RECIPROCITY LAWS FOR DEDEKIND COTANGENT SUMS

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ABSTRACT. Let $a_0,\ldots,a_d$ be pairwise coprime positive integers, $m_0,\ldots,m_d$ nonnegative integers, and $z_0,\ldots,z_d$ complex numbers. We study expressions of the form

$$\frac{1}{a_0^{m_0+1}}\sum_k\prod_{j=1}^{d}\cot^{(m_j)}\pi\big(a_j\frac{k+z_0}{a_0}-z_j\big),$$

the summation is taken over all $k \pmod{a_0}$ for which the summand is well defined. These sums generalize and unify various arithmetic sums introduced and studied by Dedekind, Apostol, Carlitz, Zagier, Berndt, Meyer, Sczech, Dieter and Beck. Special cases of these sums appear in various areas such analytic and algebraic number theory, topology, algebraic and combinatorial geometry, and algorithmic complexity. In this paper, without any additional assumption on the parameters $a_0,\ldots,a_d$ and $z_0,\ldots,z_d$, we present a simple proof for the reciprocity formula for these generalized Dedekind cotangents sums. We recover the previous known results and improve them. As applications, we give explicit formulae for sums of secant and cosecant values in terms of Apostol-Bernoulli numbers.

1. Introduction and preliminaries

Let d be a positive integer $\geqslant 2$, $a_0,\ldots,a_d$ be positive integers, $m_0,\ldots,m_d$ be non-negative integers and $z_0,\ldots,z_d$ complex numbers. We set

$$\vec{A_i}=(a_i\,;a_0,\ldots,\widehat{a_i},\ldots,a_d),\ \vec{M_i}=(m_i\,;m_0,\ldots,\widehat{m_i},\ldots,m_d),\ \vec{Z_i}=(z_i\,;z_0,\ldots,\widehat{z_i},\ldots,z_d)$$

where the notation $\widehat{x_i}$ means that the term x_i is omitted. Let $\cot^{(m)}$ denote, as usual, the m^{th}-derivative of the cotangent function. For $i=0,\ldots,d$, we consider generalized multiple Dedekind sums defined by

$$C_d(\vec{A_i},\vec{M_i},\vec{Z_i}) := \frac{1}{a_i^{m_i+1}}\sum_k{}^{*i}\prod_{\substack{j=0\\ j\neq i}}^{d}\cot^{(m_j)}\pi\big(a_j\frac{k+z_i}{a_i}-z_j\big) \tag{1}$$

where the summation $\sum_k^{*i}$ is over all $k \pmod{a_i}$ for which $\cot^{(m_j)}\pi\big(a_j\frac{k+z_i}{a_i}-z_j\big)$ is defined.

Throughout this paper, we take

$$x_i=\Re e\,(z_i)\in[0,1[\,(i=0,\ldots,d)$$

without loss of generality in the definition (1) because the function $x\mapsto\cot^{(m_j)}(\pi x)$ is 1-periodic.
The Dedekind sums and its generalizations have intrigued mathematicians from various areas such as analytic [1, 23, 25] and algebraic [31, 38] number theory, topology [27, 28, 32, 41], algebraic [17, 34, 40] and combinatorial geometry [11, 24, 33], and algorithmic complexity [29]. The Dedekind cotangent sums include as special cases various generalized Dedekind sums introduced by Rademacher [35], Apostol [2], Carlitz [19], Zagier [41], Berndt [13], Meyer, Sczech [32], and Dieter [26]. The most fundamental and important theorems for any of the generalized Dedekind sums are the reciprocity laws: an appropriate sum of generalized Dedekind sums (usually permuting the arguments in a cyclic fashion) gives a simple

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rational expression.

In [41], Zagier studies the following Dedekind sums

$$d(a_0;a_1,\dots,a_d) = (-1)^{d/2}\sum_{k=1}^{a_0-1}\prod_{j=1}^{d}\cot\pi\big(a_j\frac{k}{a_0}\big).$$

With the the notation introduced in (1), these sums can be written in the form

$$d(a_0;a_1,\dots,a_d) = (-1)^{d/2}\, a_0\, C_d(\overrightarrow{A_0},\overrightarrow{M_0},\overrightarrow{Z_0}),\ \text{with}\ \overrightarrow{M_0}=\overrightarrow{Z_0}=\overrightarrow{0}\ .$$

Theorem 1 (Zagier [41])**.** *Let d be an even integer, $a_0,\dots,a_d$ be pairwise coprime positive integers. Then we have*

$$\sum_{i=0}^{d} C_d(\overrightarrow{A_i},\overrightarrow{0},\overrightarrow{0}) = (-1)^{d/2}\left(1-\frac{\ell_d(a_0,\dots,a_d)}{a_0\dots a_d}\right) \tag{2}$$

where $\ell_d(a_0,\dots,a_d)$ is the polynomial defined as the coefficient of t^d in the power series expansion of

$$\prod_{j=0}^{d}\frac{a_jt}{\tanh(a_jt)} = \prod_{j=0}^{d}\left(1+\frac{1}{3}a_j^2t^2-\frac{1}{45}a_j^4t^4+\dots\right).$$

Remark. We quote from Zagier [41, §3, p.158] that

$$\ell_d(a_o,\dots,a_d) = L_k(p_1,\dots,p_k),$$

where $k=d/2$ and p_i $(i=1,\dots,k)$ is the i-th elementary symmetric polynomial in $a_0^2,\dots,a_d^2$, and L_k is the k-th Hirzebruch L-polynomial. The first few polynomials L_k are

$$L_0=1,\ L_1(p_1)=p_1/3,\ L_2(p_1,p_2)=(-p_1^2+7p_2)/45,$$
$$L_3(p_1,p_2,p_3)=(2p_1^3-13p_1p_2+62p_3)/945.$$

Take $d=2$, we obtain the classical Dedekind sums :

$$\begin{aligned} C_2(\overrightarrow{A_0},\overrightarrow{0},\overrightarrow{0}) &= \frac{1}{a_0}\sum_{k=1}^{a_0-1}\cot\left(\frac{\pi k a_1}{a_0}\right)\cot\left(\frac{\pi k a_2}{a_0}\right) \\ &=:\ 4\mathfrak{s}(a_0;a_1,a_2) \end{aligned} \tag{3}$$

where $\mathfrak{s}(a_0;a_1,a_2)$ is the classical Dedekind-Rademacher sum [8, 9, 10, 13, 26]. This sum was introduced by Rademacher as a generalization of the well- known Dedekind sums :

$$\mathfrak{s}(a_0;a_1) = \mathfrak{s}(a_0;a_1,1) = \frac{1}{4}C_2(\overrightarrow{A_0},\overrightarrow{0},\overrightarrow{0}). \tag{4}$$

The first results about reciprocity law for the original Dedekind sums are :

Theorem 2 (Dedekind [23])**.** *If $a_0,a_1\in\mathbb{N}$ are relatively prime then*

$$\mathfrak{s}(a_0;a_1)+\mathfrak{s}(a_1;a_0) = -\frac{1}{4}+\frac{1}{12}\left(\frac{a_0}{a_1}+\frac{1}{a_0a_1}+\frac{a_1}{a_0}\right)\ .$$

