N}_2}^{\alpha}(\Xi) = N^{\alpha}\Xi \text{ and } \widetilde{\mathcal{N}}^{\alpha}(\Xi) = \Xi N^{\alpha} + N^{\alpha}\Xi.$$

Proof. Let $\Xi \in \mathcal{L}(\mathcal{F}_{\theta_1}^*(E_1'), \mathcal{F}_{\theta_2}(E_2'))$, $f \in \mathcal{F}_{\theta_1}^*(E_1')$ and $g \in \mathcal{F}_{\theta_2}^*(E_2')$. Then, by applying the kernel theorem, we get

$$\begin{aligned}
\langle\langle g, \widetilde{\mathcal{N}_1}^{\alpha}(\Xi)f\rangle\rangle &= \langle\langle g, \mathcal{K}^{-1}(\mathcal{N}_1^{\alpha}\mathcal{K}(\Xi))f\rangle\rangle \\
&= \langle\langle f \otimes g, \mathcal{N}_1^{\alpha}\mathcal{K}(\Xi)\rangle\rangle \\
&= \langle\langle \mathcal{N}_1^{\alpha}(f \otimes g), \mathcal{K}(\Xi)\rangle\rangle \\
&= \langle\langle (N^{\alpha} \otimes I)(f \otimes g), \mathcal{K}(\Xi)\rangle\rangle \\
&= \langle\langle N^{\alpha}(f) \otimes g, \mathcal{K}(\Xi)\rangle\rangle \\
&= \langle\langle g, \Xi N^{\alpha}(f)\rangle\rangle,
\end{aligned}$$

which implies that $\widetilde{\mathcal{N}_1}^{\alpha}(\Xi) = \Xi N^{\alpha}$. Similarly, we obtain

$$\langle\langle g, \widetilde{\mathcal{N}_2}^{\alpha}(\Xi)f\rangle\rangle = \langle\langle g, N^{\alpha}\Xi f\rangle\rangle$$

to acquire $\widetilde{\mathcal{N}_2}^{\alpha}(\Xi) = N^{\alpha}\Xi$. Finally, we have

$$\widetilde{\mathcal{N}}^{\alpha}(\Xi) = \widetilde{\mathcal{N}_1}^{\alpha}(\Xi) + \widetilde{\mathcal{N}_2}^{\alpha}(\Xi) = \Xi N^{\alpha} + N^{\alpha}\Xi,$$

which completes the proof. □

4. Quantum fractional O-U semigroups with infinitesimal generator the Quantum fractional number operator

For $s,t \geq 0$, we define $Q^{\alpha}_{s,t}$ as follows:

$$Q^{\alpha}_{s,t}\psi(a,b) := \sum_{l,k=0}^{\infty} \langle a^{\otimes l} \otimes b^{\otimes k}, e^{-sl^{\alpha}-tk^{\alpha}}\psi_{l,k}\rangle, \tag{4.1}$$

where $\psi(a,b) = \sum_{l,k=0}^{\infty} \langle a^{\otimes l} \otimes b^{\otimes k},\ \psi_{l,k}\rangle \in \mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$. In the sequel, we will denote $Q^{\alpha}_{t,t}$ by Q^{α}_t.

Proposition 3. *The families $\{Q^{\alpha}_t, t \geq 0\}$, $\{Q^{\alpha}_{t,0}, t \geq 0\}$ and $\{Q^{\alpha}_{0,t}, t \geq 0\}$ are strongly continuous semigroups from $\mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$ into itself.*

Proof. First of all, it is obvious that $Q^{\alpha}_0 = I$ and $Q^{\alpha}_t Q^{\alpha}_s = Q^{\alpha}_{t+s}$, $\forall s,t \geq 0$. But we know that $|e^x - 1| \leq |x|e^{|x|}$, for all $x \in \mathbb{R}$. Therefore, for $t \leq 1$, we have

$$\begin{aligned} |Q^{\alpha}_t \psi(a,b) - \psi(a,b)| &\leq \sum_{l,k=0}^{\infty} \left(e^{-t(l^{\alpha}+k^{\alpha})} - 1\right) |a|^l_{-r_1} |b|^k_{-r_2} |\psi_{l,k}|_{r_1,r_2} \\ &\leq t \sum_{l,k=0}^{\infty} (l^{\alpha}+k^{\alpha}) e^{l^{\alpha}+k^{\alpha}} |a|^l_{-r_1} |b|^k_{-r_2} |\psi_{l,k}|_{r_1,r_2} \\ &\leq t \sum_{l,k=0}^{\infty} 2^{\alpha(l+k)} e^{l+k} |a|^l_{-r_1} |b|^k_{-r_2} |\psi_{l,k}|_{r_1,r_2}. \end{aligned}$$

