SOME PERTURBED NEWTON TYPE INEQUALITIES
FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In the present paper, we derive an identity for the case of twice-differentiable functions whose second derivatives are convex. By using this equality, we establish some perturbed Newton type inequalities for twice-differentiable convex functions. These type of inequalities are investigated by using the well-known Riemann-Liouville fractional integrals. Furthermore, we provide our results by using special cases of obtained theorems Finally, some conclusions of research are given in the last section.

1. Introduction

The inequality theory on convex functions is an important topic in many mathematical areas with numerous number of applications. In this context, many researchers have established the Hermite-Hadamard inequality, Simpson type inequality and Newton type inequality with great interest to generalise and extend it to the case of different classes of functions such as \( s \)-convex functions, quasi-convex functions, log-convex, etc. In recent years, Fractional calculus has increased interest because of the its demonstrated applications in a range of the inequality theory on convex functions. Owing to the importance of fractional calculus, mathematicians have established distinct fractional integral inequalities. It can be obtained the bounds of new formulas by using the Hermite-Hadamard, Simpson type inequality, and Newton type inequality.

For the first time using the Riemann-Liouville fractional integrals, some Hermite-Hadamard type inequalities were established in the paper [23]. In addition, some new Hermite-Hadamard type inequalities and midpoint type inequalities by using Riemann-Liouville fractional integrals to the case of differentiable convex functions were proved in the paper [22]. Some new estimates of Hermite-Hadamard type inequalities to the case of Riemann-Liouville fractional integrals and conformable fractional integrals were presented in the paper [14]. Moreover, Sarikaya and Ertugral [24] proved a new class of fractional integrals, known generalized fractional, and they used these integrals to establish the general version of Hermite-Hadamard type inequalities for convex functions. In addition to this, some variants of Simpson and Ostrowski type for differentiable convex functions by generalized fractional integrals were proved in the paper [2]. We suggest interested readers to recent studies, (see [1,12–14,19,20] and the references therein) for a good understanding of improving of fractional integral inequalities.

Simpson’s inequalities are inequalities that are created from Simpson’s following rules:

i. Simpson’s quadrature formula (Simpson’s 1/3 rule) is formulated as follows:

\[
\int_a^b f(x)\,dx \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
\]

ii. Simpson’s second formula or Newton-Cotes quadrature formula (Simpson’s 3/8 rule) is formulated as follows:

\[
\int_a^b f(x)\,dx \approx \frac{b-a}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right].
\]

For four times continuously differentiable functions, the classical Simpson and Newton type inequalities are expressed as follows:

\[ K e y \ w o r d s \ a n d \ p h r a s e s. \] Newton type Inequalities; fractional calculus; convex functions.

\[ 2010 \ M a t h e m a t i c s \ S u b j e c t \ C l a s s i f i c a t i o n. \] 26A33, 26D07, 26D10, 26D15.
Theorem 1. Suppose that \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable function on \((a, b)\), and let \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then, one has the inequality

\[
\left| \frac{1}{6} f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right| - \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b - a)^4.
\]

Theorem 2. If \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable function on \((a, b)\), and let \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then, one has the inequality

\[
\left| \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a + b}{3} \right) + 3f \left( \frac{a + 2b}{3} \right) + f(b) \right] \right| - \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{1}{6480} \|f^{(4)}\|_\infty (b - a)^4.
\]

It is established to some fractional Simpson type inequalities to the case of mappings whose second derivatives in absolute value are convex in the paper \([8]\). Moreover, new inequalities of Simpson type and their application to quadrature formulae in numerical analysis are presented in the paper \([5]\). Furthermore, some of Simpson type inequalities with the aid of the Riemann-Liouville fractional integrals and in the case of general convex functions were given in the paper \([21]\). For more information about Simpson type inequalities and some features of Riemann–Liouville fractional integrals and various fractional integral operators, we refer the reader to Refs. \([3, 18, 26]\), and the references therein.

Simpson’s second rule has the rule of three-point Newton-Cotes quadrature, hence evaluations for three steps quadratic kernel are sometimes called Newton type results. These results are known as Newton-type inequalities in the literature. Many mathematicians have been investigated to Newton type inequalities. For instance, some new integral inequalities of Newton type for functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex were established in the paper \([4]\). Moreover, some Newton type inequalities to the case of differentiable convex functions through the well-known Riemann-Liouville fractional integrals were proved and some fractional Newton type inequalities for functions of bounded variation were obtained in the paper \([9]\). In addition, some Newton type inequalities were presented by using Riemann-Liouville fractional integrals and the authors prove some inequalities of Riemann-Liouville fractional Newton type for functions of bounded variation in the paper \([25]\). Furthermore, new Newton type inequalities based on convexity were given and some applications for special cases of real functions are also proved in the paper \([6]\). It can be referred to \([10, 11, 16, 17]\) and the references therein to the case of further informations concerned with Newton type of inequalities involving convex differentiable functions.