Theorem 3 (Rademacher [35])**.** *If $a_0,a_1,a_2\in\mathbb{N}$ are pairwise coprime then*

$$\mathfrak{s}(a_0;a_1,a_2)+\mathfrak{s}(a_1;a_2,a_0)+\mathfrak{s}(a_2;a_0,a_1) = -\frac{1}{4}+\frac{1}{12}\left(\frac{a_1a_2}{a_0}+\frac{a_2a_0}{a_1}+\frac{a_0a_1}{a_2}\right)\ .$$

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In a previous work, we have studied this reciprocity law in a general context. Let us recall the definition of the k-th Bernoulli numbers B_k given by

$$\frac{z}{e^z-1}=\sum_{k\geqslant 0}\frac{B_k}{k!}z^k. \tag{5}$$

Theorem 4 (Bayad-Raouj [9])**.** *Let d be a positive integer, $a_0,\dots,a_d$ be pairwise coprime positive integers and $m_0,\dots,m_d$ be non-negative integers. Assume that the integer*

$$M=d+m_0+\cdots+m_d \text{ is even.}$$

Then we have

$$\sum_{i=0}^{d}(-1)^{m_i}m_i!\sum_{\substack{\ell_0,\dots,\widehat{\ell_i},\dots,\ell_d\geqslant 0\\ \ell_0+\cdots+\widehat{\ell_i}+\cdots+\ell_d=m_i}}\Big(\prod_{\substack{j=0\\ j\neq i}}^{d}\frac{a_j^{\ell_j}}{\ell_j!}\Big)C_d(\vec{A_i},\vec{M_i}+\vec{L_i},\vec{0})=\begin{cases}R+(-1)^{d/2} & \text{if } m_0=\cdots=m_d=0,\\ R & \text{otherwise}\end{cases}$$

with $\vec{L_i}=(\mathbf{0}\,;l_0,\dots,\widehat{l_i},\dots,l_d)$,

$$R=-\frac{(-1)^{M/2}2^M}{\prod_{i=0}^{d}a_i^{m_i+1}}\sum_{\substack{j_0,\dots,j_d\geqslant 0\\ j_0+\cdots+j_d=M/2}}\prod_{i=0}^{d}a_i^{2j_i}A_{i,j_i} \tag{6}$$

and

$$A_{i,j_i}=\begin{cases}\frac{B_{2j_i}}{(2j_i-1-m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \geqslant (m_i+1)/2,\\ (-1)^{m_i}m_i! & \text{if } j_i=0,\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Remark. According to the proof of the above theorem in [9], the condition on the parity of M is not necessary. In fact, one can take $R=0$ when M is odd.

On the other hand, when $(z_0,\dots,z_d)\neq(0,\dots,0)$, the multiple generalized Dedekind sums (1), is studied independently by Dieter ($d=2$, $m_0=m_1=m_2=0$) and Beck (for arbitrary d, $m_0,\dots,m_d$). We review Dieter's and Beck's results. Let

$$\delta(x)=\begin{cases}1 & \text{if } x\in\mathbb{Z},\\ 0 & \text{otherwise}\end{cases}\quad \textit{and}\; c_2(z)=\begin{cases}-\cot'(\pi z) & \text{if } z\notin\mathbb{Z},\\ \frac{1}{3} & \text{otherwise.}\end{cases}$$

Theorem 5 (Dieter [26])**.** *Let $a_0,a_1,a_2\in\mathbb{N}$ be pairwise relatively prime, and define A,B,C by*

$$Aa_1a_2+Ba_2a_0+Ca_0a_1=1\;.$$

Let z_0,z_1,z_2 be real numbers with $0\leqslant z_0,z_1,z_2<1$ and set

$$x'=a_2z_1-a_1z_2\;,\; y'=a_0z_2-a_2z_0\;,\; z'=a_1z_0-a_0z_1.$$

Then

$$C_2(\vec{A_0},\vec{0},\vec{Z_0})+C_2(\vec{A_1},\vec{0},\vec{Z_1})+C_2(\vec{A_2},\vec{0},\vec{Z_2})=-1+R_0+R_1+R_2$$

where

$$\begin{aligned} R_0 &= \frac{a_0}{a_1a_2}\delta(x')c_2\left(Ba_0y'-(Ca_0+Aa_2)z'\right),\\ R_1 &= \frac{a_1}{a_0a_2}\delta(y')c_2\left(Ca_1z'-(Aa_1+Ba_0)x'\right),\\ R_2 &= \frac{a_2}{a_0a_1}\delta(z')c_2\left(Aa_2x'-(Ba_2+Ca_1)y'\right). \end{aligned}$$

Theorem 6 (Beck [10]). *Let $a_0,\dots,a_d\in\mathbb{N}$, $m_0,\dots m_d\in\mathbb{N}_0$, $z_0,\dots,z_d\in\mathbb{C}$. Under the assumption that for all distinct $i,j\in\{0,\dots,d\}$ and all $m,n\in\mathbb{Z}$,*

$$\frac{m+z_i}{a_i}-\frac{n+z_j}{a_j}\notin\mathbb{Z}, \tag{8}$$

we have

$$\sum_{n=0}^{d}(-1)^{m_n}m_n!\sum_{\substack{l_0,\dots,\widehat{l_n},\dots,l_d\geqslant 0\\ l_0+\cdots+\widehat{l_n}+\cdots+l_d=m_n}}\frac{a_0^{l_0}\cdots\widehat{a_n^{l_n}}\cdots a_d^{l_d}}{l_0!\cdots\widehat{l_n!}\cdots l_d!}\,C_d(\vec{A_i},\vec{M_i+L_i},\vec{Z_i})=\begin{cases}(-1)^{d/2} & \text{if all } m_k=0 \text{ and } d \text{ even},\\ 0 & \text{otherwise}\end{cases}$$

with $\vec{L_i}=(\mathbf{0}\,;l_0,\dots,\widehat{l_i},\dots,l_d)$.

In this paper, we establish a generalization of Dieter's result (Theorem 5) and Beck's result (Theorem 6) by removing the condition (8). New illustrations to secant and cosecant sums will be also given.

2. Statement of main results

We first recall the definitions of Apostol-Bernoulli polynomials and numbers, and then we introduce new invariants necessary to state our main results.