Thus, for all $q_1 > r_1, q_2 > r_2$ and $h_1, h_2, h_1', h_2' > 0$ such that

$$\max\left\{2^{\alpha}\frac{h_1}{h_1'}e^2\|i_{q_1,r_1}\|_{HS},\ 2^{\alpha}\frac{h_2}{h_2'}e^2\|i_{q_2,r_2}\|_{HS}\right\} < 1,$$

we have

$$\|Q^{\alpha}_t\psi - \psi\|_{(\theta_1,\theta_2);(r_1,r_2);(h_1',h_2')} \leq t \times K\|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1',h_2')},$$

where

$$K := K(r_1,r_2,q_1,q_2) = \left\{\left(1 - 2^{\alpha}\frac{h_1}{h_1'}e^2\|i_{q_1,r_1}\|_{HS}\right)\left(1 - 2^{\alpha}\frac{h_2}{h_2'}e^2\|i_{q_2,r_2}\|_{HS}\}\right)\right\}^{-1}.$$

The last inequality implies the strong continuity of $\{Q^{\alpha}_t,\ t \geq 0\}$. Similarly, we complete the continuity of the semigroups $\{Q^{\alpha}_{s,0}, s \geq 0\}$ and $\{Q^{\alpha}_{0,t}, t \geq 0\}$. □

Also, we can easily observe that the linear operator $(Q^{\alpha}_{s,t})_{s,t\geq 0}$ is continuous from $\mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$ into itself. As a result, according to [28] and from (4.1), we introduce

$$\widetilde{Q_{s,t}}^{\alpha} := \mathscr{K}^{-1} Q^{\alpha}_{s,t} \mathscr{K} \in \mathscr{L}(\mathscr{L}(\mathscr{F}^*_{\theta_1}(E_1'), \mathscr{F}_{\theta_2}(E_2'))).$$

Lemma 1. *The operators $-\widetilde{\mathscr{N}}^{\alpha}$, $-\widetilde{\mathscr{N}_1}^{\alpha}$ and $-\widetilde{\mathscr{N}_2}^{\alpha}$ are the infinitesimal generators of the strongly continuous semigroups $\{\widetilde{Q_t}^{\alpha}, t \geq 0\}$, $\{\widetilde{Q_{t,0}}^{\alpha}, t \geq 0\}$ and $\{\widetilde{Q_{0,t}}^{\alpha}, t \geq 0\}$, respectively.*

Proof. To verify that $-\widetilde{\mathscr{N}}^{\alpha}$ is the infinitesimal generator of $\{\widetilde{Q_t}^{\alpha},\ t\geq 0\}$, it is sufficient to verify that $-\mathscr{N}^{\alpha}$ is the infinitesimal generator of $\{Q_t^{\alpha},\ t\geq 0\}$, we use the fact that

$$\left(\frac{Q_t^{\alpha}\psi-\psi}{t}+\mathscr{N}^{\alpha}\psi\right)\sim (Q_{l,k}^{\alpha})_{l,k\in\mathbb{N}},$$

where $Q_{l,k}^{\alpha}$ is given by

$$Q_{l,k}^{\alpha}=\left\{\frac{e^{-t(l^{\alpha}+k^{\alpha})}+t(l^{\alpha}+k^{\alpha})-1}{t}\right\}\psi_{l,k},\quad l,k\in\mathbb{N}.$$

Clearly, for $p_1,p_2\geq 0$ and $l,k\in\mathbb{N}$, we have

$$\left|Q_{l,k}^{\alpha}\right|_{r_1,r_2}\leq\left|\frac{e^{-t(l^{\alpha}+k^{\alpha})}-1+t(l^{\alpha}+k^{\alpha})}{t}\right||\psi_{l,k}|_{r_1,r_2}.$$

Since $|e^t-1-t|\leq t^2e^{|t|}$, for all $t\in\mathbb{R}$ and $l,k\in\mathbb{N}$, we get

$$\left|Q_{l,k}^{\alpha}\right|_{r_1,r_2}\leq |t|(l^{\alpha}+k^{\alpha})^2e^{t(l^{\alpha}+k^{\alpha})}|\psi_{l,k}|_{r_1,r_2}.$$