This paper purposes to prove some perturbed Newton type inequalities in the case of twice-differentiable convex functions. These inequalities are established with the help of the Riemann-Liouville fractional integrals. The general structure of the paper contains three chapters including an introduction. The other sections of the paper proceeds as follows: In Section 2, we will establish an integral equality. Moreover, it will be proved some perturbed Newton type inequalities in the case of differentiable convex functions with the aid of the Riemann-Liouville fractional integrals. Furthermore, we show that our main inequalities reduce to Newton type inequalities which are proved in earlier published papers. In Section 3, some conclusions and guidelines of research are given as an opinion at the end of the paper.

2. Main Results

In this section, we will give an integral identity. Moreover, perturbed Newton type inequalities for twice-differentiable convex functions will be established by using the Riemann-Liouville fractional integrals. Throughout the paper, we shall present fundamental fractional integral notations and several fractional integrals needed in the sequel as follows:

Definition 1. \([7, 15]\) Let us consider \( \mathfrak{F} \in L_1(\mu, \eta) \). The Riemann–Liouville integrals \( \mathcal{J}_\mu^\alpha \mathfrak{F} \) and \( \mathcal{J}_\eta^\alpha \mathfrak{F} \) of order \( \alpha > 0 \) with \( \mu \geq 0 \) are defined by

\[
\mathcal{J}_\mu^\alpha \mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_\mu^x (x - \xi)^{\alpha - 1} \mathfrak{F}(\xi)d\xi, \quad x > \mu
\]
and
\[ \mathcal{J}_{\eta}^\alpha \mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\eta} (\xi - x)^{\alpha - 1} \mathfrak{F}(\xi) d\xi, \quad x < \eta, \]
respectively. Here, \( \Gamma(\alpha) \) is the Gamma function and its defined as
\[ \Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha - 1} du. \]

Let us also consider that \( \mathcal{J}_{\eta}^\alpha \mathfrak{F}(x) = \mathcal{J}_{\eta}^\alpha \mathfrak{F}(x) = \mathfrak{F}(x) \).

**Lemma 1.** Let \( \mathfrak{F}: \lbrack \mu, \eta \rbrack \rightarrow \mathbb{R} \) denote an absolutely continuous function \((\mu, \eta)\) such that \( \mathfrak{F}'' \in L_1 \lbrack \mu, \eta \rbrack \). Then, it follows
\[
\frac{(\eta - \mu) (1 - \alpha)}{12 (\alpha + 1)} [\mathfrak{F}'(\mu) + \mathfrak{F}'(\eta)] + \frac{1}{8} \left[ \mathfrak{F}(\mu) + 3 \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + 3 \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + \mathfrak{F}(\eta) \right] \\
= - \frac{3^{\alpha - 1} \Gamma(\alpha + 1)}{(\eta - \mu)^2} \left[ \mathcal{J}_{\mu}^\alpha \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + \mathcal{J}_{\mu + \eta}^\alpha \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + \mathcal{J}_{\mu + 2\eta}^\alpha \mathfrak{F}(\eta) \right] \\
= - \frac{(\eta - \mu)^2}{27 (\alpha + 1)} [I_1 + I_2 + I_3],
\]
where
\[
I_1 = \int_0^1 \left( \xi^{\alpha + 1} - \frac{5 (\alpha + 1)}{8} \xi + \frac{3 - \alpha}{8} \right) \mathfrak{F}'' \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) d\xi,
\]
\[
I_2 = \int_0^1 \left( \xi^{\alpha + 1} - \frac{a + 1}{2} \xi + \frac{3 - 1}{8} \right) \mathfrak{F}'' \left( \frac{1 + \xi}{3} \mu + \frac{2 - \xi}{3} \eta \right) d\xi,
\]
\[
I_3 = \int_0^1 \left( \xi^{\alpha + 1} - \frac{3 (\alpha + 1)}{8} \xi + \frac{3 (\alpha - 1)}{4} \right) \mathfrak{F}'' \left( \frac{\xi}{3} \mu + \frac{3 - \xi}{3} \eta \right) d\xi.
\]