2.1. Apostol-Bernoulli polynomials and numbers.

For this subsection we refer to [3], [7, Section 4] . Let λ be complex number. The n^{th} Apostol-Bernoulli polynomials are defined by the generating function :

$$\frac{t\,e^{xt}}{\lambda\,e^{t}-1}=\sum_{n\geqslant 0}B_n(x;\lambda)\frac{t^n}{n!}\qquad (|t+\log\lambda|<2\pi). \tag{9}$$

The number $B_n(0;\lambda)$ is called the n^{th} Apostol-Bernoulli number. When $\lambda=1$, $B_n(0;1)=B_n$ is the n^{th} Bernoulli number. For $\lambda\neq 1$, we have $B_0(0;\lambda)=0$,$B_1(0;\lambda)=\frac{1}{\lambda-1}$, and for $n\geqslant 2$ we have

$$\lambda B_n(0;\lambda)=n\sum_{j=0}^{n-1}(-1)^j j!S(n-1,j)\left(\frac{\lambda}{\lambda-1}\right)^{j+1}$$

where $S(n,j)$ are the Stirling numbers of second kind. In the sequel we use the following numbers

$$\overline{B}_n(0;\lambda)=\begin{cases}B_n(0;\lambda) & \text{if } n\neq 1,\\ B_1(0;\lambda)+1/2 & \text{if } n=1.\end{cases} \tag{10}$$

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2.2. **Invariants :** $\Delta_{i,j}, \Gamma_{i,j,l}, \mathcal{P}$**.** Let $a_0, \dots, a_d$ be pairwise coprime positive integers. Let us set

$$\Delta_{ij} := a_i z_j - a_j z_i \qquad (0 \leqslant i, j \leqslant d), \tag{11}$$

and for any non-empty set $I \subset \{0, \dots, d\}$ with $|I| \geqslant 2$ we denote

$$\Delta_I := (\Delta_{ij})_{i,j \in I}$$

which, by definition, is antisymmetric matrix. For distinct indices i, j and ℓ, let A_i, A_j and A_ℓ be integers such that

$$A_\ell a_i a_j + A_i a_j a_\ell + A_j a_\ell a_i = 1. \tag{12}$$

We introduce

$$\Gamma_{i,j,\ell} \;\; := \;\; - \begin{vmatrix} 0 & a_j & a_\ell \\ A_i & A_j & A_\ell \\ \Delta_{\ell i} & \Delta_{j\ell} & 0 \end{vmatrix} = -A_i a_\ell \Delta_{j\ell} + (A_j a_\ell + A_\ell a_j)\Delta_{\ell i}, \tag{13}$$

and for $\bar{I} := \{0, \dots, d\} \smallsetminus I \neq \emptyset$,

$$\Gamma_{\bar{I}} := (\Gamma_{s_I, s'_I, \ell})_{\ell \in \bar{I}} \text{ where } s_I := \min(I), s'_I := \min\left(I \smallsetminus \{s_I\}\right).$$

The following set

$$\mathcal{P} := \left\{ I \subset \{0, \dots, d\} : |I| \geqslant 2,\ \Delta_I \in \mathscr{M}_{|I|}(\mathbb{Z}) \ \text{ and } \ \Gamma_{\bar{I}} \in (\mathbb{R} \smallsetminus \mathbb{Z})^{|\bar{I}|} \text{ if } \bar{I} \neq \emptyset \right\} \tag{14}$$

will be considered.
By Lemma 5 below we have

$$\begin{aligned} \mathcal{P} &= \left\{ I \subset \{0, \dots, d\} : |I| \geqslant 2, \forall i \neq j \in I, \Delta_{i,j} \in \mathbb{Z}, \text{ and } \text{ if } \bar{I} \neq \emptyset, (\Delta_{i,\ell}, \Delta_{j,\ell}) \notin \mathbb{Z} \times \mathbb{Z}, \forall \ell \in \bar{I} \right\} \\ &= \left\{ I \subset \{0, \dots, d\} : |I| \geqslant 2,\ \Delta_I \in \mathscr{M}_{|I|}(\mathbb{Z}) \text{ and } \text{ if } \bar{I} \neq \emptyset, (\Delta_{s_I,\ell}, \Delta_{s'_I,\ell}) \notin \mathbb{Z} \times \mathbb{Z}, \forall \ell \in \bar{I} \right\}. \end{aligned}$$

The properties of the invariants $\Delta_{i,j}, \Gamma_{i,j,l}, \mathcal{P}$ will be studied in section 3.

2.3. **Main result.** We set

$$C_{j_k}(m_k) = \begin{cases} (-1)^{m_k} m_k! & \text{if } j_k = 0, \\ \frac{(-1)^{j_k} 2^{2j_k - 1} B_{2j_k}}{(2j_k - 1 - m_k)!\, j} & \text{if } j_k \text{ integer } \geqslant (m_k + 1)/2, \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

and we denote

$$\widetilde{C}_{m_k + j_k + 1}(m_k; \theta) = \frac{(2i)^{m_k + j_k + 1}}{j!(m + j + 1)} \overline{B}_{m_k + j_k + 1}(0; e^{2\pi i \theta}), \qquad \theta \in \mathbb{R}. \tag{16}$$

For $I \in \mathcal{P}$, let

$$R_I := -\sum_{j_0, \dots, j_d}^{(1)} \prod_{i \in I} C_{j_i}(m_i) a_i^{2j_i - m_i - 1} \prod_{k \notin I} a_k^{j_k} \widetilde{C}_{m_k + j_k + 1}(m_k; \Gamma_{s_I, s'_I, k}) \tag{17}$$

where the sum $\sum\limits_{j_0, \dots, j_d}^{(1)}$ is over all integers $j_0 \geqslant 0, \dots, j_d \geqslant 0$ for which

$$2\sum_{i \in I} j_i + \sum_{i \notin I} j_i = \sum_{i \in I} (m_i + 1) - 1.$$

The main result of this paper is the following reciprocity theorem for the generalized Dedekind cotangents sums (1).

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Theorem 7 (Main result)**.** *Let d be a positive integer $\geqslant 2$, $a_0,\dots,a_d$ pairwise coprime positive integers, $m_0,\dots,m_d$ non-negative integers and $z_0,\dots,z_d$ complex numbers. We have the reciprocity law*

$$\sum_{n=0}^{d}(-1)^{m_n}m_n!\sum_{l_0+\cdots+\widehat{l_n}+\cdots+l_d=m_n}\Big(\prod_{\substack{j=0\\ j\neq n}}^{d}\frac{a_j^{l_j}}{l_j!}\Big)C_d(\vec{A_n},\vec{M_n}+\vec{L_n},\vec{Z_n})=\varepsilon(d;m_0,\dots,m_d)+\sum_{I\in\mathcal{P}}R_I,$$

where $\vec{L_n}=(\mathbf{0};l_0,\dots,\widehat{l_n},\dots,l_d)$ *and*

$$\varepsilon(d;m_0,\dots,m_d)=\begin{cases}(-1)^{d/2} & \text{if } m_0=\cdots=m_d=0 \text{ and } d \text{ even},\\ 0 & \text{otherwise.}\end{cases}$$

In sections 4 and 6 we give several applications of our main result. Its proof is given in section 5.

3. Algebraic properties of the invariants $\Delta_{i,j},\Gamma_{i,j,\ell}$ and $\mathcal{P}$

Let d be a positive integer $\geqslant 2$, $a_0,\dots,a_d$ pairwise coprime positive integers, and $z_0,\dots,z_d$ complex numbers.