Furthermore, from equation (2.5) and the fact that $(l^{\alpha}+k^{\alpha})^2\leq 2^{2\alpha(l+k)}$, $l,k\in\mathbb{N}$, we obtain

$$\begin{aligned}\left|Q_{l,k}^{\alpha}\right|_{r_1,r_2} &\leq t\|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1,h_2)}(\theta_1)_l(\theta_2)_k\\ &\quad\times\left\{2^{2\alpha}h_1e^t\|i_{q_1,r_1}\|_{HS}\right\}^l\left\{2^{2\alpha}h_2e^t\|i_{q_2,r_2}\|_{HS}\right\}^k,\end{aligned}$$

where $q_1>r_1,\ q_2>r_2$ and $h_1,h_2>0$. Therefore, for $t\leq 1$, $h_1,h_2,h_1',h_2'>0,\ q_1>r_1$ and $q_2>r_2$ such that

$$\max\left\{2^{2\alpha}e^2\frac{h_1}{h_1'}\|i_{q_1,r_1}\|_{HS},\ 2^{2\alpha}e^2\frac{h_2}{h_2'}\|i_{q_2,r_2}\|_{HS}\right\}<1,$$

there exists a constant

$$C:=C(r_1,r_2,q_1,q_2)=\left\{(1-2^{2\alpha}e^2\frac{h_1}{h_1'}\|i_{q_1,r_1}\|_{HS})(1-2^{2\alpha}e^2\frac{h_2}{h_2'}\|i_{q_2,r_2}\|_{HS})\right\}^{-1}$$

such that

$$\left\|\frac{Q_t^{\alpha}\psi-\psi}{t}+\mathscr{N}^{\alpha}\psi\right\|_{(\theta_1,\theta_2);(r_1,r_2);(h_1',h_2')}\leq t\times C\|\psi\|_{(\theta_1,\theta_2);(q_1,q_2);(h_1,h_2)}.$$

Thus, we can easily see that

$$\lim_{t\to 0^+}\left\|\frac{Q_t^{\alpha}\psi-\psi}{t}+\mathscr{N}^{\alpha}\psi\right\|_{(\theta_1,\theta_2);(r_1,r_2);(h_1,h_2)}=0, \tag{4.2}$$

which implies that

$$\frac{Q_t^{\alpha}\psi-\psi}{t}\to-\mathscr{N}^{\alpha}\psi\ \text{ as } t\longrightarrow 0^+,\ in\ \mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2'),$$

i.e., $-\mathscr{N}^{\alpha}$ is the infinitesimal generator of $\{Q_t^{\alpha},\ t\geq 0\}$. Hence, by applying the topological isomorphism $\mathscr{K}$, we get the desired statement. □

Theorem 1. *The following quantum fractional Cauchy problems*

$$\begin{cases} \frac{d}{dt}\Pi_t = -\widetilde{\mathscr{N}}^{\alpha}\Pi_t, \\ \Pi_0 = \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E_1'), \mathscr{F}_{\theta_2}(E_2')), \end{cases} \tag{4.3}$$

$$\begin{cases} \frac{d}{dt}\Lambda_t = -\widetilde{\mathscr{N}_1}^{\alpha}\Lambda_t, \\ \Lambda_0 = \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E_1'), \mathscr{F}_{\theta_2}(E_2')), \end{cases} \tag{4.4}$$

$$\begin{cases} \frac{d}{dt}\Upsilon_t = -\widetilde{\mathscr{N}_2}^{\alpha}\Upsilon_t, \\ \Upsilon_0 = \Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E_1'), \mathscr{F}_{\theta_2}(E_2')), \end{cases} \tag{4.5}$$

have a unique solutions given respectively by

$$\begin{aligned} \Pi_t &= \widetilde{Q_t}^{\alpha}\Xi, \\ \Lambda_t &= \widetilde{Q_{t,0}}^{\alpha}\Xi, \\ \Upsilon_t &= \widetilde{Q_{0,t}}^{\alpha}\Xi. \end{aligned}$$

Proof. Let $\varphi \in \mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$. Note that the function

$$U(t,a,b) = Q_t^{\alpha}\psi(a,b)$$

satisfies $\lim_{t\to 0} U(t,a,b) = \psi(a,b)$ and from (4.2), we obtain

$$\begin{aligned} \frac{d}{dt}U(t,a,b) &= -\mathscr{N}^{\alpha}Q_t^{\alpha}\psi(a,b) \\ &= -\mathscr{N}^{\alpha}U(a,b). \end{aligned}$$

Therefore, the proof of the first assertion is now reached by applying the topological isomorphism $\mathscr{K}$. For the other proofs, we proceed as above. □

From the Wiener-Itô-Segal isomorphism, $L^2(\mu_1 \otimes \mu_2)$ is unitary isomorphic to $Fock(H_1 \oplus H_2)$, where

$$Fock(H_1 \oplus H_2) = \{\overrightarrow{\psi} = (\psi_{l,k})_{l,k\in\mathbb{N}} : \ \psi_{l,k} \in H_1^{\widehat{\otimes} l}\widehat{\otimes} H_2^{\widehat{\otimes} k}, \|\overrightarrow{\psi}\|_{Fock} < \infty\},$$

where

$$\|\overrightarrow{\psi}\|^2_{Fock} = \sum_{l,k=0}^{\infty} l!k!|\psi_{l,k}|_0^2$$

and μ_j is the standard Gaussian measure on X_j uniquely specified by its characteristic function

$$e^{-\frac{1}{2}|\xi|^2_{j,0}} = \int_{X_j'} e^{i\langle x,\xi\rangle_j} d\mu_j(x), \ \xi \in X_j, \ j \in \{1,2\}.$$