**Proof.** By using the fundamental rules of integration by parts, the following equalities can be obtained
\[
I_1 = \int_0^1 \left( \xi^{\alpha + 1} - \frac{5 (\alpha + 1)}{8} \xi + \frac{3 - \alpha}{8} \right) \mathfrak{F}'' \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) d\xi \\
= - \frac{3}{\eta - \mu} \left( \xi^{\alpha + 1} - \frac{5 (\alpha + 1)}{8} \xi + \frac{3 - \alpha}{8} \right) \mathfrak{F}' \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) \bigg|_0^1 \\
+ \frac{3}{\eta - \mu} \int_0^1 (\alpha + 1) \xi^\alpha \frac{5 (\alpha + 1)}{8} \mathfrak{F}' \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) d\xi \\
= \frac{3}{\eta - \mu} \left[ \frac{3 - \alpha}{8} \mathfrak{F}' \left( \frac{2 \mu + \eta}{3} \right) + \frac{3 (\alpha - 1)}{4} \mathfrak{F}'(\mu) \right] \\
+ \frac{3}{\eta - \mu} \left[ - \frac{3}{\eta - \mu} (\alpha + 1) \xi^\alpha \frac{5 (\alpha + 1)}{8} \mathfrak{F}' \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) \bigg|_0^1 \\
+ \frac{3 \alpha (\alpha + 1)}{\eta - \mu} \int_0^1 \xi^{\alpha - 1} \mathfrak{F} \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) d\xi \right] \\
= \frac{3}{\eta - \mu} \left[ \frac{3 - \alpha}{8} \mathfrak{F}' \left( \frac{2 \mu + \eta}{3} \right) + \frac{3 (\alpha - 1)}{4} \mathfrak{F}'(\mu) \right].
\]


\[- \frac{9}{(\eta - \mu)^2} \left[ \frac{5(\alpha + 1)}{8} \tilde{b} \left( \frac{2\mu + \eta}{3} \right) + \frac{3(\alpha + 1)}{8} \tilde{b}(\mu) \right] \]

\[+ \frac{9\alpha (\alpha + 1)}{(\eta - \mu)^2} \int_{\alpha}^{1} \xi^{\alpha - 1} \tilde{g} \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) d\xi. \]

Similar to the procedures we have done above, the following \( I_2 \) and \( I_3 \) are equals to (2.3)

\[ I_2 = \frac{3}{\eta - \mu} \left[ \frac{3(\alpha + 1)}{4} \tilde{b}'(\eta) + \frac{1 - 3\alpha}{8} \tilde{b}' \left( \frac{\mu + 2\eta}{3} \right) \right] \]

\[ - \frac{9}{(\eta - \mu)^2} \left[ \frac{3(\alpha + 1)}{8} \tilde{b}(\eta) + \frac{5(\alpha + 1)}{8} \tilde{b} \left( \frac{\mu + 2\eta}{3} \right) \right] \]

\[+ \frac{9\alpha (\alpha + 1)}{(\eta - \mu)^2} \int_{\alpha}^{1} \xi^{\alpha - 1} \tilde{g} \left( \frac{\xi}{3} \mu + \frac{3 - \xi}{3} \eta \right) d\xi, \]

and (2.4)

\[ I_3 = \frac{3}{\eta - \mu} \left[ \frac{3(\alpha + 1)}{4} \tilde{b}'(\eta) + \frac{1 - 3\alpha}{8} \tilde{b}' \left( \frac{\mu + 2\eta}{3} \right) \right] \]

\[ - \frac{9}{(\eta - \mu)^2} \left[ \frac{3(\alpha + 1)}{8} \tilde{b}(\eta) + \frac{5(\alpha + 1)}{8} \tilde{b} \left( \frac{\mu + 2\eta}{3} \right) \right] \]

\[+ \frac{9\alpha (\alpha + 1)}{(\eta - \mu)^2} \int_{\alpha}^{1} \xi^{\alpha - 1} \tilde{g} \left( \frac{\xi}{3} \mu + \frac{3 - \xi}{3} \eta \right) d\xi, \]

respectively. If we sum up the equalities from (2.2) and (2.4), then we get (2.5)

\[ I_1 + I_2 + I_3 \]

\[= \frac{9(\alpha - 1)}{4(\eta - \mu)} \left[ \tilde{b}'(\mu) + \tilde{b}'(\eta) \right] - \frac{27(\alpha + 1)}{8(\eta - \mu)^2} \left[ \tilde{b}(\mu) + 3\tilde{b} \left( \frac{2\mu + \eta}{3} \right) + 3\tilde{b} \left( \frac{\mu + 2\eta}{3} \right) + \tilde{b}(\eta) \right] \]

\[+ \frac{9\alpha (\alpha + 1)}{(\eta - \mu)^2} \int_{\alpha}^{1} \xi^{\alpha - 1} \left[ \tilde{g} \left( \frac{2 + \xi}{3} \mu + \frac{1 - \xi}{3} \eta \right) + \tilde{g} \left( \frac{1 + \xi}{3} \mu + \frac{2 - \xi}{3} \eta \right) + \tilde{g} \left( \frac{\xi}{3} \mu + \frac{3 - \xi}{3} \eta \right) \right] d\xi. \]