Lemma 1. *Let i,j be two different positive integers in $\{0,\dots,d\}$. We assume that there exist $(k_i,k_j)\in[0,a_i[\times[0,a_j[$ such that $\frac{k_i+z_i}{a_i}=\frac{k_j+z_j}{a_j}$. Then, for all $\ell\in\{0,\dots,d\}$, we have*

$$a_\ell\frac{k_i+z_i}{a_i}-z_\ell\equiv\Gamma_{i,j,\ell}\pmod 1.$$

Proof. We see that $\Delta_{j\ell}$, $\Delta_{\ell i}$ and Δ_{ij} satisfy the following dependency relation :

$$a_i\Delta_{j\ell}+a_j\Delta_{\ell i}+a_\ell\Delta_{ij}=0. \tag{18}$$

Therefore, the relation $\frac{k_i+z_i}{a_i}=\frac{k_j+z_j}{a_j}$ implies that

$$k_ia_j\equiv\Delta_{ij}\pmod{a_i}.$$

Since a_0,a_1 and a_2 are pairwise coprime, we have the equality (12). Hence

$$k_i\equiv A_ia_\ell\Delta_{ij}\pmod{a_i}$$

and

$$\begin{aligned}a_\ell\frac{(k_i+z_i)}{a_i}-z_\ell &= \frac{1}{a_i}(a_\ell k_i+a_\ell z_i-a_iz_\ell)\\ &= \frac{1}{a_i}(a_\ell^2A_i\Delta_{ij}+\Delta_{\ell i})\pmod 1\\ &= \frac{1}{a_i}(-a_\ell A_ia_i\Delta_{j\ell}-a_\ell A_ia_j\Delta_{\ell i}+\Delta_{\ell i})\pmod 1\\ &= a_\ell A_i\Delta_{j\ell}+(A_ja_\ell+A_\ell a_j)\Delta_{\ell i}\pmod 1.\end{aligned}$$

This completes the proof of the lemma. □
Consider the function

$$f(z)=\prod_{i=0}^{d}f_i(z)\text{ where } f_i(z)=\cot^{(m_i)}\pi(a_iz-z_i).$$

Lemma 2. *Let $I\subset\{0,\dots,d\}$ with $|I|\geqslant 2$. Then*

(i) *The matrix $\Delta_I\in\mathscr{M}_{|I|}(\mathbb{Z})$ if and only if there exist a unique w_I common pole of all $f_i,(i\in I)$ with $\Re e\,(w_I)\in[0,1[$;*

(ii) *if* $I \subsetneq \{0,\dots,d\}$ *and* $\Delta_I \in \mathscr{M}_{|I|}(\mathbb{Z})$, *we have*

$$\Gamma_{\bar{I}} \in (\mathbb{R} \smallsetminus \mathbb{Z})^{|\bar{I}|} \Longleftrightarrow \{k=0,\dots,d : w_I \text{ pole of } f_k\} = I.$$

Proof. We prove (i). It's clear that if the functions $f_i (i \in I)$ have the common pole $w = w_I$, then for all pair (i,j) of distinct integers of I, there exist $(k_i,k_j) \in \mathbb{Z}^2$ such that

$$w = \frac{k_i + z_i}{a_i} = \frac{k_j + z_j}{a_j}.$$

Therefore $\Delta_{ij} = a_i z_j - a_j z_i \in \mathbb{Z}$.
Conversely, we assume that for all pair (i,j) of distinct integers of I, $\Delta_{ij} = a_i z_j - a_j z_i \in \mathbb{Z}$. We shall prove that all $f_i (i \in I)$ have a common pole. By Bezout's identity, there exist $(k_i,k_j) \in \mathbb{Z}^2$ such that

$$k_i a_j - k_j a_i = a_i z_j - a_j z_i.$$

Therefore $w = \frac{k_i + z_i}{a_i} = \frac{k_j + z_j}{a_j}$ is a common pole to f_i and f_j. From Lemma 1, we have $a_\ell w - z_\ell$ is an integer for all $\ell \in I \smallsetminus \{i,j\}$. This implies that w is a common pole of f_ℓ for all $\ell \in I$.
Let us now prove the uniqueness of the pole w. In fact, we check that if (i,j) is a 2-uplet of distinct integers in $\{0,\dots,d\}$, then f_i and f_j have at most one common pole. If we take w, w' two common poles of f_i and f_j then

$$\begin{aligned} w &= \frac{k_i + z_i}{a_i} = \frac{k_j + z_j}{a_j}, \\ w' &= \frac{k_i' + z_i}{a_i} = \frac{k_j' + z_j}{a_j} \end{aligned}$$

with k_i and k_i' integers in $[0, a_i[$, k_j and k_j' integers in $[0, a_j[$. Thus

$$\frac{k_i - k_i'}{a_i} = \frac{k_j - k_j'}{a_j}$$

equivalently

$$a_i (k_j - k_j') = a_j (k_i - k_i').$$

Therefore $k_i = k_i'$ and $k_j = k_j'$ since a_i and a_j are coprime.
This prove the property (i). The part (ii) is an immediate consequence of (i) and Lemma 1.□

Remark. By use of Lemma 1 and Lemma 2, we have

$$\Gamma_{s_I, s_I', \ell} \equiv \Gamma_{i,j,\ell} \pmod 1$$

for all $I \in \mathcal{P} \smallsetminus \{\{0,\dots,d\}\}$ and for all $(i,j,\ell) \in I \times I \times \bar{I}$, with $i \neq j$. Therefore in the formula (17) and the equation (14), we can use any two different $i \neq j$ in I, instead of s_I and s_I'.

Lemma 3. *Let* I_1 *and* I_2 *are two elements of* $\mathcal{P}$. *Then*

$$|I_1 \cap I_2| \geqslant 2 \Longleftrightarrow I_1 = I_2.$$

In other words, for any distinct elements I_1 *and* I_2 *of* $\mathcal{P}$, *we have* $|I_1 \cap I_2| \leqslant 1$.

Proof. We put $I = I_1 \cap I_2$. By the Lemma 2, there exist a common pole w_I for $f_i, i \in I$. By the uniqueness of w_I, we obtain $w_I = w_{I_1}$ and $w_I = w_{I_2}$. Thus, by ii) of Lemma 1, the set $\{k = 0,\dots,d : w_I \text{ pole of } f_k\}$ is equal to I_1 and also to I_2. □

Lemma 4. *Let* $d \geqslant 2$ *and* i,j,k *three distinct integers in* $[0,d]$. *Assume that* $(\Delta_{i,j}\ ,\ \Gamma_{i,j,k}) \in \mathbb{Z}^2$. *Then*

(i) $(\Delta_{i,k},\Delta_{j,k}) \in \mathbb{Z}\times\mathbb{Z}$.
(ii) *There exists unique set $I \in \mathcal{P}$ such that : $i,j,k \in I$.*

Proof. First the uniqueness is obvious by Lemma 2. From definitions (12), (11) and (13) we have

$$A_k a_i a_j + A_i a_j a_k + A_j a_k a_i = 1, \tag{19}$$

$$a_i \Delta_{j,k} + a_j \Delta_{k,i} + a_k \Delta_{i,j} = 0, \tag{20}$$

$$\Gamma_{i,j,k} = -A_i a_k \Delta_{j,k} + (A_j a_k + A_k a_j)\Delta_{k,i} \in \mathbb{Z}. \tag{21}$$