Remark 2. *Let $\psi \in L^2(\mu_1 \otimes \mu_2)$. Then, we have*

$$\begin{aligned} \|\overrightarrow{Q_t^{\alpha}\psi}\|^2_{Fock} &= \sum_{l,k=0}^{\infty} l!k!e^{-2t(l^{\alpha}+k^{\alpha})}|\psi_{l,k}|_0^2 \\ &\leq \sum_{l,k=0}^{\infty} l!k!|\psi_{l,k}|_0^2 = \|\overrightarrow{\psi}\|^2_{Fock}. \end{aligned}$$

Therefore, $\{Q_t^\alpha, t \geq 0\}$ *is continuous from* $L^2(\mu_1 \otimes \mu_2)$ *into itself.*

Definition 2. *A continuous semigroup* $Q = (Q_t)_{t\geq 0}$ *on* $\mathscr{L}(L^2(\mu_1 \otimes \mu_2))$ *satisfies the Feller property if there is a separable* $\mathbb{C}^*$*-subalgebra* $\mathscr{A}$ *of* $L^2(\mu_1 \otimes \mu_2)$ *on which* Q *is strongly continuous:*

$$\lim_{t\to 0} \|Q_t\psi - \psi\|_{L^2(\mu_1\otimes\mu_2)} = 0, \text{ for all } \psi \in \mathscr{A},$$

and if it leaves such a $\mathbb{C}^*$*-subalgebra invariant:*

$$Q_t(\mathscr{A}) \subset \mathscr{A}, \text{ for all } t \geq 0.$$

Theorem 2. *The semigroup* $\{Q_t^\alpha,\ t \geq 0\}$ *satisfies the Feller property.*

Proof. Since the Young functions θ_1 and θ_2 satisfy the following conditions

$$\lim_{x\longrightarrow\infty} \frac{\theta_1(x)}{x^2} < \infty \text{ and } \lim_{x\longrightarrow\infty} \frac{\theta_2(x)}{x^2} < \infty,$$

then

$$\overline{\mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')} = L^2(\mu_1 \otimes \mu_2).$$

Moreover, for $\psi \in L^2(\mu_1 \otimes \mu_2)$, we have

$$\begin{aligned} \|\overrightarrow{Q_t^\alpha\psi - \psi}\|^2_{Fock} &= \sum_{n,m=0}^{\infty} n!m!|e^{-t(n^\alpha+m^\alpha)} - 1|^2|\psi_{n,m}|_0^2 \\ &= \sum_{n,m=0}^{\infty} n!(\theta_1)_n^2\delta_1^n((\theta_1)_n^{-2}\delta_1^{-n})m!(\theta_2)_m^2\delta_2^m((\theta_2)_m^{-2}\delta_2^{-m}) \\ &\quad \times|e^{-t(n^\alpha+m^\alpha)} - 1|^2|\psi_{n,m}|_0^2. \end{aligned}$$

Using the fact that $|e^a - 1| \leq |a|e^{|a|}$ for all $a \in \mathbb{R}$, we obtain

$$\begin{aligned} \|\overrightarrow{Q_t^\alpha\psi - \psi}\|^2_{Fock} &= \sum_{n,m=0}^{\infty} n!(\theta_1)_n^2\delta_1^n((\theta_1)_n^{-2}\delta_1^{-n})m!(\theta_2)_m^2\delta_2^m((\theta_2)_m^{-2}\delta_2^{-m}) \\ &\quad \times t^2(n^\alpha+m^\alpha)^2e^{2t(n^\alpha+m^\alpha)}|\psi_{n,m}|_0^2. \end{aligned}$$

On the other hand, using the inequality $(n^\alpha + m^\alpha) \leq 2^{\alpha(n+m)}$, we have

$$t^2(n^\alpha+m^\alpha)^2e^{2t(n^\alpha+m^\alpha)} \leq t^2 4^{\alpha(n+m)}e^{2t(n^\alpha+m^\alpha)}.$$

It is obvious that, for $t \leq 1$,

$$n!(\theta_i)_n^2\delta_i^n(t4^{\alpha n}e^{2tn^\alpha}) \leq tn!(\theta_i)_n^2\delta_i^n 4^{\alpha n}e^{2n^\alpha} = tn!(\theta_i)_n^2(\delta_i 4^\alpha e^{2n^{\alpha-1}})^n,\ i \in \{1,2\}.$$