Let us consider the change of variable formulas for the equality (2.5). Then, equality (2.5) can be rewritten as follows:

(2.6) \[ I_1 + I_2 + I_3 \]

\[= \frac{9(\alpha - 1)}{4(\eta - \mu)} \left[ \tilde{b}'(\mu) + \tilde{b}'(\eta) \right] - \frac{27(\alpha + 1)}{8(\eta - \mu)^2} \left[ \tilde{b}(\mu) + 3\tilde{b} \left( \frac{2\mu + \eta}{3} \right) + 3\tilde{b} \left( \frac{\mu + 2\eta}{3} \right) + \tilde{b}(\eta) \right] \]

\[+ \frac{3^{\alpha + 2}(\alpha + 1) \Gamma(\alpha + 1)}{(\eta - \mu)^{\alpha + 2}} \left[ \mathcal{J}_\alpha^\beta \tilde{b} \left( \frac{2\mu + \eta}{3} \right) + \mathcal{J}_\alpha^{\beta + \mu} \tilde{b} \left( \frac{\mu + 2\eta}{3} \right) + \mathcal{J}_\alpha^{\mu + 2\eta} \tilde{b}(\eta) \right]. \]
Let us multiply the both sides of (2.6) by $-\frac{(\eta - \mu)^2}{2(\alpha + 1)}$. Then, the desired result of the equality (2.1) is obtained. This is the end of the proof of Lemma 1.

\[ \Box \]

**Remark 1.** If we choose $\alpha = 1$ in Lemma 1, then Lemma 1 reduces to

\[
\frac{1}{8} \left[ \mathfrak{F} (\mu) + 3 \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + 3 \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + \mathfrak{F} (\eta) \right] - \frac{1}{(\eta - \mu)^2} \int_0^{\eta} \mathfrak{F} (\xi) \, d\xi = -\frac{(\eta - \mu)^2}{54} [I_1 + I_2 + I_3],
\]

where

\[
\begin{align*}
I_1 &= \int_0^{\frac{\eta}{2}} \left( \xi^2 - \xi^2 + \frac{1}{4} \right) \mathfrak{F}'' \left( \frac{2\xi^2 - 2\xi + \frac{1}{4}}{2} \right) \, d\xi = -54 \int_0^{\frac{\eta}{2}} \left( \xi^2 - \frac{\xi^2}{2} + \frac{1}{8} \right) \mathfrak{F}'' (\xi + (1 - \xi) \mu) \, d\xi, \\
I_2 &= \int_0^{\frac{\eta}{2}} \left( \xi^2 - \frac{\xi}{2} + \frac{1}{4} \right) \mathfrak{F}'' \left( \frac{2\xi^2 - 2\xi + \frac{1}{4}}{2} \right) \, d\xi = -54 \int_0^{\frac{\eta}{2}} \left( \xi^2 - \frac{\xi^2}{2} + \frac{1}{8} \right) \mathfrak{F}'' (\xi + (1 - \xi) \mu) \, d\xi, \\
I_3 &= \int_0^{\frac{\eta}{2}} \left( \xi^2 - \frac{\xi}{2} + \frac{1}{4} \right) \mathfrak{F}'' \left( \frac{\xi^2 - 3\xi + 2}{2} \right) \, d\xi = -54 \int_0^{\frac{\eta}{2}} \left( \xi^2 - \frac{\xi^2}{2} + \frac{1}{8} \right) \mathfrak{F}'' (\xi + (1 - \xi) \mu) \, d\xi,
\end{align*}
\]

which is given by [6, Lemma 2.1].

**Theorem 3.** Let us consider that the assumptions of Lemma 1 are valid and the function $|\mathfrak{F}''|$ is convex on $[\mu, \eta]$. Then, the following inequality holds:

\[
(2.7) \quad \left| \frac{(\eta - \mu)^2}{12 \alpha (\alpha + 1)} [\mathfrak{F}' (\mu) + \mathfrak{F}' (\eta)] + \frac{1}{8} \left[ \mathfrak{F} (\mu) + 3 \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + 3 \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + \mathfrak{F} (\eta) \right] + \frac{3^\alpha - 1}{(\eta - \mu)^2} \left[ J_\mu^\alpha \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + J_{\mu+2\alpha}^\alpha \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + J_{\mu+2\alpha}^\alpha \mathfrak{F} (\eta) \right] \right| \leq \frac{(\eta - \mu)^2}{81 \alpha (\alpha + 1)} \left[ (2 \Omega_1 (\alpha) + \Omega_2 (\alpha) + \Omega_4 (\alpha) + \Omega_5 (\alpha) + \Omega_6 (\alpha)) |\mathfrak{F}'' (\mu)| \right. \\
+ \left. (\Omega_1 (\alpha) + 2 \Omega_2 (\alpha) + 3 \Omega_3 (\alpha) - \Omega_4 (\alpha) - \Omega_5 (\alpha) - \Omega_6 (\alpha)) |\mathfrak{F}'' (\eta)| \right].
\]