Multiply by $\Delta_{i,k}$ (resp. $\Delta_{j,k}$) the equation 19 , and use the equations (21) and (20), we obtain

$$\begin{aligned}
\Delta_{i,k} &= a_i\,(A_k a_j + A_j a_k)\,\Delta_{i,k} + A_i a_j a_k \Delta_{i,k},\\
&\equiv -a_i A_i a_k \Delta_{j,k} + A_i a_j a_k \Delta_{i,k} \pmod 1\\
&\equiv -A_i a_k \left(a_i \Delta_{j,k} + a_j \Delta_{k,i}\right) \pmod 1\\
&\equiv A_i a_k^2 \Delta_{i,j} \pmod 1\\
&\equiv 0 \pmod 1;
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{j,k} &= a_j\left(A_i a_k \Delta_{j,k}\right) + a_i\,(A_j a_k + A_k a_j)\,\Delta_{j,k},\\
&\equiv a_j\,(A_j a_k + A_k a_j)\,\Delta_{k,i} + a_i\,(A_j a_k + A_k a_j)\,\Delta_{j,k} \pmod 1,\\
&\equiv (A_j a_k + A_k a_j)\left(a_i \Delta_{j,k} + a_j \Delta_{k,i}\right) \pmod 1\\
&\equiv -a_k\,(A_j a_k + A_k a_j)\,\Delta_{i,j} \pmod 1\\
&\equiv 0 \pmod 1.
\end{aligned}$$

Therefore $\Delta_{i,j}, \Delta_{k,i}, \Delta_{j,k} \in \mathbb{Z}$. This implies, by the use of Lemma 2, the existence of $I \in \mathcal{P}$ such that $i,j,k \in I$.

An immediate consequence of the preceding lemma is the following result.

Lemma 5. *Assume that $\mathcal{P} \neq \emptyset$ and $I \subset \{0,\cdots,d\}$ with $|\, I \,| \geqslant 2$. Then we have*

$$I \in \mathcal{P} \iff \begin{cases} \Delta_{i,j} \in \mathbb{Z},\ \forall\, i,j \in I, \\ (\Delta_{i,k},\Delta_{j,k}) \notin \mathbb{Z}\times\mathbb{Z},\ \forall i \neq j \in I, k \notin I. \end{cases}$$

Remark. We note that

(i) $\mathcal{P} = \emptyset \iff \Delta_{i,j} \notin \mathbb{Z}$, for all distinct $0 \leqslant i,j \leqslant d$;
(ii) $\mathcal{P} = \{I\}$, with $2 \leqslant |\, I \,| \leqslant d+1 \iff \Delta_{i,j} \in \mathbb{Z}$ for any $i,j \in I$, and $\Delta_{i,j} \notin \mathbb{Z}$ for any $0 \leqslant i < j \leqslant d$ such that $\{i,j\} \not\subset I$;
(iii) in general, we have: $\mathcal{P} = \{I_1,\ldots,I_m\}$, with $2 \leqslant |\, I_k \,| \leqslant d+1$ $(k = 1,\ldots,m)$ and $|I_k \cap I_{k'}| \leqslant 2 (1 \leqslant k < k' \leqslant m) \iff \forall k \in \{1,\ldots,m\} : \Delta_{i,j} \in \mathbb{Z}$ for any $i,j \in I_k$ and $\Delta_{i,j} \notin \mathbb{Z}$ for any $0 \leqslant i < j \leqslant d$ such that $\{i,j\} \not\subset I_k$.

4. Further illustrations and examples

In this section we apply our study to some few special cases.

4.1. Case $\mathcal{P}=\emptyset$. In this case, $\Delta_{i,j}\notin\mathbb{Z}$ for all distinct $0\leqslant i,j\leqslant d$. Then, since the integers $a_0,\dots,a_d$ are pairwise coprime, $\frac{k_i+z_i}{a_i}-\frac{k_j+z_j}{a_j}\notin\mathbb{Z}$ for all $k_i,k_j\in\mathbb{Z}$ and for all distinct $0\leqslant i,j\leqslant d$.
Furthermore, Theorem 7 gives

$$\sum_{n=0}^{d}(-1)^{m_n}m_n!\sum_{l_0+\cdots+\widehat{l_n}+\cdots+l_d=m_n}\Big(\prod_{\substack{i=0\\ i\neq n}}^{d}\frac{a_i^{l_i}}{l_i!}\Big)C_d(\vec{A_n},\vec{M_n}+\vec{L_n},\vec{Z_n})=$$

$$\begin{cases}(-1)^{d/2} & \text{if } m_0=\cdots=m_d=0, \text{ and } d \text{ even},\\ 0 & \text{otherwise}\end{cases}$$

with $\vec{L_n}=(\mathbf{0}\,;l_0,\dots,\widehat{l_n},\dots,l_d)$.
This is exactly Beck's result [10](Theorem 6).

4.2. Case $\mathcal{P}=\{I\}$ with $I=\{0,\dots,d\}$. From Theorem 7 we obtain

$$\sum_{n=0}^{d}(-1)^{m_n}m_n!\sum_{l_0+\cdots+\widehat{l_n}+\dots l_d=m_n}\Big(\prod_{\substack{i=0\\ i\neq n}}^{d}\frac{a_i^{l_i}}{l_i!}\Big)C_d(\vec{A_n},\vec{M_n}+\vec{L_n},\vec{Z_n})= \tag{22}$$

$$\begin{cases}(-1)^{d/2}\left[1-2^d\displaystyle\sum_{j_0+\cdots+j_d=d/2}\prod_{i=0}^{d}\frac{B_{2j_i}}{(2j_i)!}a_i^{2j_i-1}\right] & \text{if } m_0=\cdots=m_d=0 \text{ and } d \text{ even},\\ R & \text{otherwise}\end{cases}$$

where

$$R=\begin{cases}-\displaystyle\sum_{j_0+\cdots+j_d=M/2}\prod_{i=0}^{d}C_{j_i}(m_i)a_i^{2j_i-m_i-1} & \text{if } M:=d+\sum_{i=0}^{d}m_i \text{ even},\\ 0 & \text{otherwise.}\end{cases} \tag{23}$$

and $\vec{L_n}=(\mathbf{0}\,;l_0,\dots,\widehat{l_n},\dots,l_d)$.

The formula (22) is a reformulation of the Zagier's result [41, Equation (47), p.158] : the polynomial function $\ell_d(a_0,\dots,a_d)$ is explicitly given in terms of Bernoulli numbers

$$\ell_d(a_0,\dots,a_d)=2^d\sum_{j_0+\cdots+j_d=d/2}\prod_{i=0}^{d}\frac{B_{2j_i}}{(2j_i)!}a_i^{2j_i}.$$

4.3. Case $d=2$. By using Lemma 2 and Lemma 3, we describe all possibilities for $\mathcal{P}$, and give the corresponding reciprocity laws. We have

(i) $\mathcal{P}=\emptyset$ if and only if $\Delta_{i,j}\notin\mathbb{Z}$, for all distinct $0\leqslant i\,,j\leqslant 2$.
We note that for $\mathcal{P}=\emptyset$ the Theorem 7 is completely described in the subsection 4.1 for any $d\geqslant 2$.

(ii) $\mathcal{P}=\{I\}$ with $I=\{0,1,2\}$ if and only if $\Delta_{0,1},\Delta_{0,2}\in\mathbb{Z}$. Then the term R_I in Theorem 7 is given by the formula 17.