From Lemma 4.1 in [25], there exist constant numbers $a_i > 0$ and $b_i > 0$ such that

$$(\theta_i)_n^2 n!(\delta_i 4^\alpha)^n \leq a_i^2(\frac{2b_i e}{n})^n(\delta_i 4^\alpha)^n n! = a_i^2(2b_i\delta_i 4^\alpha)^n\frac{\sqrt{n}e^n n!}{n^n\sqrt{n}}.$$

Using the Stirling formula, the last fraction tends to $\sqrt{2\pi}$ as $n \longrightarrow \infty$. Therefore, for $\delta_i 4^\alpha < (2b_i)^{-1}$, we have

$$M_i^2 \equiv \sup_{n\geq 0}(\theta_i)_n^2\delta_i^n 4^\alpha n! < \infty,\ i \in \{1,2\}.$$

Thus,

$$\|\overrightarrow{Q_t^\alpha \psi - \psi}\|_{Fock} = tM_1M_2\|\psi\|_{\theta_1,\theta_2,(\delta_1,\delta_2),(r_1,r_2)},$$

for all $\delta_1, \delta_2 > 0, r_1, r_2 \geq 0$. Consequently, the restriction of Q_t^α to $\mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$ is strongly continuous and we have

$$Q_t^\alpha \psi \in \mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2'),\ \psi \in \mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2').$$

□

Remark 3. *Similarly as above, we can prove that the family* $\{Q_{s,0}^\alpha, s \geq 0\}$ *and* $\{Q_{0,t}^\alpha, t \geq 0\}$ *satisfy the Feller property.*

5. Quantum fractional potentials associated to quantum fractional O-U semigroups

In this section, motivated by the results obtained in Section 4, we study the quantum fractional potentials associated to quantum fractional O-U semigroups. Moreover, we give the solutions of the quantum fractional Poisson equations.

Definition 3. *Let* $\psi \in \mathscr{F}_{\theta_1,\theta_2}(E_1' \oplus E_2')$ *and* $\lambda, \lambda_1, \lambda_2 > 0$. *We define the following operators*

$$H_\lambda^\alpha \psi(a,b) = \int_0^\infty e^{-t\lambda} Q_t^\alpha \psi(a,b)dt, \tag{5.1}$$

$$H_{\lambda_1,\lambda_2}^{g,\alpha} \psi(a,b) = H_{\lambda_1}^{-,\alpha} \psi(a,b) + H_{\lambda_2}^{+,\alpha} \psi(a,b), \tag{5.2}$$

where $H_{\lambda_1}^{-,\alpha}$ *and* $H_{\lambda_2}^{+,\alpha}$ *are given by*

$$H_{\lambda_1}^{-,\alpha} \psi(a,b) = \int_0^\infty e^{-t\lambda_1} Q_{t,0}^\alpha \psi(a,b)dt,$$

$$H_{\lambda_2}^{+,\alpha} \psi(a,b) = \int_0^\infty e^{-t\lambda_2} Q_{0,t}^\alpha \psi(a,b)dt.$$

It is obvious that $e^{-t\lambda} Q_t^\alpha \psi$, $e^{-t\lambda_1} Q_{t,0}^\alpha \psi$ and $e^{-t\lambda_2} Q_{0,t}^\alpha \psi$ are bounded. However, when $\lambda \to 0$, $\lambda_1 \to 0$ and $\lambda_2 \to 0$, we will lose the integrability of the integrand in the right hand side of equations (5.1) and (5.2). For this reason, we introduce the following normalized operators:

$$(M^\alpha \psi)(a,b) = \int_0^\infty Q_t^\alpha (\psi - \psi_{0,0})(a,b)dt, \tag{5.3}$$

$$(M^{g,\alpha} \psi)(a,b) = M^{-,\alpha} \psi(a,b) + M^{+,\alpha} \psi(a,b), \tag{5.4}$$

where

$$(M^{-,\alpha} \psi)(a,b) = \int_0^\infty Q_{t,0}^\alpha (\psi - \psi_{0,0} - \psi_0^-)(a,b)dt,$$

$$(M^{+,\alpha} \psi)(a,b) = \int_0^\infty Q_{t,0}^\alpha (\psi - \psi_{0,0} - \psi_0^+)(a,b)dt,$$

$\psi(a,b) = \sum_{i,j=0}^\infty \langle a^{\otimes i} \otimes b^{\otimes j}, \psi_{i,j}\rangle$, $\psi_0^-(a,b) = \sum_{j=0}^\infty \langle b^{\otimes j}, \psi_{0,j}\rangle$ and $\psi_0^+(a,b) = \sum_{i=0}^\infty \langle a^{\otimes i}, \psi_{i,0}\rangle$. By a simple calculus, one can show that

$$Q_t^\alpha (\psi - \psi_{0,0})(a,b) = \sum_{i,j=0,(i,j)\neq(0,0)}^\infty \langle a^{\otimes i} \otimes b^{\otimes j}, e^{-t(i^\alpha + j^\alpha)} \psi_{i,j}\rangle,$$

$$Q^{\alpha}_{t,0}(\psi-\psi_{0,0}-\psi_0^-)(a,b)=\sum_{i=1,j=0}^{\infty}\langle a^{\otimes i}\otimes b^{\otimes j},e^{-ti^{\alpha}}\psi_{i,j}\rangle,$$

$$Q^{\alpha}_{0,t}(\psi-\psi_{0,0}-\psi_0^-)(a,b)=\sum_{i=0,j=1}^{\infty}\langle a^{\otimes i}\otimes b^{\otimes j},e^{-tj^{\alpha}}\psi_{i,j}\rangle.$$