Here,

\[
(2.8) \quad \begin{align*}
\Omega_1 &= \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi, \quad \Omega_4 = \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi, \\
\Omega_2 &= \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi, \quad \Omega_5 = \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi, \\
\Omega_3 &= \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi, \quad \Omega_6 (\alpha) = \int_0^{\frac{\eta}{2}} \left( \xi^{\alpha+1} - \frac{3(\alpha+1)}{8} \xi + \frac{3-\alpha}{8} \right) \, d\xi.
\end{align*}
\]

**Proof.** Let us take modulus in Lemma 1. Then, it follows

\[
(2.9) \quad \left| \frac{(\eta - \mu)^2}{12 \alpha (\alpha + 1)} [\mathfrak{F}' (\mu) + \mathfrak{F}' (\eta)] + \frac{1}{8} \left[ \mathfrak{F} (\mu) + 3 \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + 3 \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + \mathfrak{F} (\eta) \right] + \frac{3^\alpha - 1}{(\eta - \mu)^2} \left[ J_\mu^\alpha \mathfrak{F} \left( \frac{2\mu + \eta}{3} \right) + J_{\mu+2\alpha}^\alpha \mathfrak{F} \left( \frac{\mu + 2\eta}{3} \right) + J_{\mu+2\alpha}^\alpha \mathfrak{F} (\eta) \right] \right|
\]
which is established by [6, Theorem 2.1].

Remark 2. If we assign $\alpha = 1$ in Theorem 3, then Theorem 3 becomes to

$$
\left| \frac{1}{8} \left[ \mathfrak{f}(\mu) + 3\mathfrak{f}\left(\frac{2\mu + \eta}{3}\right) + 3\mathfrak{f}\left(\frac{\mu + 2\eta}{3}\right) + \mathfrak{f}(\eta) \right] - \frac{1}{(\eta - \mu)} \int_{\mu}^{\eta} \mathfrak{f}(\xi) \, d\xi \right| \leq \frac{(\eta - \mu)^2}{384} \left[ |\mathfrak{f}''(\mu)| + |\mathfrak{f}''(\eta)| \right],
$$

which is established by [6, Theorem 2.1].
Theorem 4. Assume that the assumptions of Lemma 1 hold and the function $|\tilde{g}''|^q$ is convex on $[\mu, \eta]$ for $q > 1$. Then, the following inequality

$$
\left| \frac{\mu - \eta}{12(\alpha + 1)} [\tilde{g}'(\mu) + \tilde{g}'(\eta)] + \frac{1}{8} \tilde{g}(\mu) + 3 \tilde{g} \left( \frac{2\mu + \eta}{3} \right) + 3 \tilde{g} \left( \frac{\mu + 2\eta}{3} \right) + \tilde{g}(\eta) \right|
$$

$$
- \frac{3^{\alpha-1}\Gamma(\alpha + 1)}{(\mu - \eta)\alpha} \left[ J_{\mu+3}^\alpha \tilde{g} \left( \frac{2\mu + \eta}{3} \right) + J_{2\mu+3}^\alpha \tilde{g} \left( \frac{\mu + 2\eta}{3} \right) + J_{\mu+2\eta+3}^\alpha \tilde{g}(\eta) \right]
$$

$$
\leq \frac{(\mu - \eta)^2}{27(\alpha + 1)} \left[ \varphi_1(\alpha, p) \left( \frac{5|\tilde{g}''(\mu)|^q + |\tilde{g}''(\eta)|^q}{6} \right) + \varphi_2(\alpha, p) \left( \frac{|\tilde{g}''(\mu)|^q + |\tilde{g}''(\eta)|^q}{2} \right) \right]
$$

$$
+ \varphi_3(\alpha, p) \left( \frac{|\tilde{g}''(\mu)|^q + 5|\tilde{g}''(\eta)|^q}{6} \right)
$$

is valid. Here, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$
\varphi_1(\alpha, p) = \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{5(\alpha + 1)}{8} \xi + \frac{3 - \alpha}{8} \right|^p d\xi \right)^{\frac{1}{p}},
$$

$$
\varphi_2(\alpha, p) = \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{\alpha + 1}{2} \xi + \frac{3\alpha - 1}{8} \right|^p d\xi \right)^{\frac{1}{p}},
$$

$$
\varphi_3(\alpha, p) = \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{3(\alpha + 1)}{8} \xi + \frac{3\alpha - 1}{4} \right|^p d\xi \right)^{\frac{1}{p}}.
$$