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(iii) $\mathcal{P}=\{I\}$ with $I=\{i,j\}, i\neq j$ if and only if $\Delta_{i,j}\in\mathbb{Z}$ and $\Delta_{i,k}\notin\mathbb{Z}$, for $k\neq i,j$.

• In fact, if $|I|=2$, we have $I=\{0,1\},\{0,2\}$ or $\{1,2\}$ and expressions

$$R_{\{0,1\}}=-\sum_{\substack{2(j_0+j_1)+j_2=\\1+m_0+m_1}} C_{j_0}(m_0)C_{j_1}(m_1)\widetilde{C}_{m_2+j_2+1}(m_2;\Gamma_{0,1,2})a_0^{2j_0-m_0-1}a_1^{2j_1-m_1-1}a_2{}^{j_2},$$

$$R_{\{0,2\}}=-\sum_{\substack{2(j_0+j_2)+j_1=\\1+m_0+m_2}} C_{j_0}(m_0)\,C_{j_2}(m_2)\,\widetilde{C}_{m_1+j_1+1}(m_1;\Gamma_{0,2,1})\,a_0^{2j_0-m_0-1}a_2^{2j_2-m_2-1}a_1{}^{j_1},$$

$$R_{\{1,2\}}=-\sum_{\substack{2(j_1+j_2)+j_0=\\1+m_1+m_2}} C_{j_1}(m_1)C_{j_2}(m_2)\,\widetilde{C}_{m_0+j_0+1}(m_0;\Gamma_{1,2,0})\,a_1^{2j_1-m_1-1}a_2^{2j_2-m_2-1}a_0{}^{j_0}.$$

We obtain for $m_0=m_1=m_2=0$, by use of Lemma 6 that

$$\begin{aligned}R_{\{0,1\}}&=-\tfrac{a_2}{a_0a_1}\cot'(\pi\Gamma_{0,1,2}),\\R_{\{0,2\}}&=-\tfrac{a_1}{a_0a_2}\cot'(\pi\Gamma_{0,2,1}),\\R_{\{1,2\}}&=-\tfrac{a_0}{a_1a_2}\cot'(\pi\Gamma_{1,2,0}).\end{aligned}\tag{24}$$

Note that in the case $d=2$,$m_0=m_1=m_2=0$, we recover Theorem 5 proved in Dieter's paper [26]:

$$C_2(\overrightarrow{A_0},\overrightarrow{0},\overrightarrow{Z_0})+C_2(\overrightarrow{A_1},\overrightarrow{0},\overrightarrow{Z_1})+C_2(\overrightarrow{A_2},\overrightarrow{0},\overrightarrow{Z_2})=-1+R_0+R_1+R_2.$$

5. **Proof of the main result**

Let us prove a lemma connecting Apostol-Bernoulli numbers and derivatives of the cotangent.

5.1. **Apostol-Bernoulli polynomials and numbers.**

The Apostol-Bernoulli polynomials and numbers have many interesting combinatorial interpretations. Moreover, they can be written as Fourier series at $x=0$. In fact, from [7] we have

$$\lambda^x B_n(x;\lambda)=-\frac{n!}{(2\pi i)^n}\sum_k^{(*)}\frac{e^{2\pi ikx}}{(k-\frac{\log\lambda}{2\pi i})^n},\qquad(0\leqslant x<1,n\in\mathbb{N})$$

where the summation $\sum_k^{(*)}$ is over $k\in\mathbb{Z}$ if $\lambda\neq1$, and $k\in\mathbb{Z}\setminus\{0\}$ otherwise.

Lemma 6. *For any non-negative integer n, we have*

$$\cot^{(n)}(\pi z)=\frac{(2i)^{n+1}}{n+1}\overline{B}_{n+1}(0;e^{2\pi iz})\qquad(z\notin\mathbb{Z}).$$

Proof.

• **Method 1:** We have the following Taylor expansion

$$\pi z\cot(\pi z+\pi\theta)=\sum_{n\geqslant0}\frac{\cot^{(n)}(\pi\theta)}{n!}(\pi z)^{n+1}.$$

On the other hand

$$\pi z\cot(\pi z+\pi\theta)=\pi iz\frac{e^{2\pi i(z+\theta)}+1}{e^{2\pi i(z+\theta)}-1}=\pi iz+\frac{2\pi iz}{e^{2\pi i\theta}\,e^{2\pi iz)}-1}.$$

Thus

$$\pi z\cot(\pi(z+\theta))=\pi iz+\sum_{n\geqslant0}B_n(0;e^{2\pi i\theta})\frac{(2\pi iz)^n}{n!},$$

or $B_0(0;\mathrm{e}^{2\pi i\theta})=0$ for all $\theta\notin\mathbb{Z}$, therefore

$$\sum_{n\geqslant 0}\frac{\cot^{(n)}(\pi\theta)}{n!}(\pi z)^{n+1}=\pi iz+\sum_{n\geqslant 0}B_{n+1}(0;\mathrm{e}^{2\pi i\theta})\frac{(2\pi iz)^{n+1}}{(n+1)!}.$$

By comparing the coefficients of z^{n+1} in the both sides of the above equation; we obtain the lemma.

• **Method 2:** Using Fourier expansions :

$$B_n(x;\mathrm{e}^{2\pi i\theta})=\frac{-n!}{(2\pi i)^n}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi ikx}}{(k-\theta)^n}\qquad(0\leqslant x<1)$$

and

$$(-1)^n\cot^{(n)}(\pi\theta)=\frac{-n!}{\pi^{n+1}}\sum_{k\in\mathbb{Z}}^{(*)}\frac{1}{(k+\theta)^{n+1}}.$$

For $x=0$, we obtain the desired formula. □

5.2. **Proof of Theorem 7.** Let X and Y be positive real numbers. We consider the rectangular path $\gamma:=[A,B,C,D,A]$ where $A=X+Yi$, $B=X-1+Yi$. $C=\overline{B}$ and $D=\overline{A}$. We choose X and Y such that all poles of f are inside γ. By the Cauchy' residue theorem, we get

$$\sum_{w\text{ pole of }f}\mathrm{Res}(f(z),z=w)=\frac{1}{2\pi i}\int_{\gamma}f(z)\,\mathrm{d}z=\begin{cases}\frac{(-1)^{d/2}}{\pi} & \text{if all } m_i \text{ are zero and } d \text{ is even,}\\ 0 & \text{otherwise,}\end{cases}\tag{25}$$

(for details, see [9, p.340]). Let w be a pole of f. There is a nonempty set $I=I_w$ of $\{0,\dots,d\}$ such that for all $i\in I$, w is a pole of f_i, and for all $\ell\in\overline{I}=\{0,\dots,d\}\setminus I$, the function f_ℓ is analytic at the point w. Let

$$M_I=\sum_{i\in I}(m_i+1)-1.$$

It is clear that by definition $\mathrm{Res}(f(z),z=w)$ is the coefficient of $(z-w)^{M_I}$ in the Taylor expansion of

$$\prod_{i\in I}(z-w)^{m_i+1}\cot^{(m_i)}(\pi a_i(z-w))\prod_{i\in\overline{I}}\cot^{(m_i)}(\pi a_i(z-w)).$$