Remark 4. *For $i\geq 1$ and $0<\alpha<1$, we have $e^{-ti^{\alpha}}\leq e^{-t}$. Then, one can show that $Q^{\alpha}_t(\psi-\psi_{0,0})(a,b)$, $Q^{\alpha}_{t,0}(\psi-\psi_{0,0}-\psi_0^-)(a,b)$ and $Q^{\alpha}_{0,t}(\psi-\psi_{0,0}-\psi_0^-)(a,b)$ are bounded by the following integrable function*

$$C'e^{-t}\|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}e^{\theta_1(h_1'|a|_{-r_1})+\theta_2(h_2'|b|_{-r_2})}$$

where

$$C':=C'(r_1,r_2,q_1,q_2)=\left\{\left(1-e\frac{h_1}{h_1'}\|i_{q_1,r_1}\|_{HS}\right)\left(1-e\frac{h_2}{h_2'}\|i_{q_2,r_2}\|_{HS}\right)\right\}^{-1}.$$

Proposition 4. *Let $\varphi\in\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')$. Then, the operators H^{α}_{λ}, $H^{g,\alpha}_{\lambda_1,\lambda_2}$, M^{α} and $M^{g,\alpha}$ are continuous linear operators from $\mathscr{F}_{\theta_1,\theta_2}(E_1'\oplus E_2')$ into itself. Moreover, for $h_1,h_2,h_1',h_2'>0$ and $q_1>r_1$, $q_2>r_2$ such that*

$$\max\left\{\frac{h_1}{h_1'}e\|i_{q_1,r_1}\|_{HS},\ \frac{h_2}{h_2'}e\|i_{q_2,r_2}\|_{HS}\right\}<1,$$

there exists a positive constant $C'=C'(r_1,r_2,q_1,q_2)$ such that

$$\|H^{\alpha}_{\lambda}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1',h_2')}\leq\frac{1}{\lambda}C'\|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)},$$

$$\|H^{\alpha}_{\lambda_1,\lambda_2}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1',h_2')}\leq\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)C'\|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)},$$

$$\|M^{\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1',h_2')}\leq C'\|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)},$$

and

$$\|M^{g,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h_1',h_2')}\leq 2C'\|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}.$$

Proof. We know that $e^{-t\lambda}Q^{\alpha}_t\psi(a,b)$ is bounded by an integrable function and Q^{α}_t is a continuous linear operator. Then, using the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}H^{\alpha}_{\lambda}\psi(a,b)&=\sum_{l,k=0}^{\infty}\int_0^{\infty}e^{-\lambda t}\langle a^{\otimes l}\otimes b^{\otimes k},e^{-t(l^{\alpha}+k^{\alpha})}\psi_{l,k}\rangle dt\\&=\sum_{l,k=0}^{\infty}\langle a^{\otimes l}\otimes b^{\otimes k},\psi_{l,k}\rangle\int_0^{\infty}e^{-t(\lambda+l^{\alpha}+k^{\alpha})}dt\\&=\sum_{l,k=0}^{\infty}\frac{1}{\lambda+l^{\alpha}+k^{\alpha}}\langle a^{\otimes l}\otimes b^{\otimes k},\psi_{l,k}\rangle.\end{aligned}$$

Therefore, one may write

$$H^{\alpha}_{\lambda}\psi\sim\left\{\frac{1}{\lambda+l^{\alpha}+k^{\alpha}}\psi_{l,k}\right\}.$$

As above, we have

$$H_{\lambda_1}^{-,\alpha}\psi \sim \left\{\frac{1}{\lambda_1 + l^\alpha}\psi_{l,k}\right\},$$

and

$$H_{\lambda_2}^{+,\alpha}\psi \sim \left\{\frac{1}{\lambda_2 + k^\alpha}\psi_{l,k}\right\}.$$

Which gives

$$H_{\lambda_1,\lambda_2}^{g,\alpha}\psi \sim \left\{\left(\frac{1}{\lambda_1 + l^\alpha} + \frac{1}{\lambda_2 + k^\alpha}\right)\psi_{l,k}\right\}.$$