Proof. Let us apply the Hölder inequality to inequality (2.9). Then, the following inequality holds:

$$
\left| \frac{\mu - \eta}{12(\alpha + 1)} [\tilde{g}'(\mu) + \tilde{g}'(\eta)] + \frac{1}{8} \tilde{g}(\mu) + 3 \tilde{g} \left( \frac{2\mu + \eta}{3} \right) + 3 \tilde{g} \left( \frac{\mu + 2\eta}{3} \right) + \tilde{g}(\eta) \right|
$$

$$
- \frac{3^{\alpha-1}\Gamma(\alpha + 1)}{(\mu - \eta)\alpha} \left[ J_{\mu+3}^\alpha \tilde{g} \left( \frac{2\mu + \eta}{3} \right) + J_{2\mu+3}^\alpha \tilde{g} \left( \frac{\mu + 2\eta}{3} \right) + J_{\mu+2\eta+3}^\alpha \tilde{g}(\eta) \right]
$$

$$
\leq \frac{(\mu - \eta)^2}{27(\alpha + 1)} \left[ \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{5(\alpha + 1)}{8} \xi + \frac{3 - \alpha}{8} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \left| \tilde{g}'' \left( \frac{2 + \xi}{3} - \mu \frac{1 - \xi}{3} \eta \right) \right|^q d\xi \right)^{\frac{1}{q}}
$$

$$
+ \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{\alpha + 1}{2} \xi + \frac{3\alpha - 1}{8} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \left| \tilde{g}'' \left( \frac{1 + \xi}{3} - \mu \frac{2 - \xi}{3} \eta \right) \right|^q d\xi \right)^{\frac{1}{q}}
$$

$$
+ \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{3(\alpha + 1)}{8} \xi + \frac{3\alpha - 1}{4} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \left| \tilde{g}'' \left( \frac{\xi}{3} - \mu \frac{3 - \xi}{3} \eta \right) \right|^q d\xi \right)^{\frac{1}{q}}
$$

Since $|\tilde{g}''|^q$ is convex, we have

$$
\left| \frac{\mu - \eta}{12(\alpha + 1)} [\tilde{g}'(\mu) + \tilde{g}'(\eta)] + \frac{1}{8} \tilde{g}(\mu) + 3 \tilde{g} \left( \frac{2\mu + \eta}{3} \right) + 3 \tilde{g} \left( \frac{\mu + 2\eta}{3} \right) + \tilde{g}(\eta) \right|
$$
\[-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\mu)^{\alpha}} \left[ \mathcal{J}_\mu^\alpha \hat{\gamma} \left( \frac{2\mu + \eta}{3} \right) + \mathcal{J}_{\frac{\mu + 2\eta}{3}}^\alpha \hat{\gamma} \left( \frac{\mu + 2\eta}{3} \right) + \mathcal{J}_{\frac{\mu + 3\eta}{3}}^\alpha \hat{\gamma} (\eta) \right] \]

\[ \leq \frac{(\eta - \mu)^2}{27(\alpha + 1)} \left[ \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{5(\alpha + 1)}{8} \xi + \frac{3 - \alpha}{\xi} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \frac{2 + \xi}{3} |\hat{\gamma}''(\mu)|^q + \frac{1 - \xi}{3} |\hat{\gamma}''(\eta)|^q d\xi \right)^{\frac{1}{q}} + \left( \int_0^1 \left| \frac{\alpha + 1}{2} \xi + \frac{3\alpha - 1}{8} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1 + \xi}{3} |\hat{\gamma}''(\mu)|^q + \frac{2 - \xi}{3} |\hat{\gamma}''(\eta)|^q d\xi \right)^{\frac{1}{q}} \right] \]

\[ = \frac{(\eta - \mu)^2}{27(\alpha + 1)} \left[ \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{5(\alpha + 1)}{8} \xi + \frac{3 - \alpha}{\xi} \right|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1 + \xi}{3} |\hat{\gamma}''(\mu)|^q + \frac{2 - \xi}{3} |\hat{\gamma}''(\eta)|^q d\xi \right)^{\frac{1}{q}} \right] \]

which completes the proof of Theorem 4.

\[ \square \]