Then, by writing

$$\cot^{(m_i)}(\pi a_i(z-w))=\sum_{j\geqslant 0}\frac{1}{j!}(\pi a_i)^j\cot^{(m_i+j)}(\pi(a_iw-z_i))(z-w)^j\qquad(i\notin I),$$

and

$$(z-w)^{m_i+1}\cot^{(m_i)}(\pi a_i(z-w))=\sum_{j\geqslant 2}C_j(m_i)(\pi a_i)^{2j-m_i-1}(z-w)^{2j}\qquad(i\in I)$$

where the definition of $C_j(m_i)$ is given by (15), we obtain

$$\mathrm{Res}(f(z),z=w)=\tag{26}$$

$$\tfrac{1}{\pi}\sum_{2\sum_{i\in I}j_i+\sum_{i\notin I}j_i=M_I}\Big(\prod_{i\in I}C_{j_i}(m_i)a_i^{2j_i-m_i-1}\Big)\Big(\prod_{i\notin I}\frac{a_i^{j_i}}{j_i!}\Big)\prod_{i\notin I}\cot^{(m_i+j_i)}\pi(a_iw-z_i).$$

On one hand, by Lemma 2, for any $I \in \mathcal{P}$, there exists a unique pole $w = z_I$ of f such that $I = I_w$, so

$$\begin{aligned}\sum_{\substack{w \text{ pole of } f\\ |I_w|\geqslant 2}} \operatorname{Res}(f(z), z = w) &= \frac{1}{\pi}\sum_{I\in\mathcal{P}} \sum_{j_0,\dots,j_d}^{(1)} \prod_{i\in I} C_{j_i}(m_i) a_i^{2j_i - m_i - 1} \prod_{i\notin I} \frac{a_i^{j_i}}{j_i!} \prod_{i\notin I} \cot^{(m_i + j_i)} \pi(a_i w_I - z_i)\\ &= \frac{1}{\pi}\sum_{I\in\mathcal{P}} \sum_{j_0,\dots,j_d}^{(1)} \prod_{i\in I} C_{j_i}(m_i) a_i^{2j_i - m_i - 1} \prod_{i\notin I} \frac{a_i^{j_i}}{j_i!} \prod_{i\notin I} \cot^{(m_i + j_i)} (\pi \Gamma_{s_I, s_I', i})\end{aligned}$$

where we have used Lemma 1.
On the other hand, with $\vec{L_n} = (\mathbf{0}\,; l_0, \dots, \widehat{l_n}, \dots, l_d)$ we have

$$\sum_{n=0}^{d} \sum_{\substack{w \text{ pole of } f\\ I_w = \{n\}}} \operatorname{Res}(f(z), z = w) =$$

$$\frac{1}{\pi}\sum_{n=0}^{d} (-1)^{m_n} m_n! \sum_{l_0 + \dots + \widehat{l_n} + \dots l_d = m_n} \Big(\prod_{\substack{i=0\\ i\neq n}}^{d} \frac{a_i^{l_i}}{l_i!}\Big) C_d(\vec{A_n}, \vec{M_n} + \vec{L_n}, \vec{Z_n}).$$

The proof of the theorem is thus completed by using Lemma 6. □

6. New trigonometric identities

6.1. **Notations.** Let d, $m_0, \cdots, m_d$ be nonnegative integers,. We denote

$$\varepsilon(d) = \begin{cases} (-1)^{d/2} & \text{if } d \text{ even,}\\ 0 & \text{otherwise}\end{cases}$$

and

$$A_{i,j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i - 1 - m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \geqslant (m_i + 1)/2,\\ (-1)^{m_i} m_i! & \text{if } j_i = 0,\\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

For every real $\alpha > 0$, let J_α be Jordan's totient function defined for all positive integer n by

$$J_\alpha(n) := n^\alpha \sum_{m|n} \frac{\mu(m)}{m^\alpha},$$

where μ is the Mobius function. Since the arithmetical function $J_\alpha(n)/n^\alpha$ is multiplicative, we can write

$$J_\alpha(n) = n^\alpha \prod_{\substack{p|n\\ p \text{ prime}}} \left(1 - \frac{1}{p^\alpha}\right), \text{ see [36, p.11,p.219].}$$

For $\alpha = 1$, this is, of course, Euler's function φ.

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6.2. **Arithmetic sums.** In this part we fix some parameters. $m_0 = 0$, $a_1 = \cdots = a_d = 1$, $z_0 = \cdots = z_d = 0$, $M = d + m_0 = \cdots + m_d$. Then $\mathscr{P} = \{I\}$ with $I = \{0, \cdots, d\}$. By Theorem 7 we obtain :

Theorem 8. *For $a_0 \geqslant 2$ integer, we get*

$$\sum_{t=1}^{a_0-1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\frac{\pi t}{a_0}\right) = \varepsilon(d; m_0, \ldots, m_d) a_0 + S \tag{28}$$

where

$$S = \begin{cases} -(2i)^M \displaystyle\sum_{\substack{j_0,\ldots,j_d \geqslant 0 \\ j_0+\cdots+j_d = M/2}} \left(\prod_{i=0}^{d} A_{i,j_i}\right) a_0^{2j_0} & \text{if } M \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

By the Möbius inversion formula we obtain the following

Corollary 1. *For $a_0 \geqslant 2$ integer, we get*

$$\sum_{(t,a_0)=1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\frac{\pi t}{a_0}\right) = \varepsilon(d; m_0, \ldots, m_d) \varphi(a_0) + W \tag{29}$$

where

$$W = \begin{cases} -(2i)^M \displaystyle\sum_{\substack{j_0,\ldots,j_d \geqslant 0 \\ j_0+\cdots+j_d = M/2}} \left(\prod_{i=0}^{d} A_{i,j_i}\right) J_{2j_0}(a_0) & \text{if } M \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

For $m_0 = \cdots = m_d = 0$ we obtain

Corollary 2. *For $a_0 \geqslant 2$, d be an even integer, we have*

$$\sum_{t=1}^{a_0-1} \cot\left(\frac{\pi t}{a_0}\right)^d = (-1)^{d/2} a_0 - (2i)^d \sum_{j_0+\cdots+j_d = d/2} \left(\prod_{k=0}^{d} \frac{B_{2j_k}}{(2j_k)!}\right) a_0^{2j_0}, \tag{30}$$

and

$$\sum_{(t,a_0)=1} \cot\left(\frac{\pi t}{a_0}\right)^d = (-1)^{d/2} \varphi(a_0) - (2i)^d \sum_{j_0+\cdots+j_d = d/2} \left(\prod_{k=0}^{d} \frac{B_{2j_k}}{(2j_k)!}\right) J_{2j_0}(a_0), \tag{31}$$

(By convention, theses sums vanish if d is odd).

Different proofs and different formulations of the identity (30) have been given by Berndt in [14, (2.16),p.364], Chu and Marini [21, p. 137] and D. Cvijović and J. Klinowski [22].