For $r_1, r_2 \geq 0$, we deduce that

$$\begin{aligned} |H_\lambda^\alpha \psi(a,b)| &\leq \sum_{l,k=0}^{\infty} \frac{1}{\lambda + l^\alpha + k^\alpha} |a|_{-r_1}^{l} |b|_{-r_2}^{k} |\psi_{l,k}|_{r_1,r_2} \\ &\leq \sum_{l,k=0}^{\infty} \frac{1}{\lambda} |a|_{-r_1}^{l} |b|_{-r_2}^{k} |\psi_{l,k}|_{r_1,r_2}. \end{aligned}$$

Therefore, one can show that, there exists a positive constant $C' = C'(r_1, r_2, q_1, q_2)$ such that

$$\|H_\lambda^\alpha \psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} \leq \frac{1}{\lambda} C' \|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}.$$

Similarly, we have

$$\|H_{\lambda_1}^{-,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} \leq \frac{1}{\lambda_1} C' \|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}$$

and

$$\|H_{\lambda_2}^{+,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} \leq \frac{1}{\lambda_2} C' \|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}.$$

Hence, we obtain

$$\begin{aligned} &\|H_{\lambda_1,\lambda_2}^{g,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} \\ &\leq \|H_{\lambda_1}^{-,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} + \|H_{\lambda_2}^{+,\alpha}\psi\|_{(\theta_1,\theta_2),(r_1,r_2),(h'_1,h'_2)} \\ &= \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) C' \|\psi\|_{(\theta_1,\theta_2),(q_1,q_2),(h_1,h_2)}. \end{aligned}$$

By a similar calculus, we complete the proof. □

Definition 4. *For $\lambda, \lambda_1, \lambda_2 > 0$, we define the quantum fractional λ-potential, the generalized quantum fractional (λ_1, λ_2)-potential on $\mathscr{L}(\mathscr{F}_{\theta_1}^*(E'_1), \mathscr{F}_{\theta_2}(E'_2))$ as follows:*

$$\widetilde{H_\lambda}^\alpha := \mathscr{K}^{-1} H_\lambda^\alpha \mathscr{K},$$

$$\begin{aligned} \widetilde{H_{\lambda_1,\lambda_2}^{g}}^\alpha &:= \mathscr{K}^{-1} H_{\lambda_1,\lambda_2}^{g,\alpha} \mathscr{K} \\ &= \mathscr{K}^{-1} H_{\lambda_1}^{-,\alpha} \mathscr{K} + \mathscr{K}^{-1} H_{\lambda_2}^{+,\alpha} \mathscr{K} \\ &= \widetilde{H_{\lambda_1}^{-}}^\alpha + \widetilde{H_{\lambda_2}^{+}}^\alpha. \end{aligned}$$

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Similarly, we define the generalized quantum fractional normalized potential on $\mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1),\mathscr{F}_{\theta_2}(E'_2))$ as follows:

$$\widetilde{M}^{\alpha} := \mathscr{K}^{-1} M^{\alpha} \mathscr{K},$$

$$\begin{aligned}\widetilde{M^g}^{\alpha} &:= \mathscr{K}^{-1} M^{g,\alpha} \mathscr{K}\\ &= \mathscr{K}^{-1} M^{-,\alpha} \mathscr{K} + \mathscr{K}^{-1} M^{+,\alpha} \mathscr{K}\\ &= \widetilde{M^-}^{\alpha} + \widetilde{M^+}^{\alpha}.\end{aligned}$$

For $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1),\mathscr{F}_{\theta_2}(E'_2))$ such that $\mathscr{K}(\Xi) = \psi \sim (\psi)_{n,m}$, we consider the following quantum fractional Poisson equations

$$(\lambda I + \widetilde{\mathscr{N}}^{\alpha}) S = \Xi, \tag{5.5}$$

$$(\lambda_1 I + \widetilde{\mathscr{N}_1}^{\alpha}) S = \Xi, \tag{5.6}$$

$$(\lambda_2 I + \widetilde{\mathscr{N}_2}^{\alpha}) S = \Xi, \tag{5.7}$$

$$\widetilde{\mathscr{N}}^{\alpha} S = \Xi - \Xi_{0,0}, \tag{5.8}$$

$$\widetilde{\mathscr{N}_1}^{\alpha} S = \Xi - \Xi_{0,0} - \Xi^-_0, \tag{5.9}$$

and

$$\widetilde{\mathscr{N}_2}^{\alpha} S = \Xi - \Xi_{0,0} - \Xi^+_0, \tag{5.10}$$

where $\Xi_{0,0} = \psi_{0,0} I$, $\Xi^-_0 = \mathscr{K}^{-1}(\psi^-_0)$ and $\Xi^+_0 = \mathscr{K}^{-1}(\psi^+_0)$.