**Theorem 5.** Suppose that the assumptions of Lemma 1 hold and the function $|\hat{\gamma}''|^{q}$ is convex on $[\mu, \eta]$ for $q \geq 1$. Then, it follows

\[ \left| \frac{(\eta - \mu)(1 - \alpha)}{12(\alpha + 1)} [\hat{\gamma}'(\mu) + \hat{\gamma}'(\eta)] + \frac{1}{8} [\hat{\gamma}(\mu) + 3\hat{\gamma} \left( \frac{2\mu + \eta}{3} \right) + \hat{\gamma} \left( \frac{\mu + 2\eta}{3} \right) + \hat{\gamma}(\eta)] \right| \]

\[ - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\mu)^{\alpha}} \left[ \mathcal{J}_\mu^\alpha \hat{\gamma} \left( \frac{2\mu + \eta}{3} \right) + \mathcal{J}_{\frac{\mu + 2\eta}{3}}^\alpha \hat{\gamma} \left( \frac{\mu + 2\eta}{3} \right) + \mathcal{J}_{\frac{\mu + 3\eta}{3}}^\alpha \hat{\gamma} (\eta) \right] \]

\[ \leq \frac{(\eta - \mu)^2}{27(\alpha + 1)} \left[ (\Omega_1(\alpha))^{1-\frac{1}{q}} \left( \frac{2\Omega_1(\alpha) + \Omega_4(\alpha)}{3} |\hat{\gamma}''(\mu)|^q + (\Omega_1(\alpha) - \Omega_4(\alpha)) |\hat{\gamma}''(\eta)|^q \right)^{\frac{1}{q}} + (\Omega_2(\alpha))^{1-\frac{1}{q}} \left( \frac{\Omega_2(\alpha) + \Omega_5(\alpha)}{3} |\hat{\gamma}''(\mu)|^q + (2\Omega_2(\alpha) - \Omega_5(\alpha)) |\hat{\gamma}''(\eta)|^q \right)^{\frac{1}{q}} \right] \]
If we apply the power-mean inequality to inequality (2.9), then we obtain

Proof. If we apply the power-mean inequality to inequality (2.9), then we obtain

\[
\left| \frac{(\eta - \mu)(1 - \alpha)}{12(\alpha + 1)} \left[ 3\Omega_\alpha (\alpha) \left| \Omega'' (\mu) \right|^q + \left( 3\Omega_\alpha (\alpha) - \Omega_\delta (\alpha) \right) \left| \Omega'' (\eta) \right|^q \right] \right| \leq \frac{3^\alpha \Gamma (\alpha + 1)}{(\eta - \mu)^\alpha} \left[ \mathcal{J}_\mu^\alpha + 3\mathcal{J}_\mu^\alpha + 3\mathcal{J}_\mu^\alpha + \mathcal{J}_\delta (\eta) \right]\]

\[
\leq \frac{(\eta - \mu)^2}{27(\alpha + 1)} \left[ \left( \int_0^1 \xi^\alpha + 1 - 5(\alpha + 1) - 3 - \alpha \right| \xi^\alpha \left( \frac{2 + \xi}{3} - \frac{1 - \xi}{3} \right) \right]^\frac{1}{\alpha} \left| \mathcal{D}'' (\xi^\alpha \left( \frac{2 + \xi}{3} - \frac{1 - \xi}{3} \right) \right| \right]\]

\[
\times \left( \int_0^1 \xi^\alpha + 1 - \frac{\alpha + 1}{2} \xi + \frac{3\alpha - 1}{8} \right| \xi^\alpha \left( \frac{1 + \xi}{3} - \frac{2 - \xi}{3} \right) \right]^\frac{1}{\alpha} \left| \mathcal{D}'' (\xi^\alpha \left( \frac{1 + \xi}{3} - \frac{2 - \xi}{3} \right) \right| \right]\]

\[
\times \left( \int_0^1 \xi^\alpha + 1 - 3(\alpha + 1) - 3(\alpha - 1) \right| \xi^\alpha \left( \frac{3 - \xi}{4} \right) \right]^\frac{1}{2} \left| \mathcal{D}'' (\xi^\alpha \left( \frac{3 - \xi}{4} \right) \right| \right]\]