For $m_0 = 0, m_1 \cdots = m_d = 0$ we obtain

Corollary 3.

$$\sum_{t=1}^{a_0-1} \frac{1}{\sin\left(\frac{\pi t}{a_0}\right)^{2d}} = -2^{2d} \sum_{j_0+\cdots+j_d = d} \left(\prod_{k=0}^{d} \frac{(2j_k - 1) B_{2j_k}}{(2j_k)!}\right) a_0^{2j_0}; \tag{32}$$

$$\sum_{(t,a_0)=1} \frac{1}{\sin\left(\frac{\pi t}{a_0}\right)^{2d}} = -2^{2d} \sum_{j_0+\cdots+j_d = d} \left(\prod_{k=0}^{d} \frac{(2j_k - 1) B_{2j_k}}{(2j_k)!}\right) J_{2j_0}(a_0). \tag{33}$$

The sum (33) can be used to give an elementary new proof for Stark's conjecture on the rational field $\mathbb{Q}$. See Sczech [39].

6.3. **Arithmetic sums with parameter.** We fix $m_0 = 0$, $a_1 = \cdots = a_d = 1$, $z_1 = \cdots = z_d = 0$, $z_0 \notin \mathbb{Z}$, and $M = d + m_0 = \cdots + m_d$. Then $\mathscr{P} = \{I\}$ with $I = \{1, \cdots, d\}$. We will apply Theorem 7 and Möbius inversion formula to obtain the following identities.

Theorem 9. *For $a_0 \geqslant 2$ integer and $z_0 \in \mathbb{C} \setminus \mathbb{Z}$, $m_1, \cdots, m_d$ be nonnegative integers, we get*

$$\frac{1}{a_0} \sum_{t=0}^{a_0-1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi \frac{z_0 + t}{a_0}\right) = \varepsilon(d; m_0, \ldots, m_d) + R(a_0, z_0), \tag{34}$$

with

$$R(a_0, z_0) = -(2i)^M \sum_{j_0+2(j_1 \cdots + j_d) = M-1} \left(\prod_{i=1}^{d} A_{i,j_i}(m_i)\right) \frac{\overline{B}_{j_0+1}(0; e^{2\pi i z_0})}{(j_0+1)!} a_0^{j_0}. \tag{35}$$

$$\sum_{(t,a_0)=1} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi \frac{z_0 + t}{a_0}\right) = \varepsilon(d; m_0, \ldots, m_d)\varphi(a_0) + V, \tag{36}$$

$$V = -(2i)^M \sum_{j_0+2(j_1 \cdots + j_d) = M-1} \left(\prod_{i=1}^{d} A_{i,j_i}(m_i)\right) \left(\sum_{m|a_0} \mu(m)(a_0/m)^{j_0+1} \frac{\overline{B}_{j_0+1}(0; e^{2\pi i z_0/m})}{(j_0+1)!}\right).$$

Proof. Let us prove the identity (36). We consider the function

$$g(a_0) = \sum_{t \bmod a_0} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi z_0 + \frac{\pi t}{a_0}\right).$$

We have the equalities

$$\begin{aligned} g(a_0) &= \sum_{m|a_0} \sum_{\substack{(t,a_0)=m \\ 1 \leqslant t \leqslant a_0}} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi z_0 + \frac{\pi t}{a_0}\right) \\ &= \sum_{m|a_0} \sum_{\substack{(t/m, a_0/m)=1 \\ 1 \leqslant t/ \leqslant a_0/m}} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi z_0 + \frac{\pi t/m}{a_0/m}\right) \\ &= \sum_{r|a_0} \sum_{\substack{(k,r)=1 \\ 1 \leqslant k \leqslant r}} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi z_0 + \frac{\pi k}{r}\right). \end{aligned}$$

By writting $g(a_0) = \sum_{r|a_0} f(r)$ with $f(r) = \sum_{\substack{(k,r)=1 \\ 1 \leqslant k \leqslant r}} \prod_{i=1}^{d} \cot^{(m_i)}\left(\pi z_0 + \frac{\pi k}{r}\right)$ and by Möbius inversion formula, we see

$$\begin{aligned} f(a_0) &= \sum_{m|a_0} \mu(m) g\left(\frac{a_0}{m}\right) \\ &= \sum_{m|a_0} \mu(m) \frac{a_0}{m}\, \varepsilon(d; m_0, \ldots, m_d) + \sum_{m|a_0} \mu(m) \frac{a_0}{m}\, R(a_0/m, a_0 z_0) \\ &= \varepsilon(d; m_0, \ldots, m_d)\varphi(a_0) - (2i)^M \sum_{j_0+2(j_1 \cdots + j_d) = M-1} \left(\prod_{i=1}^{d} A_{i,j_i}(m_i)\right) \left(\sum_{m|a_0} \mu(m)(a_0/m)^{j_0+1} \frac{\overline{B}_{j_0+1}(0; e^{2\pi i \frac{a_0}{m} z_0})}{(j_0+1)!}\right). \end{aligned}$$

Now, in the above formula replace z_0 by z_0/a_0 to obtain our desired formula. □

Corollary 4. *For $a_0 \geqslant 2$ integer and $z_0 \in \mathbb{C} \setminus \mathbb{Z}$, we get*

$$\frac{1}{a_0}\sum_{t=0}^{a_0-1}\cot\left(\pi\frac{z_0+t}{a_0}\right)^d = \varepsilon(d)+R_I\ , \tag{37}$$

with

$$R_I = -(2i)^d \sum_{j_0+2(j_1\cdots+j_d)=d-1}\left(\prod_{i=1}^{d}\frac{B_{2j_i}}{(2j_i)!}\right)\frac{\overline{B}_{j_0+1}(0;e^{2\pi i z_0})}{(j_0+1)!}a_0^{j_0}.$$

Corollary 5. *For $a_0 \geqslant 3$ odd integer and $z_0 \in \mathbb{C} \setminus \mathbb{Z}$, we get for d even*

$$\frac{1}{a_0}\sum_{t=0}^{a_0-1}\tan\left(\frac{\pi t}{a_0}\right)^d = \varepsilon(d)+R\ , \tag{38}$$

with

$$R = -(2i)^d \sum_{j_0+2(j_1\cdots+j_d)=d-1}\left(\prod_{i=1}^{d}\frac{B_{2j_i}}{(2j_i)!}\right)\frac{\overline{B}_{j_0+1}(0;-1)}{(j_0+1)!}a_0^{j_0},$$

and

$$B_n(0;-1) = \sum_{j=0}^{n-1}(-1)^{j+1}j!2^{-(j+1)}S(n-1,j)\cdot$$

Corollary 6. *For $a_0 \geqslant 2$ integer and $z_0 \in \mathbb{C} \setminus \mathbb{Z}$, we get*

$$\sum_{t=0}^{a_0-1}\frac{1}{\sin\left(\pi\frac{z_0+t}{a_0}\right)^{2d}} = -2^{2d}\sum_{j_0+2(j_1\cdots+j_d)=2d-1}\left(\prod_{i=1}^{d}\frac{(2j_i-1)B_{2j_i}}{(2j_i)!}\right)\frac{\overline{B}_{j_0+1}(0;e^{2\pi i z_0})}{(j_0+1)!}a_0^{j_0+1}.$$

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