Theorem 3. *Let $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1),\mathscr{F}_{\theta_2}(E'_2))$ such that $\mathscr{K}(\Xi) = \psi \sim (\psi)_{n,m}$. Then, the operators $\widetilde{H_\lambda}^{\alpha}\Xi$, $\widetilde{H^-_{\lambda_1}}^{\alpha}\Xi$ and $\widetilde{H^+_{\lambda_2}}^{\alpha}\Xi$ are solutions of the quantum fractional Poisson equations (5.5), (5.6) and (5.7), respectively. Moreover, $\widetilde{M}^{\alpha}\Xi$, $\widetilde{M^-}^{\alpha}\Xi$ and $\widetilde{M^+}^{\alpha}\Xi$ are solutions of the quantum fractional Poisson equations (5.8), (5.9) and (5.10), respectively.*

Proof. For $\Xi \in \mathscr{L}(\mathscr{F}^*_{\theta_1}(E'_1), \mathscr{F}_{\theta_2}(E'_2))$ such that $\mathscr{K}(\Xi) = \psi \sim (\psi)_{n,m}$, we have

$$\begin{aligned}
\mathscr{N}^{\alpha} H^{\alpha}_{\lambda} \psi(a,b) &= \mathscr{N}^{\alpha} \left(\int_0^{\infty} e^{-\lambda t} Q^{\alpha}_t \psi(a,b) dt \right) \\
&= \mathscr{N}^{\alpha} \left(\int_0^{\infty} e^{-\lambda t} \sum_{i,j=0}^{\infty} \langle a^{\otimes i} \otimes b^{\otimes j}, e^{-t(i^{\alpha}+j^{\alpha})} \psi_{i,j} \rangle dt \right) \\
&= \mathscr{N}^{\alpha} \left(\sum_{i,j=0}^{\infty} \frac{1}{\lambda + i^{\alpha} + j^{\alpha}} \langle a^{\otimes i} \otimes b^{\otimes j}, \psi_{i,j} \rangle \right) \\
&= \sum_{i,j=0}^{\infty} \frac{i^{\alpha} + j^{\alpha}}{\lambda + i^{\alpha} + j^{\alpha}} \langle a^{\otimes i} \otimes b^{\otimes j}, \psi_{i,j} \rangle \\
&= \sum_{i,j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda + i^{\alpha} + j^{\alpha}} \right) \langle a^{\otimes i} \otimes b^{\otimes j}, \psi_{i,j} \rangle \\
&= \psi(a,b) - \lambda H^{\alpha}_{\lambda} \psi(a,b).
\end{aligned}$$

Applying the topological isomorphism $\mathscr{K}$, we prove that $\widetilde{H_{\lambda}}^{\alpha}$ is a solution of equation (5.5). Similarly, we obtain

$$\mathscr{N}^{\alpha} H^{-,\alpha}_{\lambda_1} \psi(a,b) = \psi(a,b) - \lambda_1 H^{-,\alpha}_{\lambda_1} \psi(a,b),$$

and

$$\mathscr{N}^{\alpha} H^{+,\alpha}_{\lambda_2} \psi(a,b) = \psi(a,b) - \lambda_2 H^{+,\alpha}_{\lambda_2} \psi(a,b).$$

On the other hand, we have

$$\begin{aligned}
\mathscr{N}^{\alpha} M^{\alpha} \psi(a,b) &= \mathscr{N}^{\alpha} \int_0^{\infty} Q^{\alpha}_t (\psi - \psi_{0,0})(a,b) dt \\
&= \mathscr{N}^{\alpha} \int_0^{\infty} \sum_{i,j=0,(i,j)\neq(0,0)}^{\infty} \langle a^{\otimes i} \otimes b^{\otimes j}, e^{-t(i^{\alpha}+j^{\alpha})} \psi_{i,j} \rangle dt \\
&= \mathscr{N}^{\alpha} \sum_{i,j=0,(i,j)\neq(0,0)}^{\infty} \frac{1}{i^{\alpha} + j^{\alpha}} \langle x^{\otimes i} \otimes y^{\otimes j}, e^{-t(i^{\alpha}+j^{\alpha})} \psi_{i,j} \rangle \\
&= \sum_{i,j=0,(i,j)\neq(0,0)}^{\infty} \langle a^{\otimes i} \otimes b^{\otimes j}, e^{-t(i^{\alpha}+j^{\alpha})} \psi_{i,j} \rangle \\
&= \psi(a,b) - \psi_{0,0}(a,b).
\end{aligned}$$

Thus, by the topological isomorphism $\mathscr{K}$, we deduce that $\widetilde{M}^{\alpha}\Xi$ is solution of (5.8). Similarly, we get the desired statement for $\widetilde{M^{-}}^{\alpha}\Xi$ and $\widetilde{M^{+}}^{\alpha}\Xi$. □

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