From the fact that \( \left| \mathcal{D}'' \right|^q \) is convex, we have

\[
\left| \frac{(\eta - \mu)(1 - \alpha)}{12(\alpha + 1)} \left[ 3\Omega_\alpha (\alpha) \left| \Omega'' (\mu) \right|^q + \left( 3\Omega_\alpha (\alpha) - \Omega_\delta (\alpha) \right) \left| \Omega'' (\eta) \right|^q \right] \right| \leq \frac{3^\alpha \Gamma (\alpha + 1)}{(\eta - \mu)^\alpha} \left[ \mathcal{J}_\mu^\alpha + 3\mathcal{J}_\mu^\alpha + 3\mathcal{J}_\mu^\alpha + \mathcal{J}_\delta (\eta) \right]\]

\[
\leq \frac{(\eta - \mu)^2}{27(\alpha + 1)} \left[ \left( \int_0^1 \xi^\alpha + 1 - 5(\alpha + 1) - 3 - \alpha \right| \xi^\alpha \left( \frac{2 + \xi}{3} - \frac{1 - \xi}{3} \right) \right]^\frac{1}{\alpha} \left| \mathcal{D}'' (\xi^\alpha \left( \frac{2 + \xi}{3} - \frac{1 - \xi}{3} \right) \right| \right]\]
\begin{align*}
\times & \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{5}{8}(\alpha+1)\xi + \frac{3-\alpha}{8} \left\{ \frac{2+\xi}{3} |\varphi''(\mu)|^q + \frac{1-\xi}{3} |\varphi''(\eta)|^q \right\} d\xi \right) \\
& + \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{\alpha+1}{2}\xi + \frac{3\alpha-1}{8} \right| d\xi \right)^{1-\frac{1}{q}} \\
& \times \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{\alpha+1}{2}\xi + \frac{3\alpha-1}{8} \right| \left\{ \frac{1+\xi}{3} |\varphi''(\mu)|^q + \frac{2-\xi}{3} |\varphi''(\eta)|^q \right\} d\xi \right)^{\frac{1}{q}} \\
& + \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{3(\alpha+1)}{8}\xi + \frac{3(\alpha-1)}{4} \right| d\xi \right)^{1-\frac{1}{q}} \\
& \times \left( \int_0^1 \left| \xi^{\alpha+1} - \frac{3(\alpha+1)}{8}\xi + \frac{3(\alpha-1)}{4} \right| \left\{ \frac{\xi}{3} |\varphi''(\mu)|^q + \frac{3-\xi}{3} |\varphi''(\eta)|^q \right\} d\xi \right)^{\frac{1}{q}} \right].
\end{align*}

= \frac{(\eta-\mu)^2}{27(\alpha+1)} \left[ (\Omega_1(\alpha))^{1-\frac{1}{q}} \left( \frac{2\Omega_1(\alpha) + \Omega_4(\alpha)}{3} |\varphi''(\mu)|^q + \left( \Omega_1(\alpha) - \Omega_4(\alpha) \right) |\varphi''(\eta)|^q \right)^{\frac{1}{q}} \\
+ (\Omega_2(\alpha))^{1-\frac{1}{q}} \left( \frac{\Omega_2(\alpha) + \Omega_3(\alpha)}{3} |\varphi''(\mu)|^q + \left( 2\Omega_2(\alpha) - \Omega_3(\alpha) \right) |\varphi''(\eta)|^q \right)^{\frac{1}{q}} \\
+ (\Omega_3(\alpha))^{1-\frac{1}{q}} \left( \frac{\Omega_6(\alpha)}{3} |\varphi''(\mu)|^q + \left( 3\Omega_3(\alpha) - \Omega_6(\alpha) \right) |\varphi''(\eta)|^q \right)^{\frac{1}{q}} \right].

Thus, we obtain the desired result of Theorem 5. \qed

Remark 3. Let us consider \( \alpha = 1 \) in Theorem 5. Then, the following Newton type inequality holds:

\begin{align*}
\left| \frac{1}{8} \left[ \varphi(\mu) + 3\varphi \left( \frac{2\mu + \eta}{3} \right) + \varphi \left( \frac{\mu + 2\eta}{3} \right) + \varphi(\eta) \right] - \frac{1}{(\eta-\mu)} \int_{\mu}^{\eta} \varphi(\xi) d\xi \right|
\leq \frac{(\eta-\mu)^2}{54} \left[ \left( \frac{19}{192} \right) \left( \frac{125 |\varphi''(\mu)|^q + 27 |\varphi''(\eta)|^q}{24 \cdot 64} \right)^{\frac{1}{q}} \\
+ \left( \frac{27 |\varphi''(\mu)|^q + 125 |\varphi''(\eta)|^q}{24 \cdot 64} \right)^{\frac{1}{q}} \right] + \frac{1}{12} \left( \frac{|\varphi''(\mu)|^q + |\varphi''(\eta)|^q}{2} \right)^{\frac{1}{q}},
\end{align*}

which is given by [6, Theorem 2.2].
3. Conclusion

Some perturbed Newton type inequalities are established to the case of twice-differentiable convex functions by using Riemann-Liouville fractional integrals. With the help of the special cases of mentioned results above, we show that our results reduce to inequalities proved in earlier paper. In the future works, mathematicians can try to make our results more general by using a different version of convex operating classes or another type of fractional integral operators.